Example sheet 2 - solutions

1. Consider an r-uniform hypergraph on $n+1$ vertices which contains no copy of $\mathcal{H}$. Then no subset of n vertices contains a copy of $\mathcal{H}$ either. Hence, any subset of size n contains at most $ex(n, \mathcal{H})$ edges. Therefore, by averaging over subsets of size n,

$$ex(n+1, \mathcal{H}) = \frac{1}{\binom{n+1-r}{n-r}} \sum_{|U|=n} e(U) \leq \frac{n+1}{n+1-r} ex(n, \mathcal{H}).$$

Dividing either side by $\binom{n+1}{r}$ yields

$$\frac{ex(n+1, \mathcal{H})}{\binom{n+1}{r}} \leq \frac{ex(n, \mathcal{H})}{\binom{n}{r}}.$$

Therefore, since these ratios are decreasing and bounded below by 0, they must approach a limit.

2. Let X be a set of n points in the plane. Form a graph G by connecting two vertices if and only if they are distance 1 apart. It is easy to check that the graph contains no copy of $K_{2,3}$. Therefore $e(G) \leq ex(n, K_{2,3}) \leq cn^{3/2}$, as required.

3. This is a standard application of the probabilistic method. Consider the random graph $G_{n,p}$ where $p = cn^{-\frac{t-2}{m-1}}$. The expected number of edges is at least $pn^2/8$ and the expected number of copies of H is at most $p^m n^t$. For c sufficiently small, we have

$$p^m n^t \leq \frac{1}{16} pn^2.$$

Therefore, since $\mathbb{E}(\text{edges} - \text{copies of } H) \geq \frac{1}{16} pn^2$, we may remove all copies of H and still be left with a graph which has $\frac{1}{16} pn^2 = \frac{c}{16} n^{2-\frac{t-2}{m-1}}$ edges.

4. By the convexity of the function $f(x) = \binom{x}{a}$ and the fact that the average degree of B is at least $a = \epsilon|A|$, we conclude that the number of pairs (U, v) with U a subset of A of size a and v a vertex in B connected to every element of U is at least

$$\sum_{v \in B} \binom{d(v)}{a} \geq |B| \binom{\frac{1}{|B|} \sum_{v \in B} d(v)}{a} \geq |B|.$$

Since A has at most $2^{|A|}$ subsets, the pigeonhole principle implies that for some $U \subset A$ of size a there are at least $b = 2^{-|A|}|B|$ elements of B which are connected to every element of U. This yields the required copy of $K_{a,b}$.

For the second part, note that

$$\sum_{v \in B} \binom{d(v)}{s} \geq |B| \binom{\epsilon|A|}{s} \geq |B| \frac{(\epsilon|A|/2)^s}{s!},$$

where the inequalities follow from the convexity of $f(x) = \binom{x}{s}$ and the fact that $s = c(\epsilon) \log n \leq \epsilon|A|/2$. If the graph does not contain $K_{s,t}$ then we know that every subset of A of size s has at most $t-1$ common neighbours. Therefore,

$$|B| \frac{(\epsilon|A|/2)^s}{s!} \leq (t-1) \binom{|A|}{s} < t \frac{|A|^s}{s!} = n^{1/2} \frac{|A|^s}{s!}.$$

But for $c(\epsilon)$ sufficiently small, $(\epsilon/2)^s |B| \geq n^{1/2}$, so this is a contradiction.

5. Partition the set of vertices into three sets at random. This easily yields a partition for which there are at least $c\delta n^3$ triangles with one vertex in each part. We will therefore, without loss of generality, assume that we have three vertex sets V_1, V_2 and V_3 and that there are δn^3 triangles with one vertex in each part. Let E_{23} be the set of edges between V_2 and V_3 which are contained in at least $\frac{\delta}{2}n$ triangles. Note that $|E_{23}| \geq \frac{\delta}{2}n^2$. Otherwise, we would have at most $|E_{23}|n + n^2\frac{\delta}{2}n < \delta n^3$ triangles, contradicting our assumption.

