Lecture 9

The aim of this lecture is to prove that if a bipartite graph H has one side whose maximum degree is Δ, then $ex(n,H) \leq c(H)n^{2-\frac{1}{\Delta}}$. This clearly generalises the result from the last lecture that $ex(n,K_{s,t}) \leq n^{2-\frac{1}{s}}$ when $s \leq t$.

We will use a very powerful technique known as dependent random choice. Roughly speaking, if one has a bipartite graph G between sets A and B, then dependent random choice allows one to find a large subset of A' where every collection of Δ elements in A' has many joint neighbours in B. This then allows one to embed subgraphs with maximum degree Δ on one side with impunity.

Lemma 1 (Dependent random choice) *Let G be a bipartite graph with vertex sets A and B, each of size n. Suppose that the graph has density α, that is, that there are αn^2 edges. Then, for any natural number $r \geq 1$, there exists a set $A' \subset A$ of size greater than $\frac{1}{2}\alpha^r n$ such that every subset of A' of size r has at least $\frac{1}{2^r}\alpha n^{1/r}$ common neighbours in B.*

Proof For each vertex v, let $d(v)$ be its degree. For randomly chosen vertices $b_1, \cdots, b_r \in B$ (allowing repetitions), let I be the random variable giving the size of the common neighborhood. What is the expectation of I? By convexity of the function x^r, we see that

$$\begin{aligned}\mathbb{E}(I) &= \sum_{v\in A} \mathbb{P}(b_1,\ldots,b_r \in N(v)) = \sum_{v\in A}\left(\frac{d(v)}{n}\right)^r \\ &\geq \frac{n\left(\frac{\sum_{v\in A} d(v)}{n}\right)^r}{n^r} = \frac{n(\alpha n)^r}{n^r} = \alpha^r n.\end{aligned}$$

We will say that an r-tuple is bad if it has fewer than $\gamma^r|B|$ common neighbours. Let J be the random variable counting the number of bad r-tuples in the common neighborhood of $b_1,\ldots,b_r$. Note that any bad r-tuple has at most $\gamma^r|B|$ common neighbours in B. Therefore, the probability that a randomly chosen b_i is contained in this set is at most γ^r. Hence, because we made r random choices of b_i, the probability that such an r-tuple be contained in the intersection of their neighborhoods is at most γ^{r^2}. Therefore, the expected number of bad r-tuples in the common neighborhood of $b_1,\ldots,b_r$ satisfies

$$\mathbb{E}(J) \leq \gamma^{r^2}\binom{|A|}{r} \leq \gamma^{r^2}n^r.$$

By linearity of expectation, we have

$$\mathbb{E}(I-J) \geq \alpha^r n - \gamma^{r^2}n^r.$$

Choose $\gamma = \frac{1}{2}\alpha^{1/r}n^{-(r-1)/r^2}$. Then $\gamma^{r^2}n^r \leq \frac{1}{2}\alpha^r n$ and, therefore, $\mathbb{E}(I-J) \geq \frac{1}{2}\alpha^r n$. Therefore, there exists a set A_0 for which $I - J \geq \frac{1}{2}\alpha^r n$. Hence, we may remove the set of bad r-tuples from A_0 and be left with a set A' of size at least $\frac{1}{2}\alpha^r n$ which has no bad r-tuples. Since there are no bad r-tuples, every set of r elements in A has at least $\gamma^r n = \frac{1}{2^r}\alpha n^{1/r}$ neighbours in common. □

We may now prove the required estimate on $ex(n,H)$, where H has one side with bounded maximum degree Δ.

Theorem 1 *Let H be a graph between two sets U and V such that the degree of every vertex in V is at most Δ. Then there exists a constant c such that*

$$ex(n, H) \leq cn^{2-\frac{1}{\Delta}}.$$

Proof Let G be a graph of size n and suppose that G has at least $cn^{2-\frac{1}{\Delta}}$ edges. Then there is a bipartite subgraph G' of G between two sets A and B containing at least half the edges of G, that is, at least $\frac{c}{2}n^{2-\frac{1}{\Delta}}$. To see this, we simply choose the sets A and B at random, placing a vertex in each of A or B with probability $\frac{1}{2}$. The probability that a given edge lies between the two sets is then just $\frac{1}{2}$. Therefore, the expected number of edges in such a cut is $\frac{1}{2}e(G)$. In particular, there must be some cut for which we do have $\frac{1}{2}e(G)$ edges.

We now have a bipartite graph G' between two sets A and B, each of size at most n, with $\frac{c}{2}n^{2-\frac{1}{\Delta}}$ edges. An application of the dependent random choice lemma with $r = \Delta$ and $\alpha = \frac{c}{2}n^{-1/\Delta}$ tells us that there is a subset A' of A of size at least

$$\frac{1}{2}\alpha^\Delta n \geq \frac{1}{2}\left(\frac{c}{2}\right)^\Delta$$

such that every Δ-tuple in A' has at least

$$\frac{1}{2^r}\alpha n^{1/\Delta} \geq \frac{c}{2^{r+1}}$$

common neighbours.

Provided $\frac{1}{2}\left(\frac{c}{2}\right)^\Delta \geq |U|$ and $\frac{c}{2^{r+1}} \geq |V|$, we have an embedding of H. To see this, let $u_1, \ldots, u_t$ be the vertices of U. Embed them in any fashion into A'. For any given vertex $v \in V$, suppose that $u_{i_1}, \ldots, u_{i_\Delta}$ are its neighbours. Then these vertices have at least $|V|$ common neighbours in B, so even if we have embedded previous elements of V, there is still place to embed v. □

A graph H is said to be d-degenerate if every subgraph contains a vertex of degree at most d. Equivalently, H is d-degenerate if there is an ordering $\{v_1, \ldots, v_t\}$ of the vertices such that any v_j has at most d neighbours v_i with $i < j$.

An old conjecture of Burr and Erdős says that if a bipartite graph H is d-generate, then $ex(n, H) \leq cn^{2-\frac{1}{d}}$. This would be strictly stronger than the result we proved in this lecture. The best result currently known, due to Alon, Krivelevich and Sudakov, is that $ex(n, H) \leq cn^{2-\frac{1}{4d}}$.