Extremal graph theory - Example Sheet 2

1. Prove that the Turán density of an r-uniform hypergraph $\mathcal{H}$, that is, $\lim_{n\to\infty} ex(n,\mathcal{H})/\binom{n}{r}$, is well-defined.

2. Given a set of n points in the plane, prove that there are at most $cn^{3/2}$ pairs of points which are a unit distance apart.

3. Prove that if H is a bipartite graph with t vertices and m edges, then $ex(n,H) \geq cn^{2-\frac{t-2}{m-1}}$.

4. Show that a bipartite graph of density ϵ between sets A and B contains a complete bipartite subgraph $K_{a,b}$ with a subset of A of size $a = \epsilon|A|$ and a subset of B of size $b = 2^{-|A|}|B|$. Show also that if $|A| = |B| = n$, one may find a complete bipartite subgraph $K_{s,t}$ with $s = c(\epsilon)\log n$ and $t = n^{1/2}$.

5. Suppose that a graph G contains δn^3 triangles. Use the result of the previous question to show that G must also contain a blow-up $K_{t,t,t}$, that is, a graph with three vertex sets of size t such that every two vertices in different vertex sets are connected, where $t = c(\delta)\log n$.

6. Use the result of the previous question with the supersaturation property to prove the particular case of the Erős-Stone-Simonovits theorem when $\chi(H) = 3$.

7. Prove that for any $\epsilon > 0$ there exists a graph H of chromatic number 3 such that $ex(n,H) > \frac{1}{4}n^2 + c_H n^{2-\epsilon}$.

8. Show that a graph H of chromatic number t can satisfy $ex(n,H) = ex(n,K_t)$ only if it is colour-critical, that is, if there is an edge e such that $\chi(H\backslash e) < t$.

9. Let $\mathcal{H} = K^{(3)}_{t,t,t}$ be the complete tripartite 3-uniform hypergraph with three sets each of size t. Prove that $ex(n,\mathcal{H}) \leq cn^{3-\frac{1}{t^2}}$.

10. The cube Q_t is the graph on vertex set $\{0,1\}^t$ where two vertices are connected if and only if they differ in exactly one coordinate. The Ramsey number of a graph H is the smallest number n such that in any 2-colouring of the complete graph K_n on n vertices there is guaranteed to be a monochromatic copy of H. Use dependent random choice to prove that $r(Q_t) \leq 2^{ct}$.