Irredundant Families of Subcubes

David Ellis

January 2010

Abstract

We consider the problem of finding the maximum possible size of a family of $k$-dimensional subcubes of the $n$-cube $\{0, 1\}^n$, none of which is contained in the union of the others. (We call such a family ‘irredundant’). Aharoni and Holzman [1] conjectured that for $k > n/2$, the answer is $\binom{n}{k}$ (which is attained by the family of all $k$-subcubes containing a fixed point). We give a new proof of a general upper bound of Meshulam [6], and we prove that for $k \geq n/2$, any irredundant family in which all the subcubes go through either $(0, 0, \ldots, 0)$ or $(1, 1, \ldots, 1)$ has size at most $\binom{n}{k}$. We then give a general lower bound, showing that Meshulam’s upper bound is always tight up to a factor of at most $e$.

1 Introduction

Let $\{0, 1\}^n$ denote the $n$-dimensional discrete cube, the set of all 0-1 vectors of length $n$. A $k$-dimensional subcube (or $k$-subcube) of $\{0, 1\}^n$ is a subset of $\{0, 1\}^n$ of the form

$$\{x \in \{0, 1\}^n : x_i = a_i \ \forall i \in T\}$$

where $T$ is a set of $n-k$ coordinates, called the fixed coordinates, and the $a_i$’s are fixed elements of $\{0, 1\}$. The other coordinates $S = [n] \setminus T$ are called the moving coordinates. We will represent a subcube by an $n$-tuple of 0’s, 1’s and *’s, where the *’s denote moving coordinates and the 0’s and 1’s denote fixed coordinates. For example, $(*, *, *, 0, 1)$ denotes a 3-dimensional subcube of $\{0, 1\}^5$. 

1
We consider the problem of finding the maximum possible size of a family of \(k\)-subcubes of the \(n\)-cube \(\{0, 1\}^n\), none of which is contained in the union of the others. In other words, each has a vertex not contained in any of the others (which we call a ‘private’ vertex). We will call such a family ‘irredundant’, and we write \(M(n, k)\) for the maximum size of an irredundant family of \(k\)-subcubes of \(\{0, 1\}^n\).

Let \([n]\) denote the set \(\{1, 2, \ldots, n\}\). We may identify \(\{0, 1\}^n\) with \(\mathcal{P}[n]\), the set all subsets of \([n]\), by identifying a subset \(x \subset [n]\) with its characteristic vector \(\chi_x\), defined by

\[
\chi_x(i) = 1 \quad \forall i \in x, \quad \chi_x(i) = 0 \quad \forall i \notin x.
\]

We write \((0, 0, \ldots, 0) = 0\) and \((1, 1, \ldots, 1) = 1\). We will refer to \(|x \Delta y|\), the number of coordinates in which \(x\) and \(y\) differ, as the \textit{Hamming distance} between \(x\) and \(y\), and the set

\[
\{y \in \{0, 1\}^n : |x \Delta y| \leq r\}
\]

as the \textit{Hamming ball of centre} \(x\) \textit{and radius} \(r\).

Here are some natural examples of irredundant families:

The family of all translates of a fixed \(k\)-subcube,

\[
\{A + x : x \in \{0, 1\}^n\}
\]

where \(A\) is a \(k\)-subcube of \(\{0, 1\}^n\) — in other words, the collection of all the subcubes having the same moving coordinates as \(A\). This family partitions \(\{0, 1\}^n\), so every vertex is a private vertex of its subcube, and it is a maximal irredundant family; it has size \(2^{n-k}\).

The family \(\mathcal{F}_0\) of all \(k\)-subcubes containing \(0\), \(\{\mathbb{P}x : x \in [n]^{(k)}\}\). Clearly, \(x\) is a private vertex of the \(k\)-subcube \(\mathbb{P}x\); it is the unique such, since any \(y \subset x\) can be extended to a different \(k\)-set \(z \neq x\). This family has size \(\binom{n}{k}\).

For \(k \geq \frac{1}{2}n\) it is maximal, since then any \(k\)-subcube contains a \(k\)-set. Similarly, for any \(v \in Q_n\) we let \(\mathcal{F}_v\) be the collection of all \(k\)-subcubes through \(v\); we call these the ‘principal’ irredundant families. Aharoni and Holzman [1] conjectured that for \(k > n/2\), there are no larger irredundant families:

\textbf{Conjecture 1} (Aharoni-Holzman, 1991). If \(k > n/2\), any irredundant family of \(k\)-subcubes of \(\{0, 1\}^n\) has size at most \(\binom{n}{k}\)
Aharoni and Holzman (unpublished – see [6]) gave the following general upper bound on the maximum size of an irredundant family of $k$-subcubes of $\{0, 1\}^n$:

$$M(n, k) \leq \sum_{i=k}^{n} \binom{n}{i} \quad \forall k \leq n. \quad (1)$$

This may be proved using a short linear independence argument. Meshulam [6] proved the following stronger upper bound using a purely combinatorial argument:

$$M(n, k) \leq \frac{2^n}{\sum_{i=0}^{k} \binom{n}{i} \binom{n}{k}} \quad \forall k \leq n. \quad (2)$$

(Intuitively, this is saying that, if there were a partition of $\{0, 1\}^n$ into Hamming balls of radius $k$, it would be best to take the irredundant family of all $k$-subcubes containing one of the centres of the balls.) We will give a simple proof of Meshulam’s bound using Bollobás’ Inequality. A variant of this proof shows that if we choose one private vertex for each subcube in an irredundant family, then any Hamming ball of radius $k$ contains at most $\binom{n}{k}$ of these private vertices. (This immediately implies Meshulam’s bound by averaging over all Hamming balls of radius $k$.)

For $k/n > \gamma$, where $\gamma \in (\frac{1}{2}, 1)$ is fixed, Meshulam’s bound gives $M(n, k) \leq (1 + o(1)) \binom{n}{k}$, i.e. it asymptotically approaches the conjectured bound; if $\gamma \geq \gamma_0 \approx 0.8900$, it gives $M(n, k) < \binom{n}{k} + 1$ for $n$ sufficiently large, proving Conjecture 1 in this case.

We observe that equality holds in Meshulam’s bound when there is a partition of $\{0, 1\}^n$ into Hamming balls of radius $k$, i.e. in the following cases:

- $k = 1$, $n + 1$ is a power of 2
- $k = 3$, $n = 23$
- $n = 2k + 1$

When $n = 2k + 1$, the irredundant family of all $k$-subcubes containing either 0 or 1 has size $2\binom{n}{k}$.

We are then led to investigate the special case when every subcube must go through either 0 or 1; we prove by an unusual linear algebra argument
that for \( k \geq n/2 \), any irredundant family in which all \( k \)-subcubes go through either 0 or 1 has size at most \( \binom{n}{k} \).

