

## Lecture 15

In this lecture, we will prove the lower bound

$$\pi(K_s^{(r)}) \geq 1 - \left(\frac{r-1}{s-1}\right)^{r-1}.$$

Equivalently, we shall show that  $t(s, r) \leq \left(\frac{r-1}{s-1}\right)^{r-1}$ .

The construction which we shall analyse is as follows. Given  $n$  vertices, divide them into  $s-1$  roughly equal parts  $A_1, \dots, A_{s-1}$ . A set  $B$  of size  $r$  is an edge of  $\mathcal{G}$  if and only if there is some  $j$  such that  $\sum_{i=1}^k |B \cap A_{j+i}| \geq k+1$  for each  $1 \leq k \leq r-1$  (taking addition modulo  $s-1$  in the subscript). Note that for  $r=3$  and  $s=4$ , this is precisely the complement of the construction described in the previous lecture.

It turns out to be convenient to analyse the construction in rather different terms.

**Lemma 1** *A lorry driver needs to follow a certain closed route. There are several petrol stations along the route and the total amount of fuel in these stations is sufficient for the route. Then there is a starting point from which the route can be completed.*

**Proof** Suppose that the driver had enough fuel for the journey and consider the journey starting from an arbitrary point where the driver still picks up the fuel at every station, even though he doesn't need it. Then the point at which the fuel reserves are lowest during this route can be used as a starting point for another route where the fuel supply never drops below zero.  $\square$

We first prove that  $\mathcal{G}$  has the fundamental property required to show that  $T(n, s, r) \leq e(\mathcal{G})$ .

**Lemma 2** *Every subset of  $\mathcal{G}$  with  $s$  vertices contains an edge.*

**Proof** Consider any set  $S$  of size  $s$ . We will use the lorry driver model where we travel through all  $n$  vertices, going through the  $A_i$  in order. Imagine, in the lorry driver model, that every element of  $S$  represents a unit of fuel and that it takes  $\frac{s}{s-1}$  units of fuel to travel from  $A_i$  to  $A_{i+1}$ . Then  $S$  contains enough fuel for a complete circuit. Hence, by the previous lemma, there is an appropriate starting point to complete the circuit.

Let  $B$  be the first  $r$  elements of  $S$  that are encountered on this circuit. Since  $r \geq (r-1)\frac{s}{s-1}$ , the lorry can advance distance  $r-1$  using just the fuel from  $B$ . This implies that  $B$  is an edge, as  $\lceil k\frac{s}{s-1} \rceil = k+1$  for  $1 \leq k \leq r-1$ . Thus any set of size  $s$  contains an edge, as required.  $\square$

All that remains to be done is to estimate the number of edges in  $\mathcal{G}$ .

**Lemma 3**

$$e(\mathcal{G}) = (1 + o(1)) \left(\frac{r-1}{s-1}\right)^{r-1} \binom{n}{r}.$$

**Proof** We will count ordered edges  $x_1 \dots x_r$ . Each edge will be in  $A_{j+1} \cup A_{j+2} \cup \dots \cup A_{j+r-1}$  for some  $j$ . We may choose an ordered edge  $x_1 \dots x_r$  by choosing (i) a starting position  $j$ , (ii) a choice of

$A_{j+i}$ ,  $1 \leq i \leq r-1$ , in which to place each  $x_\ell$  and (iii) a vertex for each  $x_\ell$  within each assigned part. There are  $s-1$  choices in step (i) and  $\left(\frac{n}{s-1}\right)^r + O(n^{r-1})$  choices in step (iii).

In step (ii), there are  $(r-1)^r$  ways to assign the parts if we ignore the required inequalities in the intersection sizes (i.e. that there should be enough fuel for the lorry). We claim that given any assignment, there is exactly one cyclic permutation which satisfies the required inequalities. More precisely, if we assign  $b_i$  of the  $x_\ell$  to  $A_{j+i}$  for  $1 \leq i \leq r-1$ , then there is exactly one  $c$  with  $1 \leq c \leq r-1$  such that the shifted sequence  $b'_i = b_{c+i}$  (addition taken modulo  $r-1$ ) satisfies  $\sum_{i=1}^k b'_i \geq k+1$  for each  $1 \leq k \leq r-1$ . To see this, consider a lorry that makes a circuit of the  $A_{j+i}$ ,  $1 \leq i \leq r-1$ , where each of the  $x_\ell$  has a unit of fuel, but now it takes one unit of fuel to advance from  $A_{j+i}$  to  $A_{j+i+1}$ , and the lorry is required to always have one spare unit of fuel. It is clear that a valid starting point for the lorry is equivalent to a shifted sequence satisfying the required inequalities. Imagine that the driver starts with enough fuel to drive around the route and consider the journey starting from an arbitrary point. Then the point at which the fuel reserves are lowest during this route is a starting point for a route where there is always a spare unit of fuel. This is the unique point at which the fuel reserves are lowest and so it gives the unique cyclic permutation satisfying the required inequalities. To see this note that if  $i_1$  and  $i_2$  with  $i_1 \leq i_2$  both worked, we would have

$$r = \sum_i b_i = \sum_{i=i_1}^{i_2-1} b_i + \sum_{i=i_2}^{i_1-1} b_i \geq (i_2 - i_1 + 1) + ((r-1) - (i_2 - i_1) + 1) = r + 1,$$

a contradiction. We deduce that there are  $(r-1)^{r-1}$  valid assignments in step (ii).

Putting everything together gives

$$\frac{1}{r!} (s-1)(r-1)^{r-1} (1 + o(1)) \left(\frac{n}{s-1}\right)^r = (1 + o(1)) \left(\frac{r-1}{s-1}\right)^{r-1} \binom{n}{r},$$

as required. □