Extremal graph theory - Example Sheet 1

1. Show that if G is a graph with n vertices and at least $\lfloor \frac{n^2}{4} \rfloor + 1$ edges, then G contains at least $\lfloor \frac{n}{2} \rfloor$ triangles. Show that, for $n \geq 3$, this result is sharp.

2. Let G be a non-bipartite graph with more than $\frac{1}{4}(n-1)^2 + 1$ edges. Show that G contains a triangle. Show that, for all odd $n \geq 5$, there is a triangle-free non-bipartite graph with $\frac{1}{4}(n-1)^2 + 1$ edges.

3. By considering a random ordering of the vertices of a graph G, show that the size $\alpha(G)$ of the largest independent set in G satisfies

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

   Deduce Turán's theorem.

4. Let S be a set of diameter 1 in the plane, that is, no two points are at distance more than 1. Show that the number of pairs of points of S whose distance is greater than $\frac{1}{\sqrt{2}}$ is at most $\lfloor \frac{n^2}{3} \rfloor$, where $n = |S|$. Moreover, for $n \geq 2$, show that this is sharp.

5. Show that Hall's theorem does not hold for infinite graphs. That is, find an infinite bipartite graph between sets A and B for which every subset of A has at least the cardinality of A neighbours in B, but there is no way of matching A to a subset of B.

6. Suppose that the edges of the complete graph K_n have been coloured with two colours, red and blue. Show that there are at least $\frac{n(n-1)(n-5)}{24}$ monochromatic triangles.

7. Determine $\lim_{n\to\infty} ex(n,H)/\binom{n}{2}$ for each of the platonic solids.

8. Show that an ϵ-regular partition of a graph G is also an ϵ-regular partition of its complement $\overline{G}$.

9. Suppose $ex(n,H) \leq \rho\binom{n}{2}$ whenever $n \geq n_0$. Show by averaging that, for n large, any graph G on n vertices with more than $(\rho+\epsilon)\binom{n}{2}$ edges contains at least $c(\epsilon)n^{v(H)}$ copies of H.

10. Show that, for any natural number t and any $\delta > 0$ there exists an n_0 such that, for $n \geq n_0$, if G is a bipartite graph between $\{1, 2, \ldots n\}$ and $\{1, 2, \ldots, n\}$ with at least δn^2 edges, there is a complete bipartite graph $K_{t,t}$ between two sets U and V of size t, where U and V are arithmetic progressions of length t with the same common difference.