The horizontal direction &

other differences between the classical and higher Cichoń diagram

---

Tristan van der Vlugt (TU Wien)

STUK 14
University of Cambridge

November 18, 2024

The **classical Baire space** is the set ${}^{\omega}\omega$ of functions $f : \omega \to \omega$. The **classical Cantor space** is the set ${}^{\omega}2$ of functions $f : \omega \to 2$.

The sets $[s] = \{f \in {}^{\omega}\omega \mid s \subseteq f\}$ where $s \in {}^{<\omega}\omega$ form a clopen basis for the topology on ${}^{\omega}\omega$. The **Borel sets** $\mathcal{B}({}^{\omega}\omega)$ are the least family closed under complements and countable unions such that $[s] \in \mathcal{B}({}^{\omega}\omega)$ for each $s \in {}^{<\omega}\omega$. Similar for ${}^{\omega}2$.

A set X is **nowhere dense** if every open U contains an open $U' \subseteq U$ with $X \cap U' = \varnothing$. A countable union of nowhere dense sets is called a **meagre** set, and $\mathcal{M}$ is the family of meagre sets of ${}^{\omega}\omega$. Each meagre set is a subset of a Borel meagre set.

In ${}^{\omega}2$, give $[s]$ the measure $\mu([s]) = 2^{-\mathrm{ot}(s)}$, and extend μ to a **Lebesgue measure** on $\mathcal{B}({}^{\omega}2)$. Let $\mathcal{N}$ be the family of sets $N \subseteq {}^{\omega}2$ such that N is contained in a Borel set of measure 0.

Let κ be inaccessible. The **higher Baire & Cantor spaces** are the sets ${}^\kappa\kappa$ and ${}^\kappa 2$ of functions $f:\kappa\to\kappa$ or $f:\kappa\to 2$ resp. Elements of ${}^\kappa\kappa$ and ${}^\kappa 2$ are called **κ-reals**.

The sets $[s]=\{f\in{}^\kappa\kappa\mid s\subseteq f\}$ where $s\in{}^{<\kappa}\kappa$ form a clopen basis for the **bounded topology** on ${}^\kappa\kappa$. The **κ-Borel sets** $\mathcal{B}_\kappa({}^\kappa\kappa)$ are the least family closed under complements and κ-unions such that $[s]\in\mathcal{B}_\kappa({}^\kappa\kappa)$ for each $s\in{}^{<\kappa}\kappa$. Similar for ${}^\kappa 2$.

A union of κ-many nowhere dense sets is a **κ-meagre set**, and $\mathcal{M}_\kappa$ is the family of all κ-meagre subsets of ${}^\kappa\kappa$. Each κ-meagre set is a subset of a κ-Borel κ-meagre set.

Without a proper analogue of **countably** infinite sums of positive **real** numbers to the uncountable, it is unclear how to define Lebesgue measure on ${}^\kappa 2$.

Let $R \subseteq X \times Y$ be a relation, λ be a cardinal, $f : \lambda \to X$ and $g : \lambda \to Y$ be functions. We define $R^\infty, R^* \subseteq {}^\lambda X \times {}^\lambda Y$:

$f\ R^\infty\ g \Leftrightarrow \{\alpha \in \lambda \mid f(\alpha)\ R\ g(\alpha)\}$ is unbounded below λ,

$f\ R^*\ g \Leftrightarrow \left\{\alpha \in \lambda \mid f(\alpha)\ \not R\ g(\alpha)\right\}$ is bounded below λ,

$f\ \not R^*\ g$ and $f\ \not R^\infty\ g$ are the negations of the above.

For instance, we may consider the **domination** $\leq^*$ and **eventual difference** $\neq^\infty$ relations on ${}^\kappa\kappa$ or on ${}^\omega\omega$.

