Part III Algebraic Geometry

Example Sheet IV, 2022.

1. Let D be a degree 0 Weil divisor on $\mathbb{P}^1_k$. Construct a rational function f, i.e. a section of the structure sheaf over the generic point, such that the divisor associated to f is D. Deduce that every degree 0 divisor on $\mathbb{P}^1_k$ is principal.

2. Let k be a field and consider the *quadric cone* $X = \operatorname{Spec} k[x,y,z]/(xy - z^2)$. Let $Z = \mathbb{V}(x,z)$. In this problem you may assume that X is normal (or take it as a commutative algebra exercise). Prove that Z is an integral closed codimension 1 subscheme. Prove that $X \setminus Z$ is the spectrum of a unique factorization domain. Using the excision exact sequence, prove that $[Z]$ generates the class group $Cl(X)$.

3. Continuing in the context of the previous problem, prove that the class $[Z]$ is not 0 in the class group, but that $2[Z]$ is 0. Deduce that the class group of the cone $Cl(X)$ is $\mathbb{Z}/2$.

4. ($\star$) Let X be an integral separated scheme of finite type over a field k, regular in codimension 1. Prove that the class group of X and $X \times \mathbb{A}^1$ are isomorphic.

5. Assuming the conclusion of the previous question, prove that $Cl(X \times \mathbb{P}^n) = Cl(X) \oplus \mathbb{Z}$. Calculate the class group of $\mathbb{P}^1 \times \mathbb{P}^1$.

6. In lectures, we sketched the construction of a line bundle associated to a Cartier divisor. You may review this construction by examining the two definitions before Propositions 6.11 and 6.13 in Chapter II of Hartshorne. Let H be any hyperplane in $\mathbb{P}^n_k$. Prove that the sheaf associated to the Cartier divisor $[H]$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(1)$, defined earlier via graded modules[1].

7. Let f be a homogeneous polynomial of degree d in $n+1$ variables and let D be the scheme theoretic vanishing locus of f in $\mathbb{P}^n$. Let $\mathcal{I}_D$ be sheaf of ideals (i.e. the subsheaf of the structure) whose sections over U are those functions that vanish on $D \cap U$. Construct an isomorphism between $\mathcal{I}_D$ and $\mathcal{O}_{\mathbb{P}^n}(-d)$. Describe the quotient sheaf $\mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_D$.

8. Let $X = \operatorname{Spec} A$ be affine, and let
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
be an exact sequence of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that
$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$
is exact. [You may freely use the following fact about quasi-coherent sheaves: if $\mathcal{F}$ is a quasi-coherent sheaf on a scheme X, and $U \subseteq X$ is open affine, then $\mathcal{F}|_U$ can be written as a cokernel of a morphism between free sheaves on U. This is an analogue of the type of result concerning locally Noetherian schemes and locally finite type morphisms. These definitions state the existence of an affine open cover with certain properties and then one shows that the required properties hold for any affine open. You may find a proof of this fact in Hartshorne II, §5.]

9. Prove that an injective sheaf of abelian groups on a topological space is *flasque*, i.e. all restriction maps are surjective. [This has very googleable answer, so try it first before doing that.]

10. Let X be the vanishing of a homogeneous polynomial in $\mathbb{P}^n_k$, with $n \geq 3$. Using the cohomology of projective space, the exact sequence in Problem 7, and the long exact sequence in cohomology, show that $H^1(X, \mathcal{O}_X)$ vanishes.

10. Let X be the closed subscheme of $\mathbb{P}^2_k$ defined by the scheme theoretic vanishing locus of a homogeneous polynomial of degree d. Assume that $(1,0,0) \notin X$. Then show X can be covered by the two affine open subsets $U = X \cap D_+(x_1)$, $V = X \cap D_+(x_2)$. Now compute the Čech complex explicitly and show that
$$\begin{aligned} \dim H^0(X, \mathcal{O}_X) &= 1 \\ \dim H^1(X, \mathcal{O}_X) &= (d-1)(d-2)/2 \end{aligned}$$
where d is the degree of f. Compare with the degree-genus formula for a plane curve.

