# Lecture 8

We are now going to begin an in-depth study of the extremal number for bipartite graphs. We have already seen that if $H$ is a bipartite graph $ex(n,H) \leq \epsilon n^2$ for any $\epsilon > 0$. We now prove a much stronger result, due essentially to Kővári, Sós and Turán.

**Theorem 1** *For any natural numbers $s$ and $t$ with $s \leq t$, there exists a constant $c$ such that*

$$ex(n, K_{s,t}) \leq cn^{2-\frac{1}{s}}.$$

**Proof** Suppose that we have a graph $G$ with $n$ vertices, at least $cn^{2-\frac{1}{s}}$ edges and not containing $K_{s,t}$ as a subgraph. Note that the average degree of $G$ is $2cn^{1-\frac{1}{s}}$. We are going to count pairs $(v, S)$ consisting of sets $S$ of size $s$ all elements of which are connected by an edge to $v$. On the one hand, the number of such pairs is given by

$$\sum_v \binom{d(v)}{s} \geq n\binom{\frac{1}{n}\sum_v d(v)}{s} \geq n\binom{2cn^{1-\frac{1}{s}}}{s} \geq n\frac{c^s n^{s-1}}{s!} = c^s\frac{n^s}{s!},$$

for $n$ sufficiently large. On the other hand, the number of pairs $(v, S)$ is at most

$$(t-1)\binom{n}{s} \leq (t-1)\frac{n^s}{s!},$$

for otherwise there would be some set $S$ of $s$ vertices which have $t$ neighbours in common. Therefore, if we choose $c$ so that $c^s \geq t-1$, we have a contradiction. □

By being a little more careful, we could have obtained the bound

$$ex(n, K_{s,t}) \leq (1+o(1))\frac{1}{2}(t-1)^{\frac{1}{s}}n^{2-\frac{1}{s}}.$$

It is known that this bound is sharp in various specific cases. For example, when $H = K_{1,t}$ and $n$ satisfies some divisibility assumptions, there is a graph on $n$ vertices with $\frac{1}{2}(t-1)n$ edges which does not contain any copies of $H$ - simply take a graph such that every vertex has degree $t-1$.

A more interesting example is $H = K_{2,2}$. The following $K_{2,2}$-free construction, due to Erdős, Rényi and Sós, allows us to show that $ex(n, K_{2,2}) \approx \frac{1}{2}n^{3/2}$.

**Construction of $K_{2,2}$-free graph** Let $p$ be a prime and consider the graph on $n = p^2 - 1$ vertices whose vertex set is $\mathbb{Z}_p \times \mathbb{Z}_p \backslash \{(0,0)\}$ and where $(x, y)$ is joined to $(a, b)$ if and only if $ax + by = 1$.

For a fixed $(x, y)$, there are exactly $p$ solutions $(a, b)$ to $ax + by = 1$. To see this, we must split into some subcases. If $x = 0$, then there is a unique non-zero solution for $b$ and anything works for $a$. Similarly, if $y = 0$, $a$ is uniquely determined and $b$ may be anything. If both $x$ and $y$ are non-zero, it is elementary to see that any choice of $b$ gives rise to a unique choice of $a$, i.e., $a = x^{-1}(1 - by)$.

Therefore, $(x, y)$ has degree at least $p-1$ (one of the solutions could be $(a, b) = (x, y)$, which we ignore) and the graph has at least $\frac{1}{2}n(p-1) \approx \frac{1}{2}n^{3/2}$ edges. Moreover, the graph does not contain a $K_{2,2}$. Suppose otherwise and that $(a, b), (x, y), (a', b'), (x', y')$ is a $K_{2,2}$. Then the set of simultaneous equations $ux + vy = 1$ and $ux' + vy' = 1$ would have two solutions, $(u, v) = (a, b)$ and $(a', b')$, which is clearly impossible, since any two distinct lines meet in at most one point.

This construction works for $n = p^2 - 1$, but the result that $ex(n, H) \approx \frac{1}{2} n^{3/2}$ follows for all $n$ since we know that, for $n$ large, there exists a prime between $\sqrt{n} - n^{1/3}$ and $\sqrt{n}$ (though this is a very deep result). □

There is also a result of Füredi extending this result. It says that, for each $t$,

$$ex(n, K_{2,t+1}) \approx \frac{\sqrt{t}}{2} n^{3/2}.$$

There is also a construction, due to Brown, which gives a lower bound $ex(n, K_{3,3}) \geq c' n^{5/3}$. Roughly speaking, take a prime $p \equiv 3 (\text{mod } 4)$ and consider the graph on $p^3$ vertices whose vertex set is $\mathbb{Z}_p^3$, where $(x, y, z)$ is joined to $(a, b, c)$ if and only if $(a-x)^2 + (b-y)^2 + (c-z)^2 = 1$. For any given $(x, y, z)$, there will be on the order of $p^2$ elements $(a, b, c)$ to which it is connected. There are, therefore, around $c' n^{5/3}$ edges in the graph. Moreover, the unit spheres around the three distinct points $(x, y, z), (x', y', z')$ and $(x'', y'', z'')$ cannot meet in more than two points, so the graph does not contain a $K_{3,3}$. The result follows for all $n$ by a similar argument to above.

Other than the constructions mentioned, there is also an impressive construction of Alon, Kollár, Rónyai and Szabó which shows that if $t \geq (s-1)! + 1$, the upper bound we gave at the start of the lecture is essentially sharp, that is, $ex(n, K_{s,t}) \geq c' n^{2 - \frac{1}{s}}$. This includes, though without sharp multiplying constants, all the cases discussed thusfar.

Apart from this, almost the best known lower bound follows from the following random construction.

**Theorem 2** *For any $s, t \geq 2$, there exists a constant c' such that*

$$ex(n, K_{s,t}) \geq c' n^{2 - \frac{(s+t-2)}{(st-1)}}.$$

**Proof** Choose each edge in the graph randomly with probability $p = \frac{1}{2} n^{-\frac{(s+t-2)}{(st-1)}}$. The expected number of copies of $K_{s,t}$ is

$$p^{st} \binom{n}{s} \binom{n}{t} \leq p^{st} n^{s+t}.$$

Phrased differently, if $J$ is the random variable counting copies of $K_{s,t}$, then $\mathbb{E}(J) \leq p^{st} n^{s+t}$. On the other hand, the expected number of edges in the graph is $p\binom{n}{2} \geq \frac{1}{4} p n^2$. Again, if $I$ is the random variable counting the number of edges in the graph, then $\mathbb{E}(I) \geq \frac{1}{4} p n^2$. By linearity of expectation,

$$\mathbb{E}(I - J) = \mathbb{E}(I) - \mathbb{E}(J) \geq \frac{1}{4} p n^2 - p^{st} n^{s+t} \geq \frac{1}{8} p n^2 = \frac{1}{16} n^{2 - \frac{(s+t-2)}{(st-1)}}.$$

The final inequality follows since $p^{st} n^{s+t} \leq \frac{1}{8} p n^2$. This in turn follows from

$$p^{st-1} n^{s+t-2} \leq \left(\frac{1}{2}\right)^{st-1} \leq \frac{1}{8}.$$

Therefore, there exists some graph $G$ on $n$ vertices for which $I - J \geq \frac{1}{16} n^{2 - \frac{(s+t-2)}{(st-1)}}$. We may therefore remove one edge from each of the $K_{s,t}$, removing all copies of $K_{s,t}$ and still be left with a graph containing $\frac{1}{16} n^{2 - \frac{(s+t-2)}{(st-1)}}$ edges. □