Lecture 14

We will now turn our attention to hypergraphs. An r-uniform hypergraph $\mathcal{G}$ on vertex set V is a collection of subsets of V of size r. The complete r-uniform hypergraph $K_n^{(r)}$ is a hypergraph on n vertices where every r-element subset of the vertex set is an edge. Our concern will be with the following function. Given an r-uniform hypergraph $\mathcal{H}$ and a natural number n, let

$$ex(n, \mathcal{H}) = \max\{e(\mathcal{G}) : |\mathcal{G}| = n, \mathcal{H} \not\subset \mathcal{G}\}.$$

Sometimes it will be convenient to talk about the Turán density, rather than the exact extremal function, for r-uniform hypergraphs $\mathcal{H}$. This is given by

$$\pi(\mathcal{H}) = \lim_{n\to\infty} \frac{ex(n, \mathcal{H})}{\binom{n}{r}}.$$

It is not hard to show that this density is well-defined. For graphs, the Erdős-Stone-Simonovits theorem tells us that if H has chromatic number t, then $\pi(H) = 1 - \frac{1}{t-1}$. For hypergraphs, much less is known. Even in the simple case where $\mathcal{H} = K_4^{(3)}$, we only know that

$$\frac{5}{9} \leq ex(n, K_4^{(3)}) \leq 0.561666.$$

The lower bound is not hard to come by. Take three vertex sets V_1, V_2 and V_3, each of size $n/3$. We let an edge uvw be in $\mathcal{G}$ if $u, v \in V_i$ and $w \in V_{i+1}$, for $i = 1, 2, 3$, or if $u \in V_1$, $v \in V_2$, $w \in V_3$. It is straightforward to check that this contains no $K_4^{(3)}$ and that its density is $5/9$. The upper bound, on the other hand, is much more difficult to obtain, using a computational technique known as flag algebras.

Over the next two lectures we will study the general case $\pi(K_s^r)$, showing that

$$1 - \left(\frac{r-1}{s-1}\right)^{r-1} \leq \pi(K_s^{(r)}) \leq 1 - \binom{s-1}{r-1}^{-1}.$$

Note that for $r = 2$ this just reduces to Turán's theorem.

We will start with the upper bound. It will be convenient in what follows to flip the definition and to take $T(n, s, r)$ to be the minimum number of edges in an r-uniform hypergraph $\mathcal{G}$ on n vertices such that any subset with s vertices contains at least one edge. We also define a density version $t(s, r) = \lim_{n\to\infty} \binom{n}{r}^{-1} T(n, s, r)$. Note that $t(s, r) + \pi(K_s^{(r)}) = 1$. Our main result of this lecture will now be that

$$T(n, s, r) \geq \frac{n-s+1}{n-r+1} \binom{s-1}{r-1}^{-1} \binom{n}{r},$$

a result due to de Caen. That $t(s, r) \geq \binom{s-1}{r-1}^{-1}$ then follows easily.

To begin, we will prove an inequality which relates the number of copies of cliques with various sizes. Given an r-uniform hypergraph $\mathcal{G}$ on n vertices, let N_s be the number of copies of $K_s^{(r)}$ in $\mathcal{G}$.

Lemma 1

$$N_{s+1} \geq \frac{s^2 N_s}{(s-r+1)(s+1)} \left(\frac{N_s}{N_{s-1}} - \frac{(r-1)(n-s)+s}{s^2}\right),$$

provided $N_{s-1} \neq 0$.

Proof Let P be the number of pairs (S, T), where S and T are sets of size s with $|S \cap T| = s-1$, S spans a copy of $K_s^{(r)}$ and T does not. We will count P in two different ways to get a bound.

