Inequalities: a journey into linear analysis
Corrections and comments

June 18, 2009

Corrections

Page 18: Exercise 2.4 (i) Replace 4 by 2. Prove the *incremental inequality*

$$|f_{\mathbf{P}}(u+h) - f_{\mathbf{P}}(u)|^2 \leq 2(1 - \Re f_{\mathbf{P}}(h)).$$

Page 21: Proposition 3.2.1 (i) Replace $nt > -1$ by $nt < 1$.
Page 22: Line 3 Replace 'if $n^\alpha > x$,' by 'if $n^{\alpha-1} > x$,'.
Page 26: Line 10'... the points $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$...'.
Page 41: Exercise 4.1 (ii) Let $I_n = \int_0^{\pi/2} \sin^n \theta \, d\theta$.
Page 45, Line 14 Thus L^p is a vector space.
Page 50: Line 1 ...$w = (u, v) = (f^p, g^p)$.
Page 52: Statement of Corollary 5.4.2 Replace Ω_1 and Ω_2 by X_1 and X_2.
Page 60: Line 3*

$$f(x) = \int_{-\infty}^{x_j} \frac{\partial f}{\partial x_j}(t, x^j)\, dt = -\int_{x_j}^{\infty} \frac{\partial f}{\partial x_j}(t, x^j)\, dt,$$

Page 65: Line 4 In the second integral, replace dy by dx.
Page 66: Statement of Theorem 5.10.2 ...and $b \in l_{p'}(\mathbf{Z}^+)$,...
Page 68: Exercise 5.11 'Show that ψ and χ...'.
Page 71: Line 16 'Then f_A is zero outside A,'.
Page 75: Line 3 'Suppose that $\Phi(t) = \int_0^t \phi(s)\, ds$...'.
Page 113: Line 5 'measured', not 'measures'.
Pages 139: Line 7 Replace this sentence with 'If g is a non-zero simple function, then just as in the construction of F above, we can construct G

such that $G(\theta) = g$, G is continuous on $\bar{S}$, analytic on S and bounded in $B_0 + B_1$ on $\bar{S}$.'

Page 141: Lines 9*-8*If we set $\theta = p/q' = 1 - p/r$ we find that...

Page 147: Line 1 Replace 'the Hausdorff-Young inequality' by 'Theorem 9.7.1'.

Page 148: Line 5 Add a subscript p to each of the four norms in this line.

Page 149: Statement of Lemma 9.8.1 If $1 < p < \infty$, there exists $C_p > 0$ such that if $s, t \in \mathbf{R}$ then ...

Page 168: Line 6 $Q_t(f) = \ldots$.

Page 209: Lines 9* and 8* Replace 'y_B' by 'x_B^+'

Page 209: Line 4* This line is badly wrong: it should be

$$\leq (\mathbf{E}_{(N)}(\mathbf{E}_{N-1}(\|u_p + \epsilon_N r_q v_p\|_E^p)^{q/p}))^{1/q}$$

Page 212: Line 14 This should be:

$$2\,\|f\|_2^2 = 2\,\langle f, f\rangle \leq \mathcal{E}(f) + 2(\mathbf{E}(f))^2 + \sum_{i=1}^{d} \hat{f}_i^2.$$

Pages 214-215. This is a comedy of errors, which illustrates that factors of 2 can cause problems. The definition of the graph Laplacian on Page 212 has a factor of 1/2, but the definition of the gradient does not. (In the latter case, I follow Michel Ledoux: *The concentration of measure phenomenon* (AMS Surveys and Monographs 89 (2001)) page 110; other authors include a factor of 1/2.)

Page 214: Lines 8*-6* Note that

$$\begin{aligned}
\mathcal{E}(f) &= \frac{1}{2^{d+1}} \sum_{\omega} \sum_{\{\eta:\eta\sim\omega\}} (f(\omega) - f(\eta)) f(\omega) \\
&= \frac{1}{2^{d+2}} \left(\sum_{\omega} \sum_{\{\eta:\eta\sim\omega\}} (f(\omega) - f(\eta)) f(\omega) + \sum_{\eta} \sum_{\{\omega:\omega\sim\eta\}} (f(\eta) - f(\omega)) f(\eta) \right) \\
&= \tfrac{1}{4}\mathbf{E}(|\nabla f|^2).
\end{aligned}$$

Page 214: Statement of Theorem 13.4.2. Delete the minus sign in the exponent: it should be $e^{\lambda^2/4}$.

On the other hand, there is a compensating error on Page 215.