   Using the second part of the previous question, we may now find a complete graph between two sets $W_2 \subset V_2$ and $W_3 \subset V_3$, where $|W_2| = |W_3| = c(\delta)\log n$. Now consider a complete matching M between W_2 and W_3. M will have $c(\delta)\log n$ edges. Consider the bipartite graph between M and the set V_1, where m and v are joined if any only if they form a triangle together. Since every edge in M is in E_{23}, there are at least $|M|\frac{\delta}{2}n = \frac{\delta}{2}|M||V_1|$ edges in the graph. Therefore, applying the first part of the previous question, we can find a subset M' of M of size $\frac{\delta}{2}|M|$ and a subset of W_1 of V_1 of size $2^{-|M'|}n \geq n^{1/2}$, for $c(\delta)$ sufficiently small. Let X_2 and X_3 be the two endpoints of the matching M' and let X_1 be a subset of W_1 of size $|M'|$. The graph between X_1, X_2 and X_3 is the required blow-up of the triangle.

6. By supersaturation, we know that as soon as we have density $\frac{1}{2} + \epsilon$, we have δn^3 triangles. By the previous question, this implies the existence of a large blow-up. This will contain any 3-chromatic graph provided n is sufficiently large. (Note that this question and the last may be generalised to give a full proof of Erdős-Stone-Simonovits.)

7. Let $H = K_{t,t,t}$, that is, there are three vertex sets of size t and any two vertices in different parts are connected. Consider also a graph G consisting of two vertex sets U and V of size $n/2$, where U is empty and V contains a graph L containing no copy of $K_{t,t}$. We know that there exist such graphs with at least $c(n/2)^{2-2/t}$ edges. We will assume that L has this many edges and, therefore, that G has $\frac{1}{4}n^2 + c'n^{2-2/t}$ edges.

   It is elementary to check that G contains no copy of H. Indeed, any copy of H clearly cannot be contained entirely within V. Therefore, there is some vertex in U. But the neighborhood of this vertex in H contains a copy of $K_{t,t}$ and the neighborhood of this vertex in G must lie entirely inside V, so we have a contradiction.

8. Let H be a graph which is not colour-critical, that is, removing any edge still leaves one with a graph of chromatic number t. Consider the Turán graph, which consists of $t-1$ vertex subsets of size as equal as possible and add a single edge e in one of the vertex sets. We will show that this graph, which necessarily contains a copy of K_t, does not contain a copy of H. Clearly, any copy of H must contain the edge e. So for any copy we get two vertices in the same vertex subset and, by construction, every other vertex must lie in distinct vertex subsets. But then the edge e is the only obstruction to making the graph $(t-1)$-chromatic, so deleting the edge e from H will yield a graph of chromatic number $t-1$. This contradicts the definition of colour-critical.

9. Suppose that we have a hypergraph $\mathcal{G}$ with n vertices and cn^{3-1/t^2} edges not containing $K_{t,t,t}$ as a subgraph. Note that the average number of edges containing a 2-edge is $3cn^{1-1/t^2}$. We will count pairs (e, T) consisting of edges e and sets of vertices T, of size t, such that every vertex in T forms an edge with e. The number of such pairs is at least

$$\sum_e \binom{d(e)}{t} \geq \binom{n}{2}\binom{\frac{1}{\binom{n}{2}}\sum_e d(e)}{t} \geq \binom{n}{2}\binom{3cn^{1-1/t^2}}{t} \geq \binom{n}{2}c^t n^{t-1/t}t! = c'\frac{n^{t+2-1/t}}{t!}.$$

Therefore, since there are $\binom{n}{t}$ possible choices for T, there exists some T_1 of size t with common neighborhood a graph E of size at least $c'n^{2-1/t}$. Provided c and hence c' is sufficiently large, we may now apply the result that we know for ordinary graphs to find sets T_2 and T_3 of size t such that there is a complete subgraph of E between them. This completes the proof.

10. I will not write out a full proof of this. Instead, I refer the reader to the survey paper 'Dependent random choice' by Jacob Fox and Benny Sudakov. A complete proof of the required result is contained in Lemma 2.1 and Theorem 3.4, though I heartily recommend the full paper.