Example sheet 1 - solutions

1. We will prove the result by induction on n. For $n = 3$, a subgraph with three edges contains one triangle, as expected. Similarly, for $n = 4$, it is easily checked that any subgraph with 5 edges must contain two triangles.

   Assume now that a graph with $n-2$ vertices and at least $\lfloor \frac{(n-2)^2}{4} \rfloor + 1$ edges contains at least $\lfloor \frac{n-2}{2} \rfloor$ triangles. We will prove the required result also holds for n. Suppose that we have a graph G on n vertices with $\lfloor \frac{n^2}{4} \rfloor + 1 = \lfloor \frac{(n-2)^2}{4} \rfloor + 1 + (n-1)$ edges but with fewer than $\lfloor \frac{n}{2} \rfloor$ triangles. Let x and y be two vertices which are joined by an edge but are not contained in a triangle. This is certainly possible, since $3(\lfloor \frac{n}{2} \rfloor - 1) \leq \lfloor \frac{n^2}{4} \rfloor + 1$. Therefore, as usual $d(x) + d(y) \leq n$. Moreover, the neighborhoods $N(x)$ and $N(y)$ of x and y must be disjoint. We now know that the graph $H = G - \{x, y\}$ contains at least $\lfloor \frac{(n-2)^2}{4} \rfloor + 1$ edges. It must therefore contain at least $\lfloor \frac{n-2}{2} \rfloor$ triangles. But the number of edges between $N(x)$ and $N(y)$ is at most $\lfloor \frac{(n-2)^2}{4} \rfloor$. Therefore, one of $N(x)$ and $N(y)$ must contain an edge. This yields one further triangle and proves the result.

   To show that the result is sharp, we just take the bipartite graph between sets of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ and add one extra edge in the set of size $\lceil \frac{n}{2} \rceil$.

2. We will prove the result by induction on n. For $n = 4$ and $n = 5$, there are no non-bipartite triangle-free graphs with n vertices and 4 or 6 edges respectively.

   Assume now that any non-bipartite graph on $n-2$ vertices with more than $\frac{1}{4}(n-3)^2 + 1$ edges contains a triangle. Let G be a non-bipartite graph on n vertices with more than $\frac{1}{4}(n-1)^2 + 1$ vertices and assume that it contains no triangles. Let xy be an edge in G. Since G is triangle-free, $N(x)$ and $N(y)$ both form independent sets. But the union of the two sets cannot be everything, for otherwise G would have to be bipartite. Therefore $d(x) + d(y) \leq n - 1$. This implies that the number of edges in $H = G - \{x, y\}$ is more than $\frac{1}{4}(n-3)^2 + 1$. If H is not bipartite, then, by induction, H contains a triangle and we are done. Therefore, the graph H must be bipartite.

   If H is bipartite, let A and B be the sets in the partition with $|A| \geq |B|$. Neither x nor y can have neighbours in both A and B. Otherwise, we would have a triangle. Moreover, if x only has neighbours in A and y only has neighbours in B (or vice versa), the graph G is bipartite. Therefore, all of the neighbours of x and y lie in A or B. Since $|A| \geq |B|$, the maximum number of edges occurs when all neighbours of x and y are in A. In this case, we have $|A||B| + |A| + 1$ edges. Since $|B| = n - |A| - 2$, this is maximised by taking $|A| = \lfloor \frac{n-1}{2} \rfloor$.

   To show that this is sharp for odd values of n, take two sets, one of size $\frac{n-1}{2}$ and the other of size $\frac{n-3}{2}$, and place every edge between them. Then take two extra vertices x and y, join them and connect one (and only one) of them to every vertex in the piece of size $\frac{n-1}{2}$, insisting that each of x and y has at least one neighbor in this set. This yields a graph G which is not bipartite (it has a 5-cycle), contains no triangle and has $\frac{1}{4}(n-1)^2 + 1$ edges.

3. Let $<$ be a uniformly chosen ordering of V. Define

   $$I = \{v \in V : \{v, w\} \in E \Rightarrow v < w\}.$$

   Let X_v be the indicator random variable which indicates whether or not $v \in I$. That is, it takes value 1 if $v \in I$ and 0 otherwise. Let $X = \sum_{v \in V} X_v = |I|$. For each v,

   $$\mathbb{E}[X_v] = \mathbb{P}[v \in I] = \frac{1}{d(v)+1},$$

since $v \in I$ if and only if it is the smallest element among v and its neighbours. Therefore

$$\mathbb{E}[X] = \sum_{v \in V} \frac{1}{d(v)+1}.$$

In particular, there exists some ordering for which $|I| \geq \sum_{v \in V} \frac{1}{d(v)+1}$. But it is easily verified that the set of elements in I form an independent set.

To deduce Turán's theorem, suppose that G is a graph with more than $\left(1 - \frac{1}{r-1}\right) \frac{n^2}{2}$ edges. Its complement $\overline{G}$ has fewer than

$$\binom{n}{2} - \left(1 - \frac{1}{r-1}\right) \frac{n^2}{2} = \frac{1}{r-1} \frac{n^2}{2} - \frac{n}{2}$$

edges. Now the function $\sum_v \frac{1}{d(v)+1}$ will be minimised when all of the $d(v)$ have size as close as possible. Therefore, taking $d(v) = \frac{1}{r-1} n - 1 - \epsilon$ for each v, we have

$$\alpha(\overline{G}) \geq \sum_v \frac{1}{d(v)+1} \geq \frac{n}{\frac{n}{r-1} - \epsilon} > r - 1.$$

Since an independent set in $\overline{G}$ is a clique in G, this implies Turán's theorem.