Page 215: Lines 5-8

$$\begin{aligned} &\leq \frac{\lambda^2}{2.2^d}\sum_{\omega}\left(\sum_{\{\eta:\eta\sim\omega\}}(f(\eta)-f(\omega))^2\right)e^{\lambda f(\omega)} \\ &= \frac{\lambda^2}{2}\mathbf{E}(\|\nabla(f)\|_2^2\, e^{\lambda f}) \leq \frac{\lambda^2}{2}\mathbf{E}(e^{\lambda f}) = \frac{\lambda^2 H(\lambda)}{2} \end{aligned}$$

Thus, applying the logarithmic Sobolev inequality,

$$\mathrm{Ent}(e^{\lambda f}) \leq 2\mathcal{E}(e^{\lambda f/2}) = \tfrac{1}{2}\mathbf{E}(|\nabla(e^{\lambda f/2})|^2) \leq \lambda^2 H(\lambda)/4.$$

Page 216: Statement of Corollary 13.4.2. Replace $h(\omega, A)$ by $h_A(\omega)$.

Page 228: Statement of Theorem 13.10.3 Replace $e^{-\lambda^2/4}$ by $e^{-\lambda^2/2}$.

Page 228: Statement of Corollary 13.10.1 'Then f is sub-Gaussian with index 1:' Replace e^{-r^2} by $e^{-r^2/2}$.

For more details, see C. Ané et al. *Sur les inégalités de Sobolev logarithmiques* (SMF Panoramas et Synthèses 10 (2000)) Chapitre 7.

Page 229: Statement of Theorem 13.11.1 Add 'where ϵ is a Bernoulli random variable'.

Page 245: Line 4* Replace y_n by x_n.

Page 253: Proposition 15.10.3 Replace 'If $S \in S_p(H_1, H_2)$' by 'If $S \in S_p(H_2, H_3)$' and replace 'then $ST \in S_r(H_1, H_2)$' by 'then $ST \in S_r(H_1, H_3)$'.

Page 275: Line 9* Cauchy-Schwarz (Whoops! See page 17!).

Page 329: [**Sch**] Problem, Variationsrechnung.

Page 333: Legendre character 236

Page 334: trace 250-251, 257

My thanks to Nick Bingham for pointing out the errors on pages 113, 275 and 329. My thanks also to Mark Veraar for pointing out many errors, particularly those relating to the logarithmic Sobolev inequalities, and for helping me correct them. My thanks also to Bill johnson for pointing out the need to correct the proof of Theorem 9.3.1, and to Hassane Kone and Yan Zhu for pointing out other errors.

Comments

Pages 103-106 The Hardy-Riesz inequality is usually called Hardy's inequality. Its gestation, and its relation to Hilbert's absolute inequality, make a complicated and fascinating story. A beautiful account is given by

Kufner, Maligranda and Persson (2006). I am grateful to Dennis Bernstein for bringing this to my attention, and also for pointing out some of the errors listed above.

A. Kufner, L. Maligranda and L.-E. Persson (2006). The Prehistory of the Hardy Inequality, *Amer. Math. Monthly* **113** 715-732.

Page 148: Theorem 9.8.1 This is a special case of Bonami's inequality (Proposition 13.1.1) which also shows that we can take $C_p = p - 1$ and that the inequality is then sharp. A full account of uniform convexity of the L^p spaces, and the operator ideals S_p introduced in Section 15.10, is given in the magisterial paper:

Keith Ball, Eric A. Carlen and Elliott H. Lieb (1994). Sharp uniform convexity and smoothness inequalities for trace norms, *Invent. Math.* **115** 463-482.

This paper also appears in

M. Loss and M.B. Ruskai (eds) (2002-3). *Inequalities, Selecta of Elliott H. Lieb,* (Springer)

This contains a wealth of further inequalities.

Page 151: Exercise 9.3 The hint is not very helpful: here is a proof. Let $h(x) = \int k(x-y)g(y)\,d\mu(y)$. Then since $q/p'+q/r = 1$ and $p/q'+p/r = 1$

$$\begin{aligned} |h(x)| &\leq \int |k(x-y)||g(y)|\,d\mu(y) \\ &= \int |k(x-y)|^{q/p'}|g(y)|^{p/q'}|k(x-y)|^{q/r}|g(y)|^{p/r}\,d\mu(y). \end{aligned}$$

Now $1/p' + 1/q' + 1/r = 1$, and so, by Proposition 5.4.1,

$$|h(x) \leq \|k\|_q^{q/p'}\,\|g\|_p^{p/q'}\left(\int |k(x-y)|^q|g(y)|^p\,d\mu(y)\right)^{1/r},$$

so that

$$\begin{aligned} \|h\|_r &\leq \|k\|_q^{q/p'}\,\|g\|_p^{p/q'}\left(\int(\int |k(x-y)|^q|g(y)|^p\,d\mu(y)\,d\mu(x)\right)^{1/r} \\ &= \|k\|_q^{q/p'}\,\|g\|_p^{p/q'}\,\|k\|_q^{q/r}\,\|g\|_p^{p/r} = \|k\|_q\,.\,\|g\|_p\,. \end{aligned}$$