Finally, we obtain a general lower bound for all \( n \) and \( k \). A probabilistic argument shows that there exists an irredundant family of \( k \)-subcubes of \( \{0,1\}^n \) of size at least

\[
\beta(1 - \beta)^{(1-\beta)/\beta} 2^n, \tag{3}
\]

where

\[
\beta := \frac{\binom{n}{k}}{\sum_{i=0}^{k} \binom{n}{i}}.
\]

Combining this with Meshulam’s bound, we see that

\[
\beta(1 - \beta)^{(1-\beta)/\beta} 2^n \leq M(n, k) \leq \beta 2^n.
\]

The ratio between the upper and lower bound above is at most \( e \) for all \( n \) and \( k \).

If \( k = \lfloor \gamma n \rfloor \) for fixed \( \gamma \in (0, \frac{1}{2}) \), then

\[
\beta = \left( \frac{1 - 2\gamma}{1 - \gamma} \right) (1 + o(1)),
\]

so we obtain

\[
(1+o(1)) \left( \frac{\gamma}{1 - \gamma} \right)^{\frac{\gamma}{1-2\gamma}} \left( \frac{1 - 2\gamma}{1 - \gamma} \right) 2^n \leq M(n, \lfloor \gamma n \rfloor) \leq (1+o(1)) \left( \frac{1 - 2\gamma}{1 - \gamma} \right) 2^n,
\]

showing that \( M(n, \lfloor \gamma n \rfloor) \) has order of magnitude \( 2^n \).

If \( k = o(n) \), we obtain \( M(n, k) = (1 - o(1))2^n \).

## 2 Upper bounds

Aharoni and Holzman proved the following:

**Proposition 2 (Aharoni-Holzman, 1991).** For any \( k \leq n \), any irredundant family of \( k \)-subcubes of \( \{0,1\}^n \) has size at most

\[
\sum_{i=k}^{n} \binom{n}{i}.
\]
Proof. Let $C$ be a $k$-subcube of $\{0,1\}^n$; we write $0(C)$ for its set of fixed 0’s and $1(C)$ for its set of fixed 1’s. The characteristic function $\chi_C$ of $C$ can be written as a function of $(x_1, \ldots, x_n) \in \mathbb{R}^n$ as follows:

$$
\chi_C(x_1, \ldots, x_n) = \prod_{i \in 0(C)} (1 - x_i) \prod_{i \in 1(C)} x_i
$$

—for example,

$$
\chi_{(1,\ast,\ast,\ast,0)}(x_1, x_2, x_3, x_4, x_5) = x_1(1 - x_5).
$$

Now let $\mathcal{A}$ be an irredundant family of $k$-subcubes of $\{0,1\}^n$. Then

$$
\{\chi_C : C \in \mathcal{A}\}
$$

is a linearly independent subset of the vector space $\mathbb{R}[x_1, \ldots, x_n]$. To see this, for each $C \in \mathcal{A}$, choose a private vertex $w_C \in C$. Suppose

$$
\sum_{C \in \mathcal{A}} a_C \chi_C = 0
$$

for some real numbers $\{a_C : C \in \mathcal{A}\}$. Then for any $D \in \mathcal{A}$, evaluating the above on $w_D$ gives:

$$
0 = \sum_{C \in \mathcal{A}} a_C \chi_C(w_D) = a_D.
$$

It is easy to check that the set of monomials

$$
S = \{\prod_{i \in A} x_i : A \in [n]^{(\leq n-k)}\}
$$

is a basis for the vector subspace

$$
W = \langle \chi_C : C \text{ is a } k\text{-subcube of } \{0,1\}^n \rangle \subset \mathbb{R}[x_1, \ldots, x_n].
$$

Hence

$$
|\mathcal{A}| \leq \dim(W) = |S| = \sum_{l=0}^{n-k} \binom{n}{l} = \sum_{i=k}^{n} \binom{n}{i},
$$

proving the proposition. \qed
For $k = \lfloor \gamma n \rfloor$, where $\gamma \in (\frac{1}{2}, 1)$, we have:
\[
\sum_{i=k}^{n} \binom{n}{i} = \sum_{t=0}^{n-k} \binom{n}{t} \leq \frac{3\gamma - 1}{2\gamma - 1} \binom{n}{\lfloor \gamma n \rfloor},
\]
so Proposition 2 gives the correct order of magnitude.

For $n = 2k - 1$, however, it only gives $M(2k - 1, k) \leq 2^{2k-2}$, compared with $2(1 - o(1)) \binom{2k-1}{k}$ from Meshulam’s bound.

We now give a proof of Meshulam’s bound which we believe to be slightly more intuitive than the proof in [6]. The idea is that for any irredundant family $\mathcal{A}$ and any choice of private vertices, for every $x \in \{0, 1\}^n$, the private vertices chosen for the subcubes containing $x$ cannot be too closely packed around $x$. Our main tool is Bollobás’ Inequality:

**Theorem 3** (Bollobás, 1965). Let $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ be subsets of $\{1, 2, \ldots, n\}$ such that $a_i \cap b_j = \emptyset$ if and only if $i = j$. Then
\[
\sum_{i=1}^{N} \left( \frac{|a_i| + |b_i|}{|b_i|} \right)^{-1} \leq 1.
\]
Equality holds only if there exists a subset $Y \subset [n]$ and an integer $a \in \mathbb{N}$ such that $\{a_1, \ldots, a_N\} = Y^{(a)}$, and $b_i = Y \setminus a_i \forall i$.

For a proof, we refer the reader to [3].

Given an irredundant family $\mathcal{A}$, we will fix a choice of private vertices, and deduce from Theorem 3 an inequality involving the subcubes containing a fixed vertex $x \in Q_n$; we will then sum this inequality over all $x \in Q_n$ to prove bound (2).

**Theorem 4** (Meshulam, 1992). For any $k \leq n$, if $\mathcal{A}$ is an irredundant family of $k$-subcubes of $\{0, 1\}^n$, then
\[
|\mathcal{A}| \leq \frac{2^n}{\sum_{i=0}^{k} \binom{n}{i} \binom{n}{k}}.
\]

**Proof.** Let $\mathcal{A}$ be an irredundant family of $k$-subcubes of $\{0, 1\}^n$, and for each subcube $C \in \mathcal{A}$, choose a private vertex $w_C \in C$.
Claim: For any $x \in \{0, 1\}^n$,

$$\sum_{C \in \mathcal{A} : x \in C} \left( \frac{|w_C \Delta x| + n - k}{n - k} \right)^{-1} \leq 1.$$

(5)

Proof of Claim:
This is an immediate consequence of Bollobás’ Inequality. By symmetry, we may assume that $x = 0$. Let $\{C_1, \ldots, C_N\}$ be the collection of subcubes in $\mathcal{A}$ containing $0$. Each $C_i$ is of the form $\mathbb{P}v_i$ for some $k$-set $v_i$. Let $w_i = w_{C_i}$ be the private vertex chosen for $C_i$. Notice that $w_i \subset v_j$ if and only if $i = j$, i.e. $w_i \cap v_j^c = \emptyset$ if and only if $i = j$, so applying Bollobás’ Inequality gives:

$$\sum_{i=1}^{N} \left( \frac{|w_i| + |v_i^c|}{|v_i|} \right)^{-1} \leq 1,$$

i.e.

$$\sum_{i=1}^{N} \left( \frac{|w_i| + n - k}{n - k} \right)^{-1} \leq 1,$$

(6)

proving the claim.

