Lecture 12

In this lecture we will consider the extremal number for odd cycles. We already know, by the Erdős-Stone-Simonovits theorem, that $ex(n, C_{2k+1}) \approx \frac{n^2}{4}$. Here we will use the so-called stability approach to prove that, for n sufficiently large, $ex(n, C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$.

The idea behind the stability approach is to show that a C_{2k+1}-free graph with roughly the maximal number of edges is approximately bipartite. This will be the first lemma below. Then one uses this approximate structural information to prove an exact result. This will be the theorem.

Lemma 1 *For every natural number $k \geq 2$ and $\epsilon > 0$ there exists $\delta > 0$ and a natural number n_0 such that, if G is a C_{2k+1}-free graph on $n \geq n_0$ vertices with at least $\left(\frac{1}{4} - \delta\right) n^2$ edges, then G may be made bipartite by removing at most ϵn^2 edges.*

Proof We will prove the result for $\delta = \frac{\epsilon^2}{100}$ and n sufficiently large. We begin by finding a subgraph G' of G with large minimum degree. We do this by deleting vertices one at a time, forming graphs $G = G_0, G_1, \ldots, G_\ell$, at each stage removing a vertex with degree less than $\frac{1}{2}\left(1 - 4\delta^{1/2}\right)|V(G_\ell)|$, should it exist. By doing this, we delete at most $4\delta^{1/2}n$ vertices. Otherwise, we would have a C_{2k+1}-free graph G' on $n' = \left(1 - 4\delta^{1/2}\right)n$ vertices with at least

$$\begin{aligned}
e(G') &> e(G) - \sum_{i=n'+1}^{n} \frac{1}{2}\left(1 - 4\delta^{1/2}\right) i \\
&\geq \left(\frac{1}{4} - \delta\right) n^2 - \frac{1}{2}\left(1 - 4\delta^{1/2}\right)\left(\binom{n+1}{2} - \binom{n'+1}{2}\right) \\
&\geq \frac{n'^2}{4} + 2\delta^{1/2}n^2 - 4\delta n^2 - \delta n^2 - \frac{1}{2}\left(1 - 4\delta^{1/2}\right)(n - n')n \\
&= \frac{n'^2}{4} + 2\delta^{1/2}n^2 - 5\delta n^2 - 2\delta^{1/2}n^2 + 8\delta n^2 \geq \frac{n'^2}{4}(1 + \delta).
\end{aligned}$$

But, by the Erdős-Stone-Simonovits theorem, for n sufficiently large G' will therefore contain a copy of C_{2k+1}, so we've reached a contradiction. We therefore have a subgraph G' with $n' \geq (1 - 4\delta^{1/2})n$ vertices and minimum degree at least $\frac{1}{2}\left(1 - 4\delta^{1/2}\right)n'$.

Since $ex(n, C_{2k}) = o(n^2)$, we know that for n (and therefore n') sufficiently large, the graph G' will contain a cycle of length $2k$. Let $a_1 a_2 \ldots a_{2k}$ be such a cycle. Note that $N(a_1)$ and $N(a_2)$ cannot intersect, for otherwise there would be a cycle of length $2k+1$. Moreover, each of the two neighborhoods must contain a small number of edges. Indeed, if $N(a_1)$ contained more than $4kn'$ edges, then our result from Lecture 2 on extremal numbers for trees would imply that there was a path of length $2k$ in $N(a_1)$. But then the endpoints could be joined to a_1 to give a cycle of length $2k+1$. Therefore, we have two large disjoint vertex sets $N(a_1)$ and $N(a_2)$, each of size at least $\frac{1}{2}\left(1 - 4\delta^{1/2}\right)n' \geq \frac{1}{2}\left(1 - 8\delta^{1/2}\right)n$ such that each contains at most $4kn'$ edges. We can make the graph bipartite by deleting all the edges within $N(a_1)$ and $N(a_2)$ and all of the edges which have one end in the complement of these two sets. In total, this is at most

$$8kn' + 8\delta^{1/2}n^2$$

edges. Therefore, for n sufficiently large and $\delta = \frac{\epsilon^2}{100}$, we will have deleted at most ϵn^2 edges, which gives the required result. $\square$