Lecture 2

A perfect matching in a bipartite graph with two sets of equal size is a collection of edges such that every vertex is contained in exactly one of them.

Hall's (marriage) theorem is a necessary and sufficient condition which allows one to decide if a given bipartite graph contains a matching. Suppose that the two parts of the bipartite graph G are A and B. Then Hall's theorem says that if, for every subset U of A, there are at least $|U|$ vertices in B with neighbours in U then G contains a perfect matching. The condition is clearly necessary. To prove that it is sufficient we use the following notation.

For any subset X of a graph G, let $N_G(X)$ be the set of neighbours of X, that is, the set of vertices with a neighbour in X.

Theorem 1 (Hall's theorem) *Let G be a bipartite graph with parts A and B of equal size. If, for every $U \subset A$, $|N_G(U)| \geq |U|$ then G contains a perfect matching.*

Proof Let $|A| = |B| = n$. We will prove the theorem by induction on n. Clearly, the result is true for $n = 1$. We therefore assume that it is true for $n - 1$ and prove it for n.

If $|N_G(U)| \geq |U| + 1$ for every non-empty proper subset U of A, pick an edge $\{a, b\}$ of G and consider the graph $G' = G - \{a, b\}$. Then every non-empty set $U \subset A \backslash \{a\}$ satisfies

$$|N_{G'}(U)| \geq |N_G(U)| - 1 \geq |U|.$$

Therefore, there is a perfect matching between $A \backslash \{a\}$ and $B \backslash \{b\}$. Adding the edge from a to b gives the full matching.

Suppose, on the other hand, that there is some non-empty proper subset U of A for which $|N(U)| = |U|$. Let $V = N(U)$. By induction, since Hall's condition holds for every subset of U, there is a matching between U and V. But Hall's condition also holds between $A \backslash U$ and $B \backslash V$. If this weren't the case, there would be some W in $A \backslash U$ with fewer than $|W|$ neighbours in $B \backslash V$. Then $W \cup U$ would be a subset of A with fewer than $|W \cup U|$ neighbours in B, contradicting our assumption. Therefore, there is a perfect matching between $A \backslash U$ and $B \backslash V$. Putting the two matchings together completes the proof. □

A Hamiltonian cycle in a graph G is a cycle which visits every vertex exactly once and returns to its starting vertex. Dirac's theorem says that if the minimum degree $\delta(G)$ of a graph G is such that $\delta(G) \geq n/2$ then G contains a Hamiltonian cycle. This is sharp since, if one takes a complete bipartite graph with one part of size $\lceil \frac{n}{2} - 1 \rceil$ (and the other the complement of this), then it cannot contain a Hamiltonian cycle. This is simply because one must pass back and forth between the two sets.

Theorem 2 (Dirac's theorem) *If a graph G satisfies $\delta(G) \geq \frac{n}{2}$, then it contains a Hamiltonian cycle.*

Proof First, note that G is connected. If it weren't, the smallest component would have size at most $n/2$ and no vertex in this component could have degree $n/2$ or more.

Let $P = x_0 x_1 \ldots x_k$ be a longest path in G. Since it can't be extended, every neighbour of x_0 and x_k must be contained in P. Since $\delta(G) \geq n/2$, we see that $x_0 x_{i+1}$ is an edge for at least $n/2$ values of i

with $0 \leq i \leq k-1$. Similarly, $x_i x_k$ is an edge for at least $n/2$ values of i. There are at most $n-1$ values of i with $0 \leq i \leq k-1$. Therefore, since the total number of edges of the form $x_0 x_{i+1}$ or of the form $x_i x_k$ with $0 \leq i \leq k-1$ is at least n, there must be some i for which both $x_0 x_{i+1}$ and $x_i x_k$ are edges in G.

We claim that

$$C = x_0 x_{i+1} x_{i+2} \dots x_k x_i x_{i-1} \dots x_0$$

is a Hamiltonian cycle. Suppose not and that there is a set of vertices Y which are not contained in C. Then, since G is connected, there is a vertex x_j and a vertex y in Y such that $x_j y$ is in $E(G)$. But then we may define a path P' starting at y, going to x_j and then around the cycle C which is longer than P. This would contradict our assumption about P. □

A tree T is a connected graph containing no cycles. The Erdős-Sós conjecture states that if a tree T has t edges, then any graph G with average degree t must contain a copy of T. This conjecture has been proven, for sufficiently large graphs G, by Ajtai, Komlós, Simonovits and Szemerédi. Here we prove a weaker version of this conjecture.

Theorem 3 *If a graph G has average degree $2t$, it contains every tree T with t edges.*

Proof We start with a standard reduction, by showing that a graph of average degree $2t$ has a subgraph of minimum degree t. If the number of vertices in G is n, the number of edges in G is at least tn. If there is a vertex of degree less than t, delete it. This will not decrease the average degree. Moreover, the process must end, since any graph with fewer than $2t$ vertices cannot have average degree $2t$.

We now use this condition to embed the vertices of the tree greedily. Suppose we have already embedded j vertices, where $j < t+1$. We will try to embed a new vertex which is connected to some already embedded vertex. By the minimum degree condition, there are at least t possible places to embed this vertex. At most $t-1$ of these are blocked by already embedded vertices, so the embedding may always proceed. □