The inequality (5) expresses the fact that the private vertices chosen for the subcubes containing $x$ cannot be too densely packed around $x$. Summing
(5) over all \( x \in \{0, 1\}^n \), and interchanging the order of summation, we obtain:

\[
2^n \geq \sum_{x \in \{0, 1\}^n} \sum_{C \in A} \left( \frac{|w_C \Delta x| + n - k}{n - k} \right)^{-1}
\]

\[
= \sum_{C \in A} \sum_{x \in C} \left( \frac{|w_C \Delta x| + n - k}{n - k} \right)^{-1}
\]

\[
= |A| \sum_{l=0}^{k} \frac{k! (n - k)! l!}{l!(k - l)!(l + n - k)!}
\]

\[
= |A| \frac{k! (n - k)! n!}{n!} \sum_{l=0}^{k} \frac{n!}{(k - l)!(n - (k - l))!}
\]

\[
= \frac{|A|}{\binom{n}{k}} \sum_{l=0}^{k} \binom{n}{k - l}
\]

\[
= \frac{|A|}{\binom{n}{k}} \sum_{l=0}^{k} \binom{n}{l}
\]

Hence,

\[
|A| \leq \frac{2^n}{\sum_{l=0}^{k} \binom{n}{l} \binom{n}{k}}
\]

as required.

As observed by Meshulam, for \( k \geq \frac{9}{10} n \), by standard estimates, the bound above is \(< \binom{n}{k} + 1 \), implying Conjecture 1 in this case. More precisely, let

\[
H_2(\gamma) = \gamma \log_2(1/\gamma) + (1 - \gamma) \log_2(1/(1 - \gamma))
\]

denote the binary entropy function, and let \( \gamma_0 \) be the unique solution of \( H_2(\gamma_0) = \frac{1}{2} \) in \( (\frac{1}{2}, 1) \), so that \( \gamma_0 = 0.8900 \) (to 4 d.p.); then we have the following

**Corollary 5.** For \( n \) sufficiently large, and \( k \geq \gamma_0 n \), any irredundant family of \( k \)-subcubes of \( \{0, 1\}^n \) has size at most \( \binom{n}{k} \).
In fact, Meshulam proved a generalization of Theorem 4 for irredundant families of $k$-dimensional subgrids of the $n$-dimensional grid $\mathbb{Z}_m^n$. (A $k$-subgrid of $\mathbb{Z}_m^n$ is a subset of $\mathbb{Z}_m^n$ the form
\[ \{ x \in \mathbb{Z}_m^n : x_i = a_i \ \forall i \in T \}, \]
where $T$ is a set of $n - k$ coordinates, and the $a_i$'s are fixed elements of $\mathbb{Z}_m$. A family of $k$-subgrids of $\mathbb{Z}_m^n$ is said to be irredundant if none of its subgrids is contained in the union of the others.) Meshulam proved the following:

**Theorem 6** (Meshulam, 1992). Let $\mathcal{A}$ be an irredundant family of $k$-subgrids of $\mathbb{Z}_m^n$; then
\[ |\mathcal{A}| \leq \sum_{j=n-k}^{n} \binom{m}{j}^{n-k} \binom{n}{k} \]
We remark that our proof generalizes straightforwardly to prove this also.

A slight modification of our method yields a result which gives us more 'geometrical' insight into the problem:

**Theorem 7.** Let $B$ be a Hamming ball of radius $k$ in $\{0,1\}^n$. If $\mathcal{A}$ is an irredundant family of $k$-subcubes of $\{0,1\}^n$, each with a private vertex in $B$, then $|\mathcal{A}| \leq \binom{n}{k}$.

*Proof.* By symmetry, we may assume that $B = [n]^{(\leq k)}$. Let $\mathcal{A}$ be an irredundant family of $k$-subcubes, each with a private vertex in $[n]^{(\leq k)}$. For each subcube $C \in \mathcal{A}$, choose a private vertex $w_C \in [n]^{(\leq k)}$. Write $C = \{ y \in Q_n : v_C \subset y \subset u_C \}$; we will call $v_C$ the 'start vertex' of $C$ and $u_C$ its 'end vertex'. Let $C' = \{ y \in Q_n : w_C \subset y \subset u_C \}$ be the $(k - |w_C| + |v_C|)$-dimensional sub-subcube of $C$ between the private vertex and the end vertex of $C$.

**Claim:** For any vertex $x \in [n]^{(k)}$,
\[ \sum_{C \in \mathcal{A} : x \in C'} \left( \frac{|v_C| + k - |w_C|}{k - |w_C|} \right)^{-1} \leq 1 \] (7)

*Proof of Claim:* As before, this is an immediate consequence of Bollobás’ Inequality. By symmetry, we may assume that $x = [k]$. Write $\{ C \in \mathcal{A} : x \in C' \} = \{ C_1, \ldots, C_N \}$. Let $v_i = v_{C_i}$ be the start vertex of $C_i$ and $w_i = w_{C_i}$ its private
vertex. Clearly, \( v_i, w_i \subset [k] \) for every \( i \in [N] \). Notice that \( v_i \subset w_j \) if and only if \( i = j \), i.e. \( v_i \cap ([k] \setminus w_j) = \emptyset \) if and only if \( i = j \). Hence, Bollobás’ Inequality gives:

\[
\sum_{i=1}^{N} \left( \frac{|v_i| + k - |w_i|}{k - |w_i|} \right)^{-1} \leq 1
\]

and the claim is proved.