If $h \in {}^\kappa\kappa$, then an h**-slalom** is a function $\varphi \in \prod_{\alpha\in\kappa}[\kappa]^{<|h(\alpha)|}$. Let Sl_h denote the set of h-slaloms. We may consider the **localisation** relation $\in^*$ on ${}^\kappa\kappa \times \mathrm{Sl}_h$.

Let $\mathcal{I}$ be an ideal on a space X, then we define:

$$\begin{aligned}
\operatorname{cov}(\mathcal{I}) &= \min\{|\mathcal{C}| \mid \mathcal{C} \subseteq \mathcal{I} \wedge \bigcup \mathcal{C} = X\} \\
\operatorname{add}(\mathcal{I}) &= \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\} \\
\operatorname{non}(\mathcal{I}) &= \min\{|N| \mid N \subseteq X \wedge N \notin \mathcal{I}\} \\
\operatorname{cof}(\mathcal{I}) &= \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \mathcal{I} \wedge \forall I \in \mathcal{I} \exists F \in \mathcal{F} (I \subseteq F)\}
\end{aligned}$$

Given a relation $R \subseteq X \times Y$, we define:

$$\begin{aligned}
\mathfrak{D}(R, X, Y) &= \min\{|\mathcal{D}| \mid \mathcal{D} \subseteq Y \wedge \forall f \in X \exists g \in \mathcal{D} (f \; R \; g)\} \\
\mathfrak{B}(R, X, Y) &= \min\left\{|\mathcal{B}| \mid \mathcal{B} \subseteq X \wedge \forall g \in Y \exists f \in \mathcal{B} (f \; \not R \; g)\right\}
\end{aligned}$$

For instance, $\operatorname{cov}(\mathcal{I}) = \mathfrak{D}(\in, X, \mathcal{I})$ and $\operatorname{non}(\mathcal{I}) = \mathfrak{B}(\in, X, \mathcal{I})$, whereas $\operatorname{cof}(\mathcal{I}) = \mathfrak{D}(\subseteq, \mathcal{I}, \mathcal{I})$ and $\operatorname{add}(\mathcal{I}) = \mathfrak{B}(\subseteq, \mathcal{I}, \mathcal{I})$.

We will use the following shorthands:

$$\mathfrak{d} = \mathfrak{D}(\leq^*, {}^\omega\omega, {}^\omega\omega) \quad \text{and} \quad \mathfrak{b} = \mathfrak{B}(\leq^*, {}^\omega\omega, {}^\omega\omega)$$

$$\mathfrak{d}(\neq^\infty) = \mathfrak{D}(\neq^\infty, {}^\omega\omega, {}^\omega\omega) \quad \text{and} \quad \mathfrak{b}(\neq^\infty) = \mathfrak{B}(\neq^\infty, {}^\omega\omega, {}^\omega\omega)$$

$$\mathfrak{d}^h(\in^*) = \mathfrak{D}(\in^*, {}^\omega\omega, \mathrm{Sl}_h) \quad \text{and} \quad \mathfrak{b}^h(\in^*) = \mathfrak{B}(\in^*, {}^\omega\omega, \mathrm{Sl}_h)$$

$$\mathfrak{d}_\kappa = \mathfrak{D}(\leq^*, {}^\kappa\kappa, {}^\kappa\kappa) \quad \text{and} \quad \mathfrak{b}_\kappa = \mathfrak{B}(\leq^*, {}^\kappa\kappa, {}^\kappa\kappa)$$

$$\mathfrak{d}_\kappa(\neq^\infty) = \mathfrak{D}(\neq^\infty, {}^\kappa\kappa, {}^\kappa\kappa) \quad \text{and} \quad \mathfrak{b}_\kappa(\neq^\infty) = \mathfrak{B}(\neq^\infty, {}^\kappa\kappa, {}^\kappa\kappa)$$

$$\mathfrak{d}_\kappa^h(\in^*) = \mathfrak{D}(\in^*, {}^\kappa\kappa, \mathrm{Sl}_h) \quad \text{and} \quad \mathfrak{b}_\kappa^h(\in^*) = \mathfrak{B}(\in^*, {}^\kappa\kappa, \mathrm{Sl}_h)$$

Theorem *Bartoszyński (1987)*
$\mathrm{cov}(\mathcal{M}) = \mathfrak{d}(\neq^\infty)$ and $\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\neq^\infty)$, and for $h \in {}^\omega\omega$ cofinally increasing $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}^h(\in^*)$ and $\mathrm{add}(\mathcal{N}) = \mathfrak{b}^h(\in^*)$.

