Lecture 11

We now begin the proof proper of the Bondy-Simonovits theorem.

Theorem 1 *For any natural number $k \geq 2$, there exists a constant c such that*

$$ex(n, H) \leq cn^{1+1/k}.$$

Proof Suppose that G is a C_{2k}-free graph on n vertices with at least $cn^{1+1/k}$ edges. Then the average degree of G is at least $2cn^{1/k}$, so there exists some subgraph H for which the minimum degree is at least $cn^{1/k}$.

Fix an arbitrary vertex x of H. Let $i \geq 0$, let V_i be the set of vertices that are at distance i from x with respect to the graph H. In particular, $V_0 = \{x\}$ and $V_1 = N(x)$. Let $v_i = |V_i|$ and let H_i be the bipartite subgraph $H[V_{i-1}, V_i]$ induced by the disjoint sets V_{i-1} and V_i.

Claim 1 *For $1 \leq i \leq k-1$, none of the graphs $H[V_i]$ or H_{i+1} contain a bipartite cycle of length at least $2k$ with a chord.*

Proof Suppose, on the contrary, that there is a bipartite cycle with a chord F, of length at least $2k$, in $H[V_i]$. Let $Y \cup Z$ be the bipartition of $V(F)$.

Let $T \subset H$ be a breadth first-search tree beginning at x. That is, we begin at the root node x. The first layer will consist of the neighbours of x, labelling them as we uncover them. At the jth step, we look at layer $j-1$. For the first vertex in the ordering, we look at its neighbours that have not yet occurred and label them as they occur. Then we do the same in order for every vertex in the $(j-1)$st level. This will give us the jth level with all vertices labelled.

Let y be the vertex which is farthest from x in the tree T and which still dominates the set Y, that is, every vertex in Y is a descendant of y. Clearly, the paths leading from y to Y must branch at y. Pick one such branch (leading to a non-trivial subset of Y), defined by some child z of y and let A be the set of descendents of z which lie in Y. Let $B = (Y \cup Z) \backslash A$. Since $Y \backslash A \neq \emptyset$, B is not an independent set of F.

Let ℓ be the distance between x and y. Then $\ell < i$ and $2k - 2i + 2\ell < 2k \leq v(F)$. By the main lemma from the last lecture, since F is not bipartite with respect to the partition into A and B, we can find a path $P \subset F$ of length $2k - 2i + 2\ell$ which starts in $a \in A$ and ends in $b \in B$. Since the path has even length and the partition into Y and Z is bipartite, b must be in Y. Let P_a and P_b be the unique paths in T that connect y to a and b. They intersect only at y, since a is a descendent of z and b is not. Also, they each have length $i - \ell$. Therefore, the union of the paths P, P_a and P_b forms a C_{2k} in H, which contradicts our assumption.

The proof follows similarly for H_{i+1} if we take $Y = V(F) \cap V_i$. □

We also know that if a bipartite graph has minimum degree $d \geq 3$ then it contains a cycle of length at least $2d$ with an extra chord. We may therefore assume that, for $1 \leq i \leq k-1$, the average degrees $d(H[V_i])$ and $d(H_{i+1})$ of $H[V_i]$ and H_{i+1} satisfy

$$d(H[V_i]) \leq 4k - 4 \text{ and } d(H_{i+1}) \leq 2k - 2.$$

For example, if $H[V_i]$ has average degree greater than $4k-4$, it has a bipartite subgraph with average degree greater than $2k-2$ and, therefore, a bipartite subgraph with minimum degree greater than $k-1$. This would then imply that the graph contained a bipartite cycle of length at least $2k$ with a chord, which would contradict the claim. The bound for $d(H_{i+1})$ follows similarly.

We will now show inductively that, provided n is sufficiently large,

$$\frac{e(H_{i+1})}{v_{i+1}} \leq 2k$$

for every $0 \leq i \leq k-1$. For $i = 0$, this is true, since every edge in V_1 is connected to x by only one edge. Suppose that we want to prove it for some $i > 0$. Then, by induction and the bound on $d(H[V_i])$,

$$e(H_{i+1}) = \sum_{y \in V_i} d_{V_{i+1}}(y) \geq \left(\delta(H) - \frac{4k-4}{2} - 2k\right) v_i \geq \left(cn^{1/k} - 4k + 2\right) v_i \geq \frac{c}{2} n^{1/k} v_i \geq 2kv_i.$$

In particular, $V_{i+1} \neq \emptyset$ and the average degree of vertices of V_i with respect to H_{i+1} is at least $2k$. But since $d(H_{i+1}) \leq 2k-2$, we must have that the average degree of V_{i+1} with respect to H_{i+1} is at most $2k-2$, that is, $e(H_{i+1}) \leq (2k-2)v_{i+1}$, implying the required bound.

Note now that we have

$$\frac{c}{2} n^{1/k} v_i \leq e(H_{i+1}) \leq 2kv_{i+1}.$$

Therefore,

$$\frac{v_{i+1}}{v_i} \geq \frac{c}{4k} n^{1/k}.$$

This implies that

$$v_k \geq \left(\frac{c}{4k}\right)^k n.$$

This is a contradiction if $c \geq 4k$, completing the proof. □

We have shown that $ex(n, C_{2k}) \leq (4k + o(1))n^{1+1/k}$. A slightly more careful rendering of this proof, due to Pikhurko, allows one to show $ex(n, C_{2k}) \leq (k-1+o(1))n^{1+1/k}$.