Summing (7) over all \( x \in [n]^{(k)} \), and interchanging the order of summation, we obtain:

\[
\binom{n}{k} \geq \sum_{x \in [n]^{(k)}} \sum_{C \in A : x \in C'} \left( \frac{|v_C| + k - |w_C|}{k - |w_C|} \right)^{-1}
\]

For each subcube \( C \in A \), the \( (k - |w_C| + |v_C|) \)-dimensional subcube \( C' \) contains \( \binom{k - |w_C| + |v_C|}{k - |w_C|} \) vertices \( x \in [n]^{(k)} \), and for each of them contributes \( \binom{|v_C| + k - |w_C|}{k - |w_C|}^{-1} \) to the above sum, i.e. a total of 1. Hence,

\[
|A| = \sum_{C \in A} \sum_{x \in C' \cap [n]^{(k)}} \left( \frac{|v_C| + k - |w_C|}{k - |w_C|} \right)^{-1} \leq \binom{n}{k},
\]

proving the theorem. \( \square \)

We have equality in Theorem 7 if \( A \) is the family of all \( k \)-subcubes through the centre of \( B \). Notice that by fixing some choice of private vertices and averaging over all Hamming balls \( B \) of radius \( k \), Theorem 7 immediately implies Theorem 4.

When \( n = 2k + 1 \), the irredundant family of all \( k \)-subcubes containing either 0 or 1 has size \( 2 \binom{k}{k} \), so we have equality in Theorem 4 when \( n = 2k + 1 \).

We have been unable to find a counterexample to Conjecture 1. Notice that by the same projection argument as in Corollary 6 (see later), if the conjecture holds for \( n, k \) then it holds for \( n+1, k+1 \), so it suffices to consider the case \( n = 2k - 1 \). For \( n = 5, k = 3 \), the conjecture can be verified by hand, but there are exactly two extremal families up to isomorphism (permuting the
coordinates and translating): $F_0$ and the following family of ten 3-subcubes of $Q_5$, five through 0 and five through 1. The (unique) private vertices are indicated above the moving coordinates:

\[
\begin{align*}
(1,0,1,0,0) \\
(0,0,1,0,1) \\
(1,0,1,0,0) \\
(0,1,0,1,0) \\
(0,1,0,0,1) \\
(0,1,0,0,1) \\
(0,0,1,1,1) \\
(1,1,0,1,0) \\
(0,1,1,0,1) \\
(1,1,0,1,0)
\end{align*}
\]

Clearly, this family is not of the form $F_x$ for any $x \in \{0, 1\}^5$. However, we have been unable to find another such example, and we conjecture that for $n > 5$ and $k > n/2$, the only irredudant families of $k$-subcubes of $\{0,1\}^n$ with size $\binom{n}{k}$ are of the form $F_x$ for $x \in \{0,1\}^n$.

The best upper bound for $n = 2k - 1$ is still Meshulam’s bound, which in this case is:

\[
M(2k - 1, k) \leq \frac{2^{2k-1}}{2^{2k-2} + \left(\frac{2k-1}{k}\right)^{\binom{2k-1}{k}}} \left(\frac{2k-1}{k}\right)
\]

\[
= \frac{2}{1 + 2^{-2(2k-2)} \left(\frac{2k-1}{k}\right)^{\binom{2k-1}{k}}} \left(\frac{2k-1}{k}\right)
\]

\[
= \frac{2}{1 + 2(1 + o(1)) / \sqrt{(2k-1)\pi}} \left(\frac{2k-1}{k}\right)
\]

\[
= 2(1 - \Theta(1/\sqrt{k})) \left(\frac{2k-1}{k}\right).
\]

To construct a large irredundant family when $k \geq n/2$, one might try just using subcubes containing 0 or 1, so that the $k$-subcubes containing 0 have private vertices in $[n]^{(\leq k)}$, and the $k$-subcubes containing 1 have private
vertices in \([n]^{(2n-k)}\). However, a surprising linear algebra argument shows that even when \(n = 2k\), such a family has size at most \(\binom{n}{k}\):

**Theorem 8.** If \(\mathcal{A}\) is an irredundant family of \(k\)-subcubes of \(\{0, 1\}^{2k}\) which contain 0 or 1, then \(|\mathcal{A}| \leq \binom{2k}{k}\).

**Proof.** Let \(\mathcal{A}\) be an irredundant family of \(k\)-subcubes of \(\{0, 1\}^{2k}\) which all contain either 0 or 1. We may assume that \(\mathcal{A}\) is maximal with respect to this condition. For \(v \in [2k]^{(k)}\), we write

\[
\mathbb{U}v := \{y : v \subset y \subset [2k]\}
\]

for the \(k\)-subcube between \(v\) and \([2k]\).

We partition the vertices of the middle layer \([2k]^{(k)}\) into three sets:

\[
S = \{v \in [2k]^{(k)} : \mathbb{P}v, \mathbb{U}v \in \mathcal{A}\};
\]

\[
T = \{v \in [2k]^{(k)} : \text{exactly one of } \mathbb{P}v \text{ and } \mathbb{U}v \text{ is in } \mathcal{A}\};
\]

\[
R = \{v \in [2k]^{(k)} : \mathbb{P}v \notin \mathcal{A}, \mathbb{U}v \notin \mathcal{A}\}.
\]

Notice that

\[
|\mathcal{A}| = \binom{2k}{k} + |S| - |R|;
\]

we must show that \(|S| \leq |R|\).

Write \(S = \{v_1, \ldots, v_N\}\). For each \(v_i \in S\), \(\mathbb{P}v_i\) must have a private vertex \(w_i \in [2k]^{(k-1)}\). If \(|w_i| < k - 2\), then we may choose \(b_i \in [2k]^{(k-1)}\) such that \(w_i \subset b_i \subset v_i\); \(b_i\) must also be a private vertex for \(\mathbb{P}v_i\), since any subcube containing both 0 and \(b_i\) must contain \(w_i\) as well. Similarly, we may choose a private vertex \(c_i \in [2k]^{(k+1)}\) for \(\mathbb{U}v_i\). Each point of \(T\) is a private vertex for the subcube in \(\mathcal{A}\) containing it. Let \(B = \{b_1, \ldots, b_N\}\), and let \(C = \{c_1, \ldots, c_N\}\). Then we can choose all the private vertices to lie in \(T \cup B \cup C\). For each \(i\), let

\[
B_i = \{x \in [2k]^{(k)} : b_i \subset x\}, \quad C_i = \{x \in [2k]^{k} : x \subset c_i\}
\]

be the neighbourhoods of \(b_i\) and \(c_i\) in \([2k]^{(k)}\). First, we claim that

\[
\left(\bigcup_{i=1}^{N} B_i\right) \cap \left(\bigcup_{i=1}^{N} C_i\right) = S \cup R.
\]