[1] Beware that sign conventions vary when it comes to $\mathcal{O}(1)$ and $\mathcal{O}(-1)$, but only one will make any sense given the context, so it's not a serious source of confusion.

11. Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, $U = X \setminus \{(x, y)\}$ (removing the maximal ideal corresponding to the origin). By choosing a suitable affine cover of U, show that $H^1(U, \mathcal{O}_U)$ is naturally isomorphic to the infinite dimensional k-vector space with basis $\{x^i y^j \mid i, j < 0\}$. Thus in particular U is not affine.

The following questions introduce the global version of the Proj construction, parallel to the global Spec construction in the previous sheet, and allow us to introduce blowups, which are very cool. They can all be considered as "starred" questions.

12. Let X be a Noetherian scheme and $\mathcal{A}_\bullet$ be a coherent sheaf of $\mathcal{O}_X$ algebras. We make the standard assumption that $\mathcal{A}_1$ is coherent and that $\mathcal{A}_1$ locally generates the algebra $\mathcal{A}_\bullet$ [i.e. locally, the algebra of sections over an open is generated in degree 1]. By direct analogy with the construction of global Spec on the previous example sheet, construct global Proj

$$\pi : \underline{\operatorname{Proj}}\, \mathcal{A}_\bullet \to X,$$

such that over an affine scheme U, $\underline{\operatorname{Proj}}\, \mathcal{A}_\bullet$ is the usual Proj applied to the graded ring $\mathcal{A}_\bullet(U)$. Observe that the fibres of π can be described by using the usual Proj construction. Construct the global analogue of the sheaf $\mathcal{O}(1)$, i.e. construct a line bundle on $\mathcal{L}$ on $\underline{\operatorname{Proj}}\, \mathcal{A}_\bullet$ such that, for each point x of X, the restriction of $\mathcal{L}$ to the fibre $\pi^{-1}(x)$ is the "usual" $\mathcal{O}(1)$ on that fibre.

13. Let X be a noetherian scheme as above and let $\mathcal{A}_\bullet = \mathcal{O}_X[T_0, \ldots, T_n]$ be the sheaf of algebras whose value over an affine U is a polynomial algebra on $n + 1$ variables over the ring of functions on U. Give it a grading by requiring each variable to have weight 1. Show that the global Proj applied to $\mathcal{A}_\bullet$ gives

$$\underline{\operatorname{Proj}}\, \mathcal{O}_X[T_0, \ldots, T_n] = \mathbb{P}^1_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} X.$$

This is the most boring instance of the global Proj construction.

14. **Not actually a question.** Let X be Noetherian and let $\mathcal{I}$ be a coherent sheaf of ideals. Consider the graded algebra

$$\mathcal{A}_\bullet := \bigoplus_{d \geq 0} \mathcal{I}^d.$$

This algebra satisfies the simplifying assumption in Question 5, so we can apply global Proj. Define the *blowup of X along $\mathcal{I}$* to be

$$\operatorname{Bl}_{\mathcal{I}} X := \underline{\operatorname{Proj}}\, \mathcal{A}_\bullet.$$

Try to understand this construction.

15. (The blowup of affine space; hard but crucial) Let A be the ring $k[X_1, \ldots, X_n]$ and let I be the ideal $\langle X_1, \ldots, X_n \rangle$. By using the surjection

$$A[Y_1, \ldots, Y_n] \to \bigoplus_{d \geq 0} I^d$$

sending Y_i to X_i, identify the blowup of $\mathbb{A}^n_k$ at I with a closed subscheme in $\mathbb{P}^{n-1}_A$. Let $n = 2$ and let X denote the blowup above in this case. Show that the fiber of

$$X \to \mathbb{A}^2_k$$

over the point $(0, 0)$ is naturally identified with $\mathbb{P}^1$.