Theorem *Blass, Hyttinen, and Zhang (n.d.)*
$\mathrm{cov}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa(\neq^\infty)$ and $\mathrm{non}(\mathcal{M}_\kappa) = \mathfrak{b}_\kappa(\neq^\infty)$.

Theorem *Truss (1977), Miller (1981)*
$\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and
$\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

Theorem *Brendle, Brooke-Taylor, Friedman, and Montoya (2018)*
$\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}_\kappa, \mathrm{cov}(\mathcal{M}_\kappa)\}$ and
$\mathrm{cof}(\mathcal{M}_\kappa) = \max\{\mathfrak{d}_\kappa, \mathrm{non}(\mathcal{M}_\kappa)\}$.

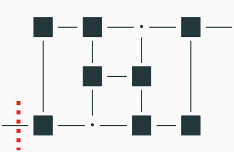
We need κ inaccessible. For example, Brendle (2022) showed that $2^{<\kappa} = 2^\kappa$ implies $\max\{\mathfrak{d}_\kappa, \mathrm{non}(\mathcal{M}_\kappa)\} \leq 2^\kappa < \mathrm{cof}(\mathcal{M}_\kappa)$.

$$
\begin{array}{ccccccccccc}
 & & \mathrm{cov}(\mathcal{N}) & \longrightarrow & \mathrm{non}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{M}) & \longrightarrow & \mathrm{cof}(\mathcal{N}) & \longrightarrow & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\aleph_1 & \longrightarrow & \mathrm{add}(\mathcal{N}) & \longrightarrow & \mathrm{add}(\mathcal{M}) & \longrightarrow & \mathrm{cov}(\mathcal{M}) & \longrightarrow & \mathrm{non}(\mathcal{N}) & &
\end{array}
$$

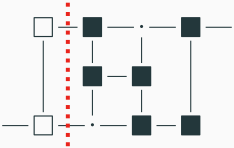
$$
\begin{array}{ccccccccc}
 & & & & \mathrm{non}(\mathcal{M}_\kappa) & \to & \mathrm{cof}(\mathcal{M}_\kappa) & \longrightarrow & \mathfrak{d}^h_\kappa(\in^*) & \longrightarrow & 2^\kappa \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & \mathfrak{b}_\kappa & \longrightarrow & \mathfrak{d}_\kappa & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\kappa^+ & \longrightarrow & \mathfrak{b}^h_\kappa(\in^*) & \longrightarrow & \mathrm{add}(\mathcal{M}_\kappa) & \to & \mathrm{cov}(\mathcal{M}_\kappa) & & & &
\end{array}
$$

Classical models with vertical separation

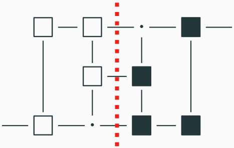
$\aleph_1 = \square < \blacksquare = \aleph_2.$ (FSI) (CSI)



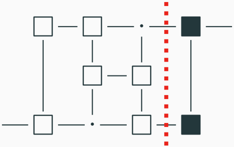
localisation
$2^{\aleph_0} = \aleph_2 +$ MA