To see this, take \(x \in (\bigcup_{i=1}^{N} B_i) \cap (\bigcup_{i=1}^{N} C_i)\); then \(b_i \subset x \subset c_j\) for some \(i\) and \(j\). Suppose \(\mathbb{P}x \in \mathcal{A}\); then \(b_i \in \mathbb{P}x\), so \(x = v_i \in S\), i.e. \(\mathbb{U}x \in \mathcal{A}\) as well. Similarly, if \(\mathbb{U}x \in \mathcal{A}\), then \(\mathbb{P}x \in \mathcal{A}\) as well. Hence, \((\bigcup_{i=1}^{N} B_i) \cap (\bigcup_{i=1}^{N} C_i) \subset S \cup R\).
Clearly, $S \subset (\bigcup_{i=1}^{N} B_i) \cap (\bigcup_{i=1}^{N} C_i)$, as $b_i \subset v_i \subset c_i$ for every $i$. If $x \in R$, then by the maximality of $\mathcal{A}$, $\mathbb{P}x$ must contain some $b_i$ (otherwise it could be added to $\mathcal{A}$ to produce a larger irredundant family), and similarly $\bigcup x$ must contain some $c_j$. Hence, $x \in (\bigcup_{i=1}^{N} B_i) \cap (\bigcup_{i=1}^{N} C_i)$. It follows that $R \subset (\bigcup_{i=1}^{N} B_i) \cap (\bigcup_{i=1}^{N} C_i)$ as well, proving the claim.

For each $i$, let $B'_i = B_i \cap R = B_i \setminus S$, and let $C'_i = C_i \cap R = C_i \setminus S$; then $B'_i, C'_i \subset R$ for each $i$. We claim that

$$|B'_i \cap C'_i| = 1 \text{ for each } i, \text{ and } |B'_i \cap C'_j| = 0 \text{ or } 2 \text{ for each } i \neq j. \quad (8)$$

To see this, first observe that for each $i$,

$$B_i \cap C_i = \{x \in [2k]^{(k)} : b_i \subset x \subset c_i\} = \{v_i, y_i\}$$

for some $y_i \in R$, and therefore

$$B'_i \cap C'_i = \{y_i\}.$$ 

For each $i \neq j$, if $b_i \not\subset c_j$, then

$$B_i \cap C_j = \emptyset$$

and therefore

$$B'_i \cap C'_j = \emptyset.$$ 

If $b_i \subset c_j$, then $B_i \cap C_j = \{x \in [2k]^{(k)} : b_i \subset x \subset c_j\}$ has size 2, and cannot contain a point of $S$, since if $b_i \subset v_i \subset c_j$, then $i = j = l$. Hence, $B'_i \cap C'_j$ also has size 2, proving (8).

We recall the following easy lemma, the $p = 2$ case of which appears in [2]:

**Lemma 9.** Let $p$ be prime. If $F_1, \ldots, F_N, G_1, \ldots, G_N \subset [m]$ are such that

$$|F_i \cap G_j| \equiv 0 \mod p \quad \forall i \neq j$$

and $$|F_i \cap G_i| \not\equiv 0 \mod p \quad \forall i,$$

then

$$N \leq m.$$
Proof. Let \( \chi_F \) be the characteristic function of \( F \subset [m] \). Consider it as an element of the \( m \)-dimensional vector space \( \mathbb{F}_p^m \) over \( \mathbb{F}_p \). Observe that \( \{ \chi_{F_1}, \ldots, \chi_{F_N} \} \) is linearly independent over \( \mathbb{F}_p \). To see this, suppose

\[
\sum_{i=1}^{N} r_i \chi_{F_i} = 0
\]

for some \( r_1, \ldots, r_N \in \mathbb{F}_p \). Taking the inner product of the above with \( \chi_{G_j} \) gives \( r_j = 0 \). Hence, \( N \leq m \) as required.

Applying the \( p = 2 \) case of this lemma to the sets \( B'_1, \ldots, B'_N, C'_1, \ldots, C'_N \subset \mathbb{R} \) shows that

\[
|S| \leq |R|,
\]

proving the theorem.

We immediately obtain the same result for all \( n \leq 2k \), by induction on \( n \) for fixed codimension \( c = n - k \), using a projection argument:

**Corollary 10.** Let \( n \leq 2k \). If \( \mathcal{A} \) is an irredundant family of \( k \)-subcubes of \( \{0,1\}^n \) which contain 0 or 1, then \( |\mathcal{A}| \leq \binom{n}{k} \).

**Proof.** Suppose the result is true for some \( n \) and \( k \) such that \( n \geq 2k \); we will prove it for \( n + 1, k + 1 \). Let \( \mathcal{A} \) be an irredundant family of \( (k+1) \)-subcubes of \( \{0,1\}^{n+1} \) which contain 0 or 1. Let \( \mathcal{A}_i = \{ C \in \mathcal{A} : C_i = * \} \) be the collection of subcubes in \( \mathcal{A} \) with coordinate \( i \) moving; since each subcube has \( k + 1 \) moving coordinates,

\[
\sum_{i=0}^{n+1} |\mathcal{A}_i| = (k + 1)|\mathcal{A}|.
\]

We will show that \( |\mathcal{A}_i| \leq \binom{n}{k} \) for each \( i \in [n+1] \), giving \( |\mathcal{A}| \leq \frac{n+1}{k+1} \binom{n}{k} = \binom{n+1}{k+1} \). Without loss of generality, \( i = n + 1 \). We project the family \( \mathcal{A}_{n+1} \) of \( (k+1) \)-subcubes onto \( \{0,1\}^n \): let \( \mathcal{A}'_{n+1} = \{ C' : C \in \mathcal{A}_{n+1} \} \), where \( C' \) is the \( k \)-subcube of \( \{0,1\}^n \) produced by projecting \( C \) onto \( \{0,1\}^n \), i.e. deleting the \( (n+1) \)-coordinate of \( C \) (which is a \( * \)). Clearly, \( \mathcal{A}'_{n+1} \) is a collection of \( |\mathcal{A}_{n+1}| \) \( k \)-subcubes of \( \{0,1\}^n \) through 0 or 1. It is also irredundant, as the projection of a private vertex of \( C \) in \( \mathcal{A}_{n+1} \) is clearly a private vertex for \( C' \) in \( \mathcal{A}'_{n+1} \). Hence, by the induction hypothesis, \( |\mathcal{A}'_{n+1}| \leq \binom{n}{k} \), giving the result.

Notice that we do not have uniqueness of the extremal families in Theorem 8 for any value of \( k \): as well as taking \( \mathcal{A} = \mathcal{F}_0 \) or \( \mathcal{F}_1 \), any family \( \mathcal{A} \) containing exactly one of \( \mathbb{P}_x, \mathbb{U}_x \) for each \( x \in [2k]^{(k)} \) is extremal. Slightly more surprisingly, we do not have uniqueness (in Corollary 10) for \( n = 5, k = 3 \) either:
consider the irredundant family of ten 3-subcubes of \( \{0, 1\}^5 \), five through 0 and five through 1, exhibited earlier.

