Part III Algebraic Geometry

Example Sheet III, 2022.

1. Let Z be the closed subscheme of $\mathbb{P}^2_{\mathbb{C}[\![t]\!]}$ given by the vanishing locus of the homogeneous ideal $(xy - tz^2)$, where x, y, z are the homogeneous coordinate functions on $\mathbb{P}^2_{\mathbb{C}[\![t]\!]}$. (i) Calculate[1] the scheme theoretic fibers of Z over each of the two points of $\operatorname{Spec} \mathbb{C}[\![t]\!]$. (ii) For each integer $n \geq 1$, inductively define Z^n to be the fiber product of Z with Z^{n-1} over $\operatorname{Spec} \mathbb{C}[\![t]\!]$. Calculate the fiber of Z^n over the closed point of $\operatorname{Spec} \mathbb{C}[\![t]\!]$. Calculate the number of irreducible components of this closed fiber as a function of n.

2. Let X be the scheme $\operatorname{Spec} \mathbb{Q}(i)$ over the base $\operatorname{Spec} \mathbb{Q}$. Is this scheme X connected? Consider the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Q}$ and calculate the base change

$$Y := \operatorname{Spec} \mathbb{Q}(i) \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \mathbb{C}.$$

Is Y connected?

3. ($\star$) Let k be a field. Let V be the 4-dimensional vector space of 2×2-matrices with entries in k. Analogously, view $k[x, y, z, w]$ as the graded ring of polynomial functions on V. Prove that the set of *rank* 1 matrices in V is Zariski closed defined by an ideal I, and therefore has the structure of an affine scheme. Prove that $I \subset k[x, y, z, w]$ is a *homogeneous* ideal. Let $Z \subset \mathbb{P}^3_k$ be the resulting closed subscheme. Prove that Z is isomorphic to $\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k$.

4. Let $X \to S$ be a closed immersion and let $Y \to S$ be any morphism. Prove that the base change $X \times_S Y \to Y$ is also a closed immersion. Deduce that closed immersions are proper. In particular, deduce that any closed subscheme $Z \subset \mathbb{P}^n_k$ is proper over $\operatorname{Spec} k$.[2]

5. Let k be a field and let A be any finitely generated k-algebra. Explicitly verify the valuative criterion of separatedness for the morphism $\operatorname{Spec} A \to \operatorname{Spec} k$. Explicitly check that if A is a polynomial ring over k, then $\operatorname{Spec} A \to \operatorname{Spec} k$ fails the valuative criterion for universal closedness.

6. Let X be the line with 2-origins over $\operatorname{Spec} k$, i.e. glue two copies of $\mathbb{A}^1_k$ to itself by identifying the complement of the origin in the obvious way. Identify the fiber product $X \times_{\operatorname{Spec} k} X$ by describing it as a gluing of affine schemes.

7. Explicitly verify that the valuative criterion for universal closedness fails for the scheme X constructed in the previous problem, for the morphism to $\operatorname{Spec} k$.

8. By using the valuative criterion of properness, prove that (i) compositions of proper morphisms are proper, and (ii) if $X \to S$ be proper and $Y \to S$ is any morphism, then the base change $X \times_S Y \to Y$ is also proper. Deduce that $\mathbb{P}^n_k \times_{\operatorname{Spec} k} \mathbb{P}^m_k$ is proper over $\operatorname{Spec} k$.

   (If you need help on this problem, look up *Segre embedding* on Wikipedia.)

9. In lecture, we have claimed that given a module M over a ring A, there is an associated sheaf M^{sh} on $\operatorname{Spec}(A)$ whose value over a distinguished open U_f is identified with the localization of M at f. We have also claimed that if $A_\bullet$ is an $\mathbb{N}$-graded ring and $M_\bullet$ is a graded $A_\bullet$–module, there is an analogous sheaf on $\operatorname{Proj}(A_\bullet)$ whose sections, over a distinguished open associated to a positive degree element, are the degree 0 elements in the localization of $M_\bullet$ at f. Give precise definitions of these sheaves and verify that they do indeed form sheaves of modules over the structure sheaf.