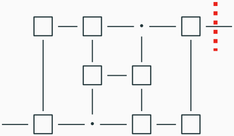
Hechler



Cohen



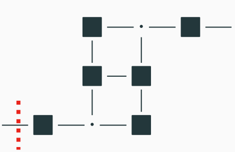
short Hechler



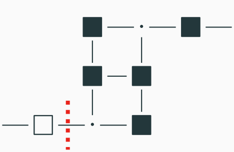
short localisation
Sacks

Higher models with vertical separation

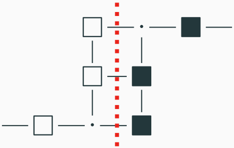
$\kappa^+ = \square < \blacksquare = \kappa^{++}$. ($<\kappa$-SI) ($\leq\kappa$-SI)



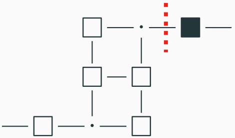
κ-localisation



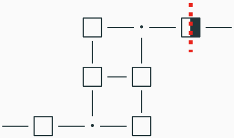
κ-Hechler



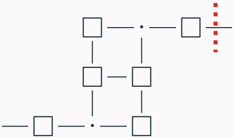
κ-Cohen



short κ-Hechler



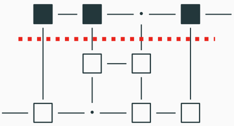
κ-Sacks



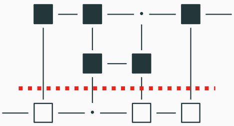
short κ-localisation

Classical models with a horizontal separation

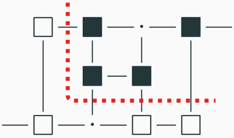
$\aleph_1 = \square < \blacksquare = \aleph_2$. (FSI) (CSI)



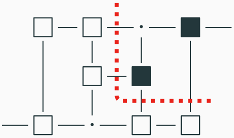
random



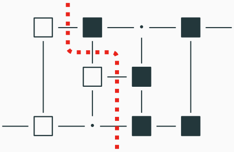
short random



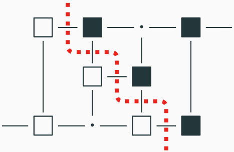
Laver
Mathias



Miller



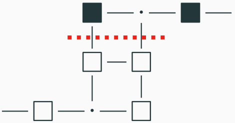
eventually different



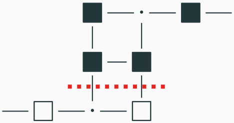
Blass-Shelah

Higher models with a horizontal separation

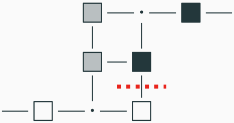
$\kappa^{+} = \square < \blacksquare = \kappa^{++}$.



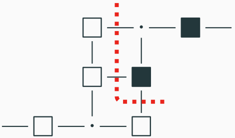
? ? ?



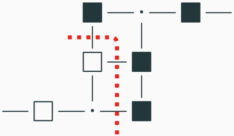
? ? ?



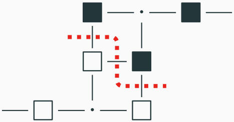
Shelah's model for $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa$



? ? ?



? ? ?



? ? ?

So what's wrong with …

… random forcing?

… Laver forcing?

… Mathias forcing?

… Miller forcing?

… eventually different forcing?

… (Blass-Shelah forcing?)

A **tree forcing** is a forcing notion $\mathbb{P}$ with conditions being trees T on ${}^{<\omega}\omega$ or ${}^{<\omega}2$, such that if G is a generic filter, then $r = \bigcup_{T\in G} \mathrm{stem}(T)$ is a generic real and $\mathbf{V}[G] = \mathbf{V}[r]$.

If B is Borel, then there is some **code** $c_B \in {}^{\omega}\omega$ describing how to construct B from the basic clopen sets $[s]$. Given a model $\mathbf{W}$ that contains c_B, we may **decode** c_B in $\mathbf{W}$ to obtain a set $(B)^{\mathbf{W}}$ that is Borel in $\mathbf{W}$.