Now suppose that k is a positive integer, and suppose that $f \in L^{2k/(2k-1)}(G)$. Let g be the k-fold convolution of f with itself. Use Young's inequality, extended to k terms, to show that $\|g\|_2 \leq \|f\|_{2k/(2k-1)}^k$. But $\hat{g} = (\hat{f})^k$, so that,

using the Plancherel theorem (Theorem 9.5.1),

$$\left\|\hat{f}\right\|_{2k} = \left\|(\hat{f})^k\right\|_2^{1/k} = \|g\|_2^{1/k} \leq \|f\|_{2k/(2k-1)} \,.$$

This is what Young (1912) proved. Thus Young proved the Hausdorff-Young inequality for L^r when r' is an even integer.

Hausdorff (1923) extended the result to all $1 < r < 2$, by using an optimization argument involving Lagrange's undetermined multiplers to show that if the result holds for r, then it holds for s, where $s' = r'/2 + 1$. Thus it holds for all r, where r is a dyadic rational greater than 2: the general result then follows by continuity.

W.H. Young (1912). Sur la généralisation du théorème de Parseval, *Comptes Rendus* **155** 30-33 and 472-475.

F. Hausdorff(1923). Eine Ausdehnung der Parsevalsche Satzes über Fourierreihen, *Math Z.* **16** 163-169.

Pages 263-265: Khinchine's inequality can be applied effectively to prove *Orlicz' inequality.*

Theorem 0.1 (Orlicz' inequality) *Suppose that $\Omega, \Sigma, \mu)$ is a probability space, that $1 \leq p < 2$ and that $\sum_{n=1}^{\infty} f_n$ is an unconditionally convergent sequence in $L^p(\mu)$, so that*

$$\sup\{\sum_{n=1}^{\infty} |\int g f_n \, d\mu| : \|g\|_{p'} \leq 1\} = \sup\{\left\|\sum_{n=1}^{\infty} b_n f_n\right\|_p : \|b\|_\infty \leq 1\} = M < \infty.$$

Then $\sum_{n=1}^{\infty} \|f_n\|_p^2 \leq 2M^2$.

Proof Let $N \in \mathbf{N}$. Then, first using Corollary 4.5.2 and then Khintchine's inequality,

$$\begin{aligned}
\sum_{n=1}^{N} \|f_n\|_p^2 &= \sum_{n=1}^{N} (\int_\Omega |f_n|^p \, d\mu)^{2/p} \\
&\leq \left(\int_\Omega \left(\sum_{n=1}^{N} |f_n|^2\right)^{p/2} d\mu\right)^{2/p} \\
&\leq 2\left(\int_\Omega \left(\mathbf{E}(|\sum_{n=1}^{N} \epsilon_n f_n|^p\right) d\mu\right)^{2/p} \\
&= 2\mathbf{E}\left(\int_\Omega |\sum_{n=1}^{N} \epsilon_n f_n|^p \, d\mu\right)^{2/p} \leq 2M^2.
\end{aligned}$$

Corollary 0.1 *Suppose that* $(f_n)_{n=1}^{\infty}$ *is an orthonormal basis for* $L^2(\mu)$*, that* $1 \leq p < 2$ *and that* $\inf_n \|f_n\|_p > 0$*. Then* (f_n) *is not an unconditional basis for* $L^p(\mu)$*.*

Proof For if $f = \sum_{n=1}^{\infty} a_n f_n$ converges, then $\sum_{n=1}^{\infty} |a_n|^2 \|f_n\|_p^2 < \infty$, and so $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Consequently, the sum $\sum_{n=1}^{\infty} a_n f_n$ converges in L^2 norm, and so $f \in L^2(\mu)$. Thus $L^2(\mu) = L^p(\mu)$, which is not possible.

Corollary 0.2 *Suppose that* $(f_n)_{n=1}^{\infty}$ *is an orthonormal basis for* $L^2(\mu)$*, that* $2 < p < \infty$ *and that* (f_n) *is bounded in* L^p*. Then* (f_n) *is not an unconditional basis for* $L^p(\mu)$*.*

Proof If it were, then, by duality, (f_n) would be an unconditional basis of $L^{p'}(\mu)$, and $\inf_n \|f_n\|_{p'}$ would be strictly positive.

Corollary 0.3 *If* G *is a compact abelian group, with Haar measure* μ*, and* $\{\gamma_n : n \in \mathbf{N}\}$ *is the set of characters of* G *then* (γ_n) *is an unconditional basis for* $L^p(\mu)$ *(where* $1 \leq p < \infty$*) if and only if* $p = 2$*.*

Proof For (γ_n) is an orthonormal basis for $L^2(\mu)$, and $\|\gamma_n\|_p = 1$ for all $1 \leq p < \infty$.