## 3 Lower bounds

### The case \( n = 2k \)

Now, returning to general irredundant families, what can we say about the case \( n = 2k \)? Meshulam’s bound gives:

\[
M(2k, k) \leq \frac{2}{1 + 2^{-2k} \binom{2k}{k}} = \frac{2}{1 + (1 + o(1))/\sqrt{2\pi k}} \binom{2k}{k} = 2(1 - \Theta(1/\sqrt{k})) \binom{2k}{k}.
\]

Our lower bound (3) no longer beats \( \mathcal{F}_0 \), since it only gives

\[
M(2k, k) \geq \beta(1 - \beta)^{(1-\beta)/\beta} 2^{2k} = (1 + o(1)) \frac{\beta}{e(1 - \beta)} 2^{2k} = (1 + o(1)) \frac{2^{2k}}{e \binom{2k}{k}}.
\]

Notice that \( \mathcal{F}_0 \) is a maximal irredundant family. We know from Theorem 8 that any irredundant family of \( k \)-subcubes in which each goes through either 0 or 1 has size at most \( \binom{2k}{k} \); we now exhibit a maximal such family \( \mathcal{B} \) which is not maximal irredundant.

Let \( \mathcal{B}_0 = \{ \mathbb{P} x : 1 \in x \} \) be the collection of \( k \)-subcubes containing the line \((*, 0, 0, ..., 0)\), and \( \mathcal{B}_1 = \{ \mathbb{U} x : n \notin x \} \) the collection containing \((1, 1, ..., 1, *)\). Consider the family \( \mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \); it has size \( |\mathcal{B}| = 2 \binom{2k-1}{k-1} = \binom{2k}{k} \); we will show that it is irredundant and not maximal. What are the \( \mathcal{B} \)-private vertices of each subcube \( C \in \mathcal{B} \)? Write \( C_i \) for the symbol (0, 1 or *) in the \( i \)-coordinate of the subcube \( C \). There are 4 different types of subcubes in \( \mathcal{B} \) to consider:

- \( C \in \mathcal{B}_0 \) with \( C_n = 0 \), e.g. \( C = (*, *, ..., *, 0, ..., 0) \) has \( \mathcal{B}_0 \)-private vertices \((*, 1, ..., 1, 0, ..., 0)\);
- \( (1, 1, ..., 1, 0, ..., 0) \in (1, 1, ..., 1, *, ..., *) \in \mathcal{B}_1 \), but \( (0, 1, ..., 1, 0, ..., 0) \in [n]^{k-1} \) so is not in any \( D \in \mathcal{B}_1 \), so is the unique \( \mathcal{B} \)-private vertex of \( C \).
• $C \in \mathcal{B}_0$ with $C_n = \ast$: e.g. $C = (\ast, \ast, \ldots, \ast, 0, \ldots, 0, \ast)$ has $\mathcal{B}_0$-private vertices 
$(\ast, 1, \ldots, 1, 0, \ldots, 0, 1)$;
this line has $k$ fixed 0’s in coordinates $\{2, \ldots, n - 1\}$ whereas each 
$D \in \mathcal{B}_1$ has at most $k - 1$ *’s in this range, hence this line is disjoint 
from $\mathcal{B}_1$ and both its vertices are the unique $\mathcal{B}$-private vertices of $C$.

• $C \in \mathcal{B}_1$ with $C_1 = 1$: e.g. $C = (1, \ast, \ldots, \ast, 1, \ldots, 1, \ast)$ has 
$\mathcal{B}$-private vertex 
$(1, 0, \ldots, 0, 1, \ldots, 1, 1)$

• $C \in \mathcal{B}_1$ with $C_1 = \ast$: e.g. $C = (\ast, \ast, \ldots, \ast, 1, \ldots, 1, \ast)$ has $\mathcal{B}$-private vertices 
$(0, 0, \ldots, 0, 1, \ldots, 1, \ast)$

Notice that 
$$\bigcup_{D \in \mathcal{B}_0} D = [n]^{(\leq k - 1)} \cup \{x \in [n]^{(k)}: 1 \in x\}$$
and 
$$\bigcup_{D \in \mathcal{B}_1} D = [n]^{(\geq k + 1)} \cup \{x \in [n]^{(k)}: n \notin x\}$$
Hence, 
$$\{0, 1\}^n \setminus \bigcup_{D \in \mathcal{B}} D = \{x \in [n]^{(k)}: 1 \notin x, n \in x\}$$
Now let $E$ be any $k$-subcube with $E_1 = 0, E_n = 1$.

**Claim:** $\mathcal{B} \cup \{E\}$ is also irredundant.

**Proof of Claim:** If $E$ has $s$ 0’s and $t$ 1’s in coordinates $\{2, \ldots, n - 1\}$, where 
$s + t = k - 2$, then setting $k - 1 - t \ast$’s = 1 and the other $t + 1$ *’s = 0, 
we find an $x \in E \cap [n]^{(k)}: 1 \notin x, n \in x$, i.e. a $\mathcal{B}$-private vertex for $E$. We 
must now check that each of the above types of subcube in $\mathcal{B}$ has a $\mathcal{B}$-private 
vertex not in $E$:

• $C \in \mathcal{B}_0$ with $C_n = 0$: disjoint from $E$, so the $\mathcal{B}$-private vertex will do.

• $C \in \mathcal{B}_0$ with $C_n = \ast$: choose the $\mathcal{B}$-private vertex with 1-coordinate 1.

• $C \in \mathcal{B}_1$ with $C_1 = 1$: disjoint from $E$, so the $\mathcal{B}$-private vertex will do.

• $C \in \mathcal{B}_1$ with $C_1 = \ast$: choose the $\mathcal{B}$-private vertex with $n$-coordinate 0.
This proves the claim. How many such subcubes can we add on? We can certainly add on the family:
\[ \mathcal{E} = \{ E : E_1 = 0, E_n = 1, E_2 = *, E_i = 0 \text{ or } * \forall i \neq 1, 2 \text{ or } n \} \]
e.g. the subcube
\[(0, *, 0, \ldots, 0, *, \ldots, *, 1) \text{ has private vertex} \]
\[(0, 1, 0, \ldots, 0, 1, \ldots, 1, 1). \]
Hence,
\[ M(2k, k) \geq \binom{2k}{k} + \binom{2k - 3}{k - 1} = (1 + \frac{1}{8} + o(1)) \binom{2k}{k} \]
but we still have a gap of \( \frac{7}{8} \) between the constants in our lower and upper bounds.