Given a tree forcing $\mathbb{P}$ with $\dot{r}$ a name for the generic real, we define an ideal $\mathcal{I}_{\mathbb{P}}$ generated by those Borel sets B such that $\mathbb{P}$ forces that $\dot{r} \notin B$ (as decoded in the extension).

For example, **random forcing** $\mathbb{B}$ can be defined with conditions being trees $T \subseteq {}^{<\omega}2$ such that the set of branches $[T]$ has positive measure. The corresponding ideal $\mathcal{I}_{\mathbb{B}}$ is equal to $\mathcal{N}$.

We can describe conditions $T \in \mathbb{B}$ as follows. If $0 < n \in \omega$, let $N_n = \{s \in {}^n2 \mid s \notin T \wedge s \restriction n-1 \in T\}$. Then $T \in \mathbb{B}$ if and only if $\sum_{n\in\omega} |N_n| \cdot 2^{-n} < 1$.

Shelah (2017) described a tree forcing notion $\mathbb{Q}_\kappa$ on ${}^\kappa2$, where $T \in \mathbb{Q}_\kappa$ if only at a "small" number of levels of inaccessible height of T at most a "negligible" set of nodes is pruned. If $\dot{r}$ names the generic κ-real, then one defines $\mathcal{N}_\kappa$ as the ideal generated by those κ-Borel sets B such that $\mathbb{Q}_\kappa \Vdash$ "$\dot{r} \notin B$".

The notions of "small" and "negligible" are defined inductively.

Let $T \in \mathbb{Q}_\kappa$ if $T \subseteq {}^{<\kappa}2$ is a tree and there are

- a node $\mathrm{stem}(T) \in T$ that is the stem of T,
- a set $S_T \subseteq \{\lambda < \kappa \mid \lambda \text{ is inaccessible}\}$ that is **nowhere stationary**, i.e. $S_T \cap \lambda$ is nonstationary for all $\lambda < \kappa$,
- a set $N_\lambda \in \mathcal{N}_\lambda$ for each $\lambda \in S_T$,

such that for all $s \supseteq \mathrm{stem}(T)$

- if $\mathrm{ot}(s) \notin S_T$, then $s \in T$ iff $s \restriction \alpha \in T$ for all $\alpha < \mathrm{ot}(s)$,
- if $\mathrm{ot}(s) \in S_T$, then $s \in T$ iff $s \notin N_\lambda$.

Let κ be weakly compact.

Random forcing & $\mathcal{N}$	**Shelah's forcing & $\mathcal{N}_\kappa$**
${}^\omega\omega$-bounding	${}^\kappa\kappa$-bounding
σ-n-linked for $n \in \omega$	κ-λ-linked for $\lambda < \kappa$
finitely closed	(strategically) $<\kappa$-closed
Orthogonal $\mathcal{M}$ and $\mathcal{N}$ sets	Orthogonal $\mathcal{M}_\kappa$ and $\mathcal{N}_\kappa$ sets
No generic in σ-centred forcing extension	No generic in κ-centred forcing extension
Symmetry (Fubini's theorem)	Asymmetry (anti-Fubini set)

Question

How to preserve the ${}^\kappa\kappa$-bounding property under iteration?

Question

Is there a side-by-side version of Shelah's forcing?

A tree $T \subseteq {}^{<\kappa}\kappa$ is **limit-closed** if it has no branch of length $<\kappa$. Call $T \subseteq {}^{<\kappa}\kappa$ a **κ-Laver tree** if T is limit closed and there exists $\mathrm{stem}(T) \in T$ such that $|\mathrm{suc}(s,T)| = \kappa$ for all $\mathrm{stem}(T) \subseteq s \in T$. For $<\kappa$-complete nonprincipal filter $\mathcal{U}$, a tree T is **guided by** $\mathcal{U}$ if $\mathrm{suc}(s,T) \in \mathcal{U}$ for all splitting nodes $s \in T$.

