Lecture 13

In the last lecture we showed that a C_{2k+1}-free graph with roughly $\frac{n^2}{4}$ edges must be approximately bipartite. We will now refine this structure to prove that the graph must be exactly bipartite for C_{2k+1}-free graphs of maximum size.

Theorem 1 *For n sufficiently large, $ex(n, C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$.*

Proof Let G be a C_{2k+1}-free graph on n vertices with the maximum number of edges. It will have at least $\lfloor \frac{n^2}{4} \rfloor$ edges. Note that it is sufficient to prove the result in the case where G has minimum degree at least $\frac{1}{2}(1 - 4\epsilon^{1/2})n$. For suppose that we knew the result under this assumption for all $n \geq n_0$. As in the previous lemma, we form a sequence of graphs $G = G_0, G_1, \ldots, G_\ell$. If there is a vertex in G_ℓ of degree less than $\frac{1}{2}(1 - 4\epsilon^{1/2})|V(G_\ell)|$, we remove it, forming $G_{\ell+1}$. This process must stop before we reach a graph G' with $n' = (1 - 4\epsilon^{1/2})n$ vertices. Otherwise, we would have a graph with n' vertices and more than $(1 + \epsilon)\frac{n'^2}{4}$ edges. It would therefore, for n sufficiently large, contain a copy of C_{2k+1}, which would be a contradiction. When we reach the required graph, we will have a graph with $n' > (1 - 4\epsilon^{1/2})n$ vertices, minimum degree at least $\frac{1}{2}(1 - 4\epsilon^{1/2})n'$ and more than $\lfloor \frac{n'^2}{4} \rfloor$ edges, so we will have a contradiction if the removal process begins at all. Hence, we may assume that the minimum degree of G at least $\frac{1}{2}(1 - 4\epsilon^{1/2})n$.

By the previous lemma, we know that G is approximately bipartite between two sets of size roughly $\frac{n}{2}$. Consider a bipartition $V(G) = A \cup B$ such that $e(A) + e(B)$ is minimised. Then $e(A) + e(B) < \epsilon n^2$, where ϵ may be taken to be arbitrarily small provided n is sufficiently large. We may assume that A and B have size $\left(\frac{1}{2} \pm \epsilon^{1/2}\right) n$. Otherwise, $e(G) < |A||B| + \epsilon n^2 < \frac{n^2}{4}$, contradicting the choice of G as having maximum size. Let $d_A(x) = |A \cap N(x)|$ and $d_B(x) = |B \cap N(x)|$ for any vertex x. Note that for any $a \in A$, $d_A(a) \leq d_B(a)$. Otherwise, we could improve the partition by moving a to B. Similarly, $d_B(b) \leq d_A(b)$ for any $b \in B$.

Let $c = 2\epsilon^{1/2}$. We claim that there are no vertices $a \in A$ with $d_A(a) \geq cn$. If $d_A(a) \geq cn$, then also $d_B(a) \geq cn$. Moreover, $A \cap N(a)$ and $B \cap N(a)$ span a bipartite graph with no path of length $2k - 1$ and, therefore, there are at most $4kn$ edges between them. For n sufficiently large, this gives $(cn)^2 - 4kn > e(A) + e(B)$ missing edges between A and B. Therefore, $e(G) < |A||B| \leq \frac{n^2}{4}$, a contradiction. Similarly, there are no vertices $b \in B$ with $d_B(b) \geq cn$.

Now suppose that there is an edge in A, say aa'. Then

$$|N_B(a) \cap N_B(a')| > d(a) - cn + d(a') - cn - |B| > \left(\frac{1}{2} - 9\epsilon^{1/2}\right) n.$$

Let $A' = A \backslash \{a, a'\}$ and $B' = N_B(a) \cap N_B(a')$. There is no path of length $2k - 1$ of the form $b_1 a_1 b_2 a_2 \ldots b_{k-1} a_{k-1} b_k$ between A' and B'. But this implies that there is no path of any type of length $2k$ (remember that since the graph is bipartite a path must alternate sides). But this implies that the number of edges between A' and B' is at most $4kn$. This then implies that the number of edges in the graph is at most

$$e(A', B') + e(A \backslash A', V(G)) + e(V(G), B \backslash B') \leq 4kn + 2n + 10\epsilon^{1/2} n^2,$$

a contradiction for n large. $\square$

More generally, there is a result of Simonovits which shows that if H is a graph with $\chi(H) = t$ and $\chi(H \backslash e) < t$, for some edge e, then $ex(n, H) = ex(n, K_t)$ for n sufficiently large. We say that such graphs are colour-critical. It is easy to verify that odd cycles are colour-critical.