Lecture 7

We are now ready to give the promised alternative proof of Erdős-Stone-Simonovits. To begin, we will need a counting lemma which generalises that given earlier for triangles.

Lemma 1 *Let $\epsilon > 0$ be a real number. Let G be a graph and suppose that $V_1, V_2, \ldots, V_r$ are subsets of $V(G)$ such that $|V_i| \geq 2\epsilon^{-\Delta}t$ for each $1 \leq i \leq r$ and the graph between V_i and V_j has density $d(V_i, V_j) \geq 2\epsilon$ and is $\frac{1}{2}\epsilon^{\Delta}\Delta^{-1}$-regular for all $1 \leq i < j \leq r$. Then G contains a copy of any graph H on t vertices with chromatic number r and maximum degree Δ.*

Proof Since the chromatic number of H is at most r, we may split $V(H)$ into r independent sets $U_1, \ldots, U_r$. We will give an embedding f of H into G so that $f(U_i) \subset V_i$ for all $1 \leq i \leq r$.

Let the vertices of H be $u_1, \ldots, u_t$. For each $1 \leq h \leq t$, let $L_h = \{u_1, \ldots, u_h\}$. For each $y \in U_j \backslash L_h$, let T_y^h be the set of vertices in V_j which are adjacent to all already embedded neighbours of y. That is, letting $N_h(y) = N(y) \cap L_h$, T_y^h is the set of vertices in V_j adjacent to every element of $f(N_h(y))$. We will find, by induction, an embedding of L_h such that, for each $y \in V(H)\backslash L_h$, $|T_y^h| \geq \epsilon^{|N_h(y)|}|V_j|$.

For $h = 0$ there is nothing to prove. We may therefore assume that L_h has been embedded consistent with the induction hypothesis and attempt to embed $u = u_{h+1} \in U_k$ into an appropriate $v \in T_u^h$. Let Y be the set of neighbours of u which are not yet embedded. We wish to find an element $v \in T_u^h \backslash f(L_h)$ such that, for all $y \in Y$, $|N(v) \cap T_y^h| \geq \epsilon|T_y^h|$. If such a vertex v exists, taking $f(u) = v$ and $T_y^{h+1} = N(v) \cap T_y^h$ will complete the proof.

Let B_y be the set of vertices in T_u^h which are bad for $y \in Y$, that is, such that $|N(v) \cap T_y^h| < \epsilon|T_y^h|$. Note that, by induction, if $y \in U_\ell$, $|T_y^h| \geq \epsilon^{\Delta}|V_\ell|$. Therefore, we must have $|B_y| < \frac{1}{2}\epsilon^{\Delta}\Delta^{-1}|V_k|$, for otherwise the density between B_y and T_y^h would be less than ϵ, contradicting the regularity assumption on G. Hence, since $|V_k| \geq 2\epsilon^{-\Delta}t$,

$$\left| T_u^h \backslash \cup_{y \in Y} B_y \right| > \epsilon^{\Delta}|V_k| - \Delta \frac{1}{2}\epsilon^{\Delta}\Delta^{-1}|V_k| \geq t.$$

Since at most $t-1$ vertices have already been embedded, an appropriate choice for $f(u)$ exists. □

In fact, there are at least $\frac{1}{2}\epsilon^{\Delta}|V_k| - t$ choices for each vertex u. Therefore, if H has d_i vertices in U_i, the lemma tells us that, for $|V_i| \gg 2\epsilon^{-\Delta}t$, we have at least

$$c_H(\epsilon) \prod_{i=1}^{r} |V_i|^{d_r}$$

copies of H, where $c_H(\epsilon)$ is an appropriate constant. Like the triangle counting lemma, we could make the constant $c_H(\epsilon)$ reflect the densities between the various V_i, but I simply wanted to note that the graph G contained a positive proportion of the total number of possible copies of H.

We are now ready to give another proof of the Erdős-Stone-Simonovits theorem. That is, we will show that for any r-chromatic graph H and n sufficiently large, $ex(n, H) \leq \left(1 - \frac{1}{r-1} + \epsilon\right)\frac{n^2}{2}$.

Alternative proof of Erdős-Stone-Simonovits Let H be a graph with t vertices, chromatic number r and maximum degree Δ. Suppose that G is a graph on n vertices with at least $\left(1 - \frac{1}{r-1} + \epsilon\right)\frac{n^2}{2}$ edges. We will show how to embed H in G. Let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_M$ be a $\frac{1}{2}\left(\frac{\epsilon}{8}\right)^{\Delta}\Delta^{-1}$-regular partition of the vertex set of G. We remove edges as in the triangle-removal lemma, removing xy if

1. $(x, y) \in X_i \times X_j$, where (X_i, X_j) is not $\frac{1}{2}\left(\frac{\epsilon}{8}\right)^{\Delta} \Delta^{-1}$-regular;

2. $(x, y) \in X_i \times X_j$, where $d(X_i, X_j) < \frac{\epsilon}{4}$;

3. $x \in X_i$, where $|X_i| < \frac{\epsilon}{16M} n$.

The total number of edges removed is at most $\frac{\epsilon}{16} n^2$ from the first condition, since if I is the set of (i, j) corresponding to non-regular pairs (X_i, X_j), we have

$$\sum_{(i,j) \in I} |X_i||X_j| \leq \frac{1}{2}\left(\frac{\epsilon}{8}\right)^{\Delta} \Delta^{-1} n^2 \leq \frac{\epsilon}{16} n^2.$$

The total number of edges removed by condition 2 is clearly at most $\frac{\epsilon}{4} n^2$ and the total number removed by condition 3 is at most $\frac{\epsilon}{16} n^2$.

Overall, we have removed at most $\frac{3\epsilon}{8} n^2$ edges. Hence, the graph G' that remains after all these edges have been removed has density at least $1 - \frac{1}{r-1} + \frac{\epsilon}{8}$. It must, therefore, contain a copy of K_r. We may suppose that this lies between sets $V_1, \dots, V_r$ (some of which may be equal). Because of our removal process, $|V_j| \geq \frac{\epsilon}{16M} n$, the graph between V_i and V_j has density at least $\frac{\epsilon}{4}$ and is $\frac{1}{2}\left(\frac{\epsilon}{8}\right)^{\Delta} \Delta^{-1}$-regular. Therefore, if

$$\frac{\epsilon}{16M} n \geq 2\left(\frac{\epsilon}{8}\right)^{-\Delta} t,$$

an application of the previous lemma with $\frac{\epsilon}{8}$ implies that G contains a copy of H. □

Because of the observation made after the previous lemma, we know that, for n large, G not only contains one copy of any given r-chromatic graph H, it must contain $cn^{v(H)}$ copies. This phenomenon, that once one passes the extremal density one gets a very large number of copies rather than one single copy, is known as supersaturation.