$\mathbb{L}_\kappa^{\mathcal{U}}$ consists of all κ-Laver trees guided by $\mathcal{U}$, ordered by inclusion. Laver-like forcings add dominating κ-reals.

Proposition *Folklore?*

If $\mathcal{U}$ is the club filter, then $\mathbb{L}_\kappa^{\mathcal{U}}$ adds a κ-Cohen generic.

Proof. Let $S_0 \sqcup S_1 = \kappa$ be a stationary partition, then $\mathrm{suc}(s,T) \cap S_i$ is nonempty for any $s \supseteq \mathrm{stem}(T)$ and $i \in 2$. □

Theorem *Khomskii, Koelbing, Laguzzi, and Wohofsky (2022)*
If $\mathbb{L}$ has κ-Laver trees as conditions and $(T)_s \in \mathbb{L}$ for every $T \in \mathbb{L}$ and $s \in T$, then $\mathbb{L}$ adds a κ-Cohen generic.

Theorem *Khomskii, Koelbing, Laguzzi, and Wohofsky (2022)*
If $\mathbb{P}$ is $<\kappa$-distributive, has limit-closed κ-trees as conditions and $(T)_s \in \mathbb{P}$ for every $T \in \mathbb{P}$ and $s \in T$, and the $\mathbb{P}$-generic κ-real is dominating, then $\mathbb{P}$ adds a κ-Cohen generic.

Corollary
κ-Mathias forcing adds a κ-Cohen generic.

Question
Is there a $<\kappa$-distributive forcing that adds a dominating κ-real without adding a κ-Cohen generic?

A tree $T \subseteq {}^{<\kappa}\kappa$ is **splitting-closed** if it is limit-closed and for every $f \in [T]$ the set $\{\alpha \in \kappa \mid f \restriction \alpha \text{ splits in } T\}$ is club.

$\mathbb{Mi}_\kappa^\mathcal{U}$ consists of all splitting-closed trees guided by $\mathcal{U}$.
Miller-like forcings add unbounded κ-reals.

Proposition *Brendle, Brooke-Taylor, Friedman, and Montoya (2018)*
If $\mathcal{U}$ is the club filter, then $\mathbb{Mi}_\kappa^\mathcal{U}$ adds a κ-Cohen generic.

Let $\mathbb{P}$ be a forcing notion and $h \in {}^\kappa\kappa$ be cofinally increasing.
$\mathbb{P}$ has the h**-Laver property** if for all $p \in \mathbb{P}$ and $\dot{f}$ with $p \Vdash$ “$\dot{f} \in {}^\kappa\kappa$” and $p \Vdash$ “$\dot{f} \leq^* g$” for some $g \in {}^\kappa\kappa$, there exists $q \leq p$ and $\varphi \in \mathrm{Sl}_h$ such that $q \Vdash$ “$\dot{f} \in^* \varphi$”.

Proposition *Brendle, Brooke-Taylor, Friedman, and Montoya (2018)*
If $\mathcal{U}$ is a $<\kappa$-complete normal ultrafilter, then $\mathbb{Mi}_\kappa^\mathcal{U}$ has the h-Laver property, for $h : \alpha \mapsto |2^\alpha|^+$.

Proposition *Mildenberger and Shelah (2021)*
Let κ be Laver indestructibly supercompact and $\overline{\mathbb{Mi}} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_n \mid n \in \omega \rangle$ be a csi where $\mathbb{P}_n \Vdash$ "$\dot{\mathbb{Q}}_n = \mathbb{Mi}_\kappa^{\mathcal{U}_n}$" for $\mathcal{U}_n$ a normal ultrafilter in $\mathbf{V}^{\mathbb{P}_n}$. Then $\overline{\mathbb{Mi}}$ does not have the h-Laver property for any $h \in {}^\kappa\kappa$.

Proposition *Basically similar to: Mildenberger and Shelah (2021)*
$\overline{\mathbb{Mi}}$