   Given a graded ring $A_\bullet$ and a graded module $M_\bullet$, define the sheaf $M_\bullet(1)$ to be graded module whose weight n piece is the weight $n+1$ piece of $M_\bullet$. Let X be $\operatorname{Proj}(A_\bullet)$. The sheaf associated to $A_\bullet(1)$ is denoted $\mathcal{O}_X(1)$. Take $A_\bullet$ to be $\mathbb{C}[x, y, z]$ with the standard grading. Calculate the sections of this sheaf over the standard distinguished open sets in $\mathbb{P}^2_{\mathbb{C}}$. Repeat the exercise for the sheaves $\mathcal{O}_X(d)$ for all integers d. For which d does $\mathcal{O}_{\mathbb{P}^2}(d)$ have non-zero global sections?

10. Let X be a Noetherian scheme, $f : \mathcal{F} \to \mathcal{G}$ a morphism of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that $\ker f, \operatorname{coker} f$ and $\operatorname{im} f$ are quasi-coherent. If $\mathcal{F}$ and $\mathcal{G}$ are coherent, show $\ker f, \operatorname{coker} f$ and $\operatorname{im} f$ are coherent.

    Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{F}$ a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_Y$-modules. Show that $f^*\mathcal{F}$ is a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_X$-modules.

    Show by example that if $\mathcal{G}$ is a coherent sheaf on X, then $f_*\mathcal{G}$ need not be a coherent sheaf on Y. [Note: $f_*\mathcal{G}$ is always quasi-coherent, but this is harder to prove.]

[1] In this context, calculate means to describe the scheme up to isomorphism using basic operations: standard rings, Spec and Proj, and gluing constructions, as explicitly as possible.

[2] This should match the fact from topology that closed subsets of a compact set are compact.

11. Let $i : Z \to X$ be a closed immersion of schemes. Recall this means i is a homeomorphism of Z onto a closed subset of X, and the map $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. We write $\mathcal{I}_{Z/X} = \ker i^\#$.

    a) Show that $\mathcal{I}_{Z/X}$ is a *sheaf of ideals* (or *ideal sheaf*) of $\mathcal{O}_X$, i.e., $\mathcal{I}_{Z/X}(U)$ is an ideal in $\mathcal{O}(U)$ for each $U \subseteq X$ open.

    b) Show that $\mathcal{I}_{Z/X}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, and is coherent if X is Noetherian.

    c) Show that there is a one-to-one correspondence between quasi-coherent sheaves of ideals of X and closed subschemes of X.

12. Let k be a field and let $H \hookrightarrow \mathbb{P}^n_k$ be the closed immersion corresponding to the inclusion of a hyperplane, i.e. the vanishing of a linear homogeneous polynomial. Let $\mathcal{I}$ be the ideal sheaf of $H \hookrightarrow \mathbb{P}^n_k$. Prove that $\mathcal{I}$ is isomorphic to the sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$.

13. (Global Spec Construction). Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$–algebras on a scheme X, i.e. a sheaf of commutative rings that is simultaneously a sheaf of $\mathcal{O}_X$-modules. Assume that the underlying sheaf of modules is quasi-coherent. We will now construct the "relative" or "global" Zariski spectrum $\pi : \underline{\operatorname{Spec}}\,\mathcal{A} \to X$ as a scheme with a map to X.

    (Set) Given a point $p \in X$, the set $\pi^{-1}(p)$ is defined to be $\operatorname{Spec}(\mathcal{A} \otimes \kappa(p))$, where $\kappa(p)$ is the residue field of X at p. Ranging over all p in X defines the points of $\underline{\operatorname{Spec}}\,\mathcal{A}$ with a natural map to X.

    (Topology) Given an affine open U in X, describe a natural bijection between $\pi^{-1}(U)$ and the spectrum of the algebra $\mathcal{A}(U)$. Use this to upgrade $\pi : \underline{\operatorname{Spec}}\,\mathcal{A} \to X$ to a continuous map of topological spaces.

    (Functions) Construct a structure sheaf on $\underline{\operatorname{Spec}}\,\mathcal{A}$ by using the identification of sets of the form $\pi^{-1}(U)$ with the spectrum of $\mathcal{A}(U)$ to endow such sets with a natural ring of functions.

    Show that all these data produce a scheme map $\underline{\operatorname{Spec}}\,\mathcal{A} \to X$.

    For any scheme X over S, for each positive integer n, describe a sheaf of algebras on X such that the construction above gives rise to $X \times_S \mathbb{A}^n_S \to X$.

    [If the sheaf of algebras is graded, one can similarly construct a global Proj scheme of X. This is a very useful thing to do; in fact, once you've constructed global Spec, you can patch it together to form global Proj like we did in the ordinary Spec/Proj case.]