On the one hand, for each $i = 1, \ldots, N_{s-1}$, let a_i be the number of copies of $K_s^{(r)}$ which contain the ith copy of $K_{s-1}^{(r)}$. Note that $\sum_{i=1}^{N_{s-1}} a_i = sN_s$. Therefore,

$$P = \sum_{i=1}^{N_{s-1}} a_i(n-s+1-a_i) = (n-s+1)\sum_{i=1}^{N_{s-1}} a_i - \sum_{i=1}^{N_{s-1}} a_i^2 \leq (n-s+1)sN_s - N_{s-1}^{-1}s^2N_s^2,$$

where the inequality follows from Cauchy-Schwarz.

On the other hand, let the copies of $K_s^{(r)}$ be $B_1, \ldots, B_{N_s}$ and let b_i be the number of $K_{s+1}^{(r)}$ containing B_i. For each B_j, there are $n-s-b_j$ ways to choose $x \notin B_j$ such that $B_j \cup \{x\}$ does not span a $K_{s+1}^{(r)}$. For any such x, there must be some $C \subseteq B_j$ of size $r-1$ such that $C \cup \{x\}$ is not an edge. Therefore, for every $y \in B_j \backslash C$, the pair $(B_j, B_j \cup \{x\} \backslash y)$ is counted by P. Hence,

$$P \geq \sum_{j=1}^{N_s}(n-s-b_j)(s-r+1) = (s-r+1)((n-s)N_s - (s+1)N_{s+1}),$$

where we used that $\sum_{j=1}^{N_s} b_j = (s+1)N_{s+1}$. Comparing the upper and lower bounds gives the result. □

For graphs, this is known as the Moon-Moser inequality. The hypergraph case is due to de Caen. From it, we may derive the following lemma.

Lemma 2

$$N_s \geq N_{s-1}\frac{r^2\binom{s}{r}}{s^2\binom{n}{r-1}}(e(G) - F(n,s,r)),$$

where $F(n,s,r) = r^{-1}((n-r+1) - \binom{s-1}{r-1}^{-1}(n-s+1))\binom{n}{r-1}$.

Proof We prove the result by induction on s. For $s = r$, we have $N_s = e(G)$. This is easily seen to accord with the inequality.

Suppose, therefore, that the inequality holds for s. We will prove it for $s+1$. By the Moon-Moser inequality and the induction hypothesis,

$$\begin{aligned}
\frac{N_{s+1}}{N_s} &\geq \frac{s^2}{(s-r+1)(s+1)}\left(\frac{N_s}{N_{s-1}} - \frac{(r-1)(n-s)+s}{s^2}\right) \\
&\geq \frac{s^2}{(s-r+1)(s+1)}\left(\frac{r^2\binom{s}{r}}{s^2\binom{n}{r-1}}(e(G)-F(n,s,r)) - \frac{(r-1)(n-s)+s}{s^2}\right) \\
&= \frac{r^2\binom{s+1}{r}}{(s+1)^2\binom{n}{r-1}}e(G) - \frac{r^2\binom{s+1}{r}}{(s+1)^2\binom{n}{r-1}}F(n,s,r) - \frac{(r-1)(n-s)+s}{(s-r+1)(s+1)}.
\end{aligned}$$

It remains to show that

$$F(n,s+1,r) \geq F(n,s,r) + \frac{((r-1)(n-s)+s)}{s-r+1}\frac{(s+1)\binom{n}{r-1}}{r^2\binom{s+1}{r}}.$$

A long but relatively straightforward computation allows us to show that equality actually holds. The result follows. □

De Caen's result now follows easily.

Theorem 1

$$T(n,s,r) \geq \frac{n-s+1}{n-r+1}\binom{s-1}{r-1}^{-1}\binom{n}{r}.$$

Proof From the previous lemma and since the $F(n,s,r)$ increase with s, we must have $e(\mathcal{G}) \leq F(n,s,r)$ for any $K_s^{(r)}$-free graph. Therefore,

$$T(n,s,r) \geq r^{-1}\binom{s-1}{r-1}^{-1}(n-s+1)\binom{n}{r-1} = \frac{n-s+1}{n-r+1}\binom{s-1}{r-1}^{-1}\binom{n}{r},$$

as required. □