1. Show that if $G$ is a graph with $n$ vertices and at least $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges, then $G$ contains at least $\left\lfloor \frac{n^2}{4} \right\rfloor$ triangles. Show that, for $n \geq 3$, this result is sharp.

2. Let $G$ be a non-bipartite graph with more than $\frac{1}{4}(n - 1)^2 + 1$ edges. Show that $G$ contains a triangle. Show that, for all odd $n \geq 5$, there is a triangle-free non-bipartite graph with $\frac{1}{4}(n - 1)^2 + 1$ edges.

3. By considering a random ordering of the vertices of a graph $G$, show that the size $\alpha(G)$ of the largest independent set in $G$ satisfies

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v) + 1}.$$ 

Deduce Turán’s theorem.

4. Let $S$ be a set of diameter 1 in the plane, that is, no two points are at distance more than 1. Show that the number of pairs of points of $S$ whose distance is greater than $\frac{1}{\sqrt{2}}$ is at most $\left\lfloor \frac{n^2}{4} \right\rfloor$, where $n = |S|$. Moreover, for $n \geq 2$, show that this is sharp.

5. Show that Hall’s theorem does not hold for infinite graphs. That is, find an infinite bipartite graph between sets $A$ and $B$ for which every subset of $A$ has at least the cardinality of $A$ neighbours in $B$, but there is no way of matching $A$ to a subset of $B$.

6. Suppose that the edges of the complete graph $K_n$ have been coloured with two colours, red and blue. Show that there are at least $n(n-1)(n-5)\frac{1}{24}$ monochromatic triangles.

7. Determine $\lim_{n \to \infty} ex(n, H)/(\binom{n}{2})$ for each of the platonic solids.

8. Show that an $\epsilon$-regular partition of a graph $G$ is also an $\epsilon$-regular partition of its complement $\overline{G}$.

9. Suppose $ex(n, H) \leq \rho\binom{n}{2}$ whenever $n \geq n_0$. Show by averaging that, for $n$ large, any graph $G$ on $n$ vertices with more than $(\rho + \epsilon)\binom{n}{2}$ edges contains at least $c(\epsilon)n^{v(H)}$ copies of $H$.

10. Show that, for any natural number $t$ and any $\delta > 0$ there exists an $n_0$ such that, for $n \geq n_0$, if $G$ is a bipartite graph between $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, n\}$ with at least $\delta n^2$ edges, there is a complete bipartite graph $K_{t,t}$ between two sets $U$ and $V$ of size $t$, where $U$ and $V$ are arithmetic progressions of length $t$ with the same common difference.