4. Let $S = \{x_1, \ldots, x_n\}$. Consider the graph G formed by joining two vertices if the distance between them is greater than $1/\sqrt{2}$. If we can show that G contains no copy of K_4, then Turán's theorem will imply that there are at most $\frac{2}{3}\frac{n^2}{2} = \frac{n^2}{3}$ edges in G, as required.

   To prove that G contains no K_4, we begin by noting that the convex hull of any four points forms either a line, a triangle or a quadrilateral. In any of these cases, there will be three points x_i, x_j and x_k such that the angle $x_i x_j x_k$ is at least 90 degrees.

   Now, consider the triangle formed by x_i, x_j and x_k. If both $d(x_i, x_j)$ and $d(x_j, x_k)$ are greater than $\frac{1}{\sqrt{2}}$, then $d(x_i, x_k)$ will be greater than 1, which contradicts the assumption about the set S. Therefore, at least one of $x_i x_j$ or $x_j x_k$ is not in G, so the graph does not contain a K_4.

   To show that it is sharp, let r be a real number with $0 < r < \left(1 - \frac{1}{\sqrt{2}}\right)/4$ and let $p = \lfloor \frac{n}{3} \rfloor$. Take an equilateral triangle with side length $1 - 2r$ and draw a circle of radius r around each of the vertices. Place $x_1, \ldots, x_p$ in the first circle, $x_{p+1}, \ldots, x_{2p}$ in the second circle and $x_{2p+1}, \ldots, x_n$ in the third circle. We may also insist that x_1 and x_n are distance 1 exactly apart to give the set diameter 1. If x_i and x_j are in different pairs, they are distance greater than $\frac{1}{\sqrt{2}}$ apart and if they are in the same set their distance is smaller than this. Therefore, there are $\lfloor \frac{n^2}{3} \rfloor$ pairs with $d(x_i, x_j) > \frac{1}{\sqrt{2}}$.

5. We take $A = B = \mathbb{N}$. We connect the vertex 1 in A to everything in B and, for $i > 1$, we connect i in A to $i - 1$ in B. This then satisfies Hall's condition but contains no matching.

6. The number of monochromatic triangles is at least

$$\frac{1}{2}\left(\sum_v \binom{r_v}{2} + \sum_v \binom{b_v}{2} - \binom{n}{3}\right),$$

where r_v and b_v are the red and blue degrees, respectively, of the vertices v over which we are summing. (To prove this formula, consider, in turn, the contribution of monochromatic and non-monochromatic triangles to the sum.) This is maximised when $r_v = b_v = (n-1)/2$ for all v. A quick calculation then implies that the number of monochromatic triangles is at least $\frac{n-5}{12}\binom{n}{2}$, as required.

7. This clearly reduces to determining the chromatic number of each of the graphs. One may easily verify that $\chi(\text{Tetrahedron}) = 4$, $\chi(\text{Cube}) = 2$, $\chi(\text{Octahedron}) = 3$, $\chi(\text{Dodecahedron}) = 3$ and $\chi(\text{Icosahedron}) = 4$.

8. This follows easily from the definition.

9. By assumption, any set of size n_0 containing more than $\rho\binom{n}{2}$ edges contains a copy of H. For at least $\frac{\epsilon}{2}\binom{n}{n_0}$ choices of a set N of size n_0, we must have that the number of edges in N is at least $\left(\rho + \frac{\epsilon}{2}\right)\binom{n_0}{2}$. If, on the contrary, this wasn't the case, we would have

$$\sum_N e(G[N]) \leq \binom{n}{n_0}\left(\rho + \frac{\epsilon}{2}\right)\binom{n_0}{2} + \frac{\epsilon}{2}\binom{n}{n_0}\binom{n_0}{2} = (\rho + \epsilon)\binom{n}{n_0}\binom{n_0}{2}.$$

On the other hand, we have

$$\sum_N e(G[N]) = \binom{n-2}{n_0-2}e(G) > \binom{n-2}{n_0-2}(\rho+\epsilon)\binom{n}{2} = (\rho+\epsilon)\binom{n}{n_0}\binom{n_0}{2},$$

which would be a contradiction. Now, every set of size n_0 with density $\rho + \frac{\epsilon}{2}$ contains a copy of H. Therefore, the number of copies of H is at least

$$\binom{n - v(H)}{n_0 - v(H)}^{-1}\frac{\epsilon}{2}\binom{n}{n_0} = \frac{\epsilon}{2}\binom{n_0}{v(H)}^{-1}\binom{n}{v(H)}.$$

The required result follows with $c(\epsilon) = \frac{\epsilon}{2}\binom{n_0}{v(H)}^{-1}$.

10. Given a bipartite graph G between $\{1, 2, \ldots n\}$ and $\{1, 2, \ldots, n\}$ of density at least δ, we may describe a subset of $[n]^2$ of density at least δ by including (i, j) if and only if there is an edge between i and j. If we now apply the multidimensional version of Szemerédi's theorem with $d = 2$ and $P = \{(i, j) : 0 \leq i, j \leq t-1\}$, we get a subset of the form $\{(u+ki, v+kj) : 0 \leq i, j \leq t-1\}$. This implies the theorem with $U = \{u + ki : 0 \leq i \leq t-1\}$ and $V = \{v + kj : 0 \leq j \leq t-1\}$ being arithmetic progressions of length t with common difference k.