Notice the sharp drop by a factor of order \( \sqrt{n} \) from \( M(n, \lfloor \gamma n \rfloor) = \Theta(2^n) \) for \( \gamma \in (0, \frac{1}{2}) \) to
\[ M(n, \lfloor n/2 \rfloor) \leq 2 \binom{n}{\lfloor n/2 \rfloor} = 2(1 + o(1)) \frac{2^n}{\sqrt{\pi n}} \]

**The case \( k < \frac{1}{2} n \)**

When \( k < \frac{1}{2} n \), we can construct an irredundant family by taking a union of \( \mathcal{F}_v \)'s: choose a maximum \((2k + 1)\)-separated subset \( S \subset \{0, 1\}^n \) (i.e. a maximum \( k \)-error correcting code) and let
\[ \mathcal{F}_S = \bigcup_{v \in S} \mathcal{F}_v \]
be the family of all \( k \)-subcubes containing a point of \( S \); then
\[ |\mathcal{F}_S| = |S| \binom{n}{k}. \]

When there is a subset \( S \subset \{0, 1\}^n \) such that the Hamming balls of radius \( k \) centred on the vertices of \( S \) partition \( \{0, 1\}^n \) (i.e. a perfect \( k \)-error correcting code),
\[ |\mathcal{F}_S| = \frac{2^n}{\sum_{i=0}^{k} \binom{n}{i} \binom{n}{k}} \]
which exactly matches Meshulam's bound.

It is known that there is a perfect \( k \)-error correcting code in \( \{0, 1\}^n \) precisely in the following cases (see [7]):
• $k = 1, n + 1$ is a power of 2 (take any Hamming code)

• $k = 3, n = 23$ (take the Golay code)

• $n = 2k + 1$ (take a ‘trivial’ code, two vertices of distance $n$ apart)

so in these cases, we have equality in Meshulam’s bound:

$$M(n, k) = \frac{2^n}{\sum_{l=0}^{k} \binom{n}{l}} \binom{n}{k}.$$ 

First, consider the case $k = 1$; a 1-subcube is simply an edge of $\{0, 1\}^n$. Meshulam’s bound is

$$M(n, 1) \leq \frac{n}{n+1} 2^n.$$ 

Kabatyanskii and Panchenko [5] proved the existence of asymptotically perfect packings of 1-balls into $\{0, 1\}^n$, namely that there is a packing of

$$\frac{2^n}{n+1} (1 - O(\ln \ln n / \ln n))$$

1-balls into $\{0, 1\}^n$. Taking all edges through the centre of each ball gives an irredundant family of size

$$\frac{n}{n+1} 2^n (1 - O(\ln \ln n / \ln n)) = 2^n (1 - O(\ln \ln n / \ln n))$$

We can in fact improve on this with the following ‘product’ construction. Let $s \in \mathbb{N}$ be maximal such that $2^s - 1 \leq n$; write $n = m + r$ where $m = 2^s - 1$. Take a perfect packing of 1-balls into $\{0, 1\}^m$ and take all edges through the centre of each ball, producing an irredundant family $\mathcal{B}$ in $\{0, 1\}^m$ of size $\frac{m}{m+1} 2^m$. Writing $\{0, 1\}^n = \{0, 1\}^m \times \{0, 1\}^r$, let $\mathcal{A}$ be the family consisting of a copy of $\mathcal{B}$ in each of the $2^r$ disjoint copies of $\{0, 1\}^r$; $|\mathcal{A}| = \frac{m}{m+1} 2^n$. Notice that $m = 2^s - 1 \geq \frac{1}{2} n$, since otherwise $2^{s+1} - 1 \leq n$, contradicting the maximality of $s$. Hence, $|\mathcal{A}| \geq \frac{n}{n+2} 2^n$, and we have

$$M(n, 1) \geq \frac{n}{n+2} 2^n \quad \forall n \in \mathbb{N},$$

so

$$M(n, 1) = 2^n (1 - \Theta(1/n)).$$
What about for $k$ fixed and $n$ growing? It is a longstanding open problem in coding theory to determine whether, for $k$ fixed, there is an asymptotically perfect packing of $k$-balls into $\{0, 1\}^n$, i.e. a packing of

$$\frac{2^n}{\sum_{i=0}^{k} \binom{n}{i}} (1 - o(1))$$

$k$-balls into $\{0, 1\}^n$; given such, by taking all $k$-subcubes through the centre of each ball, we would immediately obtain an irredundant family of size

$$\frac{\binom{n}{k}}{\sum_{i=0}^{k} \binom{n}{i}} 2^n (1 - o(1)) = 2^n (1 - o(1))$$

However, this conjecture remains unsolved for all $k > 1$.

Moreover, for $k = \Omega(n)$, the approach outlined above can only give a relatively small irredundant family. Corrádi and Katai [4] proved the following:

**Theorem 11** (Corrádi-Katai, 1969). Let $S \subset \{0, 1\}^n$ be an $(n/2)$-separated set; then

- $|S| \leq n + 1$ if $n$ is odd
- $|S| \leq n + 2$ if $n \equiv 2 \mod 4$
- $|S| \leq 2n$ if $n \equiv 0 \mod 4$

(For a proof of this, we refer the reader for example to [3] §10.)

So we see that, for example, any $(2k+1)$-separated family $S$ of vertices in $Q_{4k}$ must have $|S| \leq 8k$, and so taking all $k$-subcubes through each of these vertices only gives

$$|\mathcal{F}_S| \leq 8k \binom{4k}{k} \leq 8k \exp \left( -\frac{4k}{32} \right) 2^{4k}.$$

We now improve on this using a probabilistic method. The idea is to take a random subset $S \subset \{0, 1\}^n$ where each vertex is present independently with some fixed probability $p$; for each vertex $w \in \{0, 1\}^n$ of (Hamming) distance $k$ from $S$, we choose a $k$-subcube $C_w$ between $w$ and some vertex of $S$, giving a random irredundant family of $k$-subcubes $\mathcal{A} = \{C_w : d(w, S) = k\}$; the expected size of this family is then a lower bound for $M(n, k)$.  

19
**Theorem 12.** For any \( k \leq n \), there exists an irredundant family of \( k \)-subcubes of \( \{0, 1\}^n \) of size at least

\[
\beta (1 - \beta)^{(1 - \beta)/\beta} 2^n,
\]

where

\[
\beta = \beta_{n,k} := \frac{(n)}{\sum_{i=0}^{k} \binom{n}{i}}.
\]

**Proof.** Let \( S \) be a random set of vertices in \( \{0, 1\}^n \) where each vertex is present independently with probability \( p \) (to be chosen later). Consider the random set of vertices

\[
W = \{ x \in \{0, 1\}^n : d(x, S) = k \},
\]

where \( d(x, y) = |x \Delta y| \) denotes the Hamming distance between \( x \) and \( y \). For each \( w \in W \), choose any \( x_w \in S \) such that \( |w \Delta x_w| = k \), and let \( C_w \) be the \( k \)-subcube between \( x_w \) and \( w \), i.e.

\[
C_w = \{ y \in \{0, 1\}^n : y \Delta w \subset x_w \Delta w \}.
\]

