ALGEBRAIC GEOMETRY: FIRST QUARTER REVIEW

An **affine variety** V **in** $\mathbb{A}^n_k$ is the vanishing locus of a set of polynomials $S \subset k[X_1, \dots, X_n]$. The set V is equal to the vanishing locus of the *ideal* generated by S. In other words, *replacing a set by the ideal it generates doesn't change the vanishing locus.*

The polynomial ring is *Noetherian* as a consequence of Hilbert's basis theorem: every ideal is finitely generated. Therefore, every variety is the vanishing locus of a finite collection of hypersurfaces: if I is generated by $f_1, \dots f_r$, then

$$V = \mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_r) \subset \mathbb{A}^n_k.$$

Given an ideal I one obtains a variety $\mathbb{V}(I)$. Conversely, given a variety V inside $\mathbb{A}^n_k$, the set $I(V)$ of polynomials that vanish on all points of V forms an ideal. The two procedures are not inverse: take $I = (X^2) \subset k[X]$. This example captures the failure of these constructions to be inverse. The theorem is captured by *Hilbert's Nullstellensatz.*

The Nullstellensatz has a few different forms:

- For any ideal I, the ideal $I(\mathbb{V}(I))$ is exactly the *radical ideal*, i.e. the set

$$\{f \in k[\underline{X}] : f^m \in I, \quad \text{for some positive } m\}$$

.

- The maximal ideals of the polynomial ring are in bijection with the points of affine space: the correspondence is given by

$$(a_1, \dots, a_n) \leftrightarrow (X_1 - a_1, \dots, X_n - a_n), \quad a_i \in k.$$

- Proper ideals have nontrivial vanishing loci.

We haven't as yet proved these statements; we will. But the second statement should give an intuitive understanding of the points of the geometry.

Given a variety $V \subset \mathbb{A}^n$, the *coordinate ring* $\mathcal{O}_V$ is the quotient ring[1] $k[\underline{X}]/I(V)$. This the set of *algebraic functions* or *morphisms*

$$\varphi : V \to \mathbb{A}^1.$$

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[1]It is important here that $I(V)$ is radical, i.e. that we construct the ideal from the variety rather than the variety from the ideal.

Note that since $\mathcal{O}_V$ is a quotient of the polynomial ring $k[\underline{X}]$, we can choose (many!) polynomials in $k[\underline{X}]$ whose image in $\mathcal{O}_V$ is φ. All these lifts give the *same* function on V.

Similarly, m elements of $\mathcal{O}_V$ give morphisms

$$\psi : V \to \mathbb{A}^m.$$

If we happen to have a variety $W \subset \mathbb{A}^m$, we can ask (purely set theoretic condition!) for $\psi(V)$ to be contained in W. This is (again, by definition!) a morphism $V \to W$.

Rephrase: the data of ψ is m elements $\varphi_1, \dots, \varphi_m$ gives a ring homomorphism

$$\psi^\star : k[Y_1, \dots Y_m] \to \mathcal{O}_V.$$

The Y_i are the coordinate functions on the $\mathbb{A}^m$; the Y_i maps to φ_i for each i. The condition that the image of V lies inside W is precisely the condition that every point in the image of V satisifes all the equations of W. Unwind this – the image of V lies in W precisely when the homomorphism $\psi^\star$ *factors* as

$$\psi^\star : k[Y_1, \dots Y_m] \to k[Y_1, \dots Y_m]/I(W) = \mathcal{O}_W \to \mathcal{O}_V.$$

Abstractly, a morphism of affine varieties $V \to W$ is simply a ring homomorphism $\mathcal{O}_W \to \mathcal{O}_V$ that preserves k. The ring theoretic data is the data of how to construct a function on V when given a function on W via the *pullback* (i.e. pre-composition).

A variety V is *irreducible* if it is not the union of two smaller varieties. On the ring theory side, $\mathcal{O}_V$ is an integral domain if and only if V is irreducible. On the ideal, we're asking for the ideal defining V to be prime. Even if a variety is not irreducible, it is a *finite union of irreducible varieties*.

If V is irreducible, the coordinate ring has a fraction field, which we denote[2] $\mathcal{O}_V(\eta)$. The elements of this fraction field (aka the function field of V) are called *rational functions*. For a Zariski open $U \subset V$, i.e. the complement of the vanishing locus of an ideal, there's a ring of functions

$$\mathcal{O}_V(U) = \{h \in \mathcal{O}_V(\eta) : h \text{ can be represented as } \frac{f}{g} \text{ with } g \text{ never zero on } U\}.$$

For different open sets U, there is a containment

$$\mathcal{O}_V \subset \mathcal{O}_V(U) \subset \mathcal{O}_V(\eta).$$

Given a point $p \in V$, an element $h \in \mathcal{O}_V(\eta)$ is *defined* or *regular* at p if it can be written as $h = \frac{f}{g}$ with $g \neq 0$. Intuitively, the function value of h doesn't blowup up at p,

[2]For now, η is simply a symbol. It means, "some open set but not any open set in particular". It reflects that any given ratio of polynomials is defined on some open set, but taken together, the elements of the fraction field are not defined on any particular open set. In *scheme theory* this business of "an unspecified open set" is made formal via the notion of *generic points*.

The *local ring at* p is the subset $\mathcal{O}_{V,p} \subset \mathcal{O}_V(\eta)$ of functions defined at p. It has a maximal ideal of functions vanishing at p which we denote $\mathfrak{m}_p \subset \mathcal{O}_{V,p}$. Every element that is not in $\mathfrak{m}_p$ has an inverse in $\mathcal{O}_{V,p}$.

The functions $\mathfrak{m}_p$ are the functions that vanish to order 1 at p; the ideal $\mathfrak{m}_p^2$ are products two of these, so they all vanish to order at least 2. The quotient $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a k-vector space[3]. It is the *Zariski cotangent space*; its k-vector space dual is defined to be the *tangent space*.

A *smooth affine variety* is one whose tangent (or cotangent) space has the same dimension everywhere. The variety $\mathbb{V}(Y^2 - X^2(X+1))$ is irreducible and has a 2-dimensional cotangent space at $(0,0)$ [do this calculation!]. We will come back to smoothness and treat it in its appropriate context soon.

[3]the analogue in analysis is that this is the set of possible coefficients of the linear terms in a Taylor series expansion of a point, centered at p.