Consider the random family of \( k \)-subcubes

\[
\mathcal{A} = \{ C_w : w \in W \}.
\]

Note that the subcubes \( C_w \) are pairwise distinct: \( x_w \) is the unique point of \( S \) in \( C_w \), and \( w \) is the ‘opposite’ point, so \( C_w \) determines \( w \). Moreover, \( \mathcal{A} \) is irredundant, since \( w \) is a private vertex of \( C_w \). (If \( w \in C_{w'} \), then \( |x_{w'} \Delta w| \leq k \), so \( |x_{w'} \Delta w| = k \), so \( w \) is the unique vertex in \( C_{w'} \) of distance \( k \) from \( x_{w'} \), so \( w = w' \).) We now calculate the expectation of the random variable \( |\mathcal{A}| = |W| \). A vertex \( v \in \{0, 1\}^n \) is in \( W \) if and only if the \( (k-1) \)-ball around \( v \) contains no vertices of \( S \) but the \( k \)-ball around \( v \) does contain a vertex of \( S \); the probability of this event is

\[
(1 - p)^{\sum_{i=0}^{k-1} \binom{n}{i}} - (1 - p)^{\sum_{i=0}^{k} \binom{n}{i}}.
\]

Hence, the expected size of \( \mathcal{A} \) is

\[
E|\mathcal{A}| = 2^n \left( (1 - p)^{\sum_{i=0}^{k-1} \binom{n}{i}} - (1 - p)^{\sum_{i=0}^{k} \binom{n}{i}} \right).
\]
Let 
\[ \beta = \beta_{n,k} := \frac{\binom{n}{k}}{\sum_{i=0}^{k} \binom{n}{i}}, \quad t := (1 - p)\sum_{i=0}^{k} \binom{n}{i}; \]

then 
\[ \mathbb{E}|\mathcal{A}| = 2^n(t^{1-\beta} - t). \]

The function 
\[ f : [0, 1] \rightarrow \mathbb{R}; \quad t \mapsto t^{1-\beta} - t \]
attains its maximum of 
\[ \beta(1 - \beta)^{(1-\beta)/\beta} \]
at 
\[ t = (1 - \beta)^{1/\beta}. \]

Hence, choosing \( p \) such that 
\[ (1 - p)\sum_{i=0}^{k} \binom{n}{i} = (1 - \beta)^{1/\beta}, \]
our random irredundant family has expected size 
\[ \mathbb{E}|\mathcal{A}| = \beta(1 - \beta)^{(1-\beta)/\beta}2^n. \]

Hence, there exists an irredundant family of size at least this, proving the theorem. \( \square \)

Combining this with Meshulam’s bound, we see that 
\[ \beta(1 - \beta)^{(1-\beta)/\beta}2^n \leq M(n, k) \leq \beta 2^n. \]

The ratio between the lower and upper bound above is 
\[ g(\beta) := (1 - \beta)^{(1-\beta)/\beta}. \]

Observe that \( g'(\beta) > 0 \ \forall \beta \in (0, 1) \), so \( g \) is strictly increasing on \((0, 1)\). Note that 
\[ \ln(g(\beta)) = \frac{1-\beta}{\beta} \ln(1 - \beta) \rightarrow -1 \text{ as } \beta \rightarrow 0, \]
so \( g(\beta) \rightarrow 1/e \) as \( \beta \rightarrow 0 \); \( \ln(g(\beta)) \rightarrow 0 \) as \( \beta \rightarrow 1 \), so \( g(\beta) \rightarrow 1 \) as \( \beta \rightarrow 1 \). Hence, \( 1/e \leq g(\beta) \leq 1 \ \forall \beta \in (0, 1) \), so the ratio between the upper and lower
bounds above never exceeds $e$. We believe that the upper bound is closer to the true value, but we have been unable to improve our lower bound.

If $k = o(n)$, then $\beta = 1 - o(1)$. Let

$$\eta = 1 - \beta = \frac{\sum_{i=0}^{k-1} \binom{n}{i}}{\sum_{i=0}^{k} \binom{n}{i}};$$

then $\eta = o(1)$.

Theorem 12 implies that

$$M(n, k) \geq (1 - \eta)\eta^{n/(1-\eta)}2^n = (1 - O(\eta \ln(1/\eta)))2^n;$$

which asymptotically matches the upper bound from Meshulam’s theorem,

$$M(n, k) \leq \beta 2^n = (1 - \eta)2^n.$$

If $k = [\gamma n]$ for some $\gamma \in (0, \frac{1}{2})$, using the fact that as $l$ decreases from $k - 1$ to 0, $\binom{n}{l}$ decreases geometrically, we obtain

$$\beta_{n,[\gamma n]} = (1 + o(1))\frac{1 - 2\gamma}{1 - \gamma};$$

substituting this into (9) gives:

$$(1+o(1)) \left( \frac{\gamma}{1 - \gamma} \right)^{1 - \frac{2\gamma}{1 - \gamma}} \left( \frac{1 - 2\gamma}{1 - \gamma} \right) 2^n \leq M(n, [\gamma n]) \leq (1+o(1)) \left( \frac{1 - 2\gamma}{1 - \gamma} \right) 2^n.$$

Hence, we see that

$$M(n, [\gamma n]) = \Theta(2^n).$$

Comparing this with

$$M(n, [n/2]) = \Theta \left( \left( \frac{n}{\lfloor n/2 \rfloor} \right) \right) = \Theta(2^n/\sqrt{n}),$$

we see that $M(n, [\gamma n])$ experiences a drop in its order of magnitude at $\gamma = 1/2$. 

22
4 Conclusion

To conclude, we believe Conjecture 1 to be true, but that new ideas would be required to prove it for all $k > n/2$. The problem seems at first glance to be ideal for tackling using the methods of linear algebra, but we have only been able to obtain a sharp result using such methods under the additional constraint of all the subcubes going through 0 or 1. All the above-mentioned proofs of Meshulam’s bound involve considering separately certain subfamilies of an irredundant family, and then averaging; to prove the conjecture when $k$ is close to $n/2$, one would need to take into account how an efficient arrangement in one region of $\{0,1\}^n$ is incompatible with efficient arrangements in other parts. The fact that Meshulam’s bound is tight for $n = 2k + 1$ indicates that the ideas used to prove it will probably not help to approach the conjecture when $k$ is close to $n/2$.

If Conjecture 1 turns out to be true, it would also be of interest to determine when the only extremal families are the $F_x$’s; we conjecture this to be the case for all $n > 5$. It may also be possible to close the gap between the lower and upper bounds in (9) for $k < n/2$, though we consider it fortunate that there is only a constant gap between our ‘random’ lower bound and Meshulam’s ‘combinatorial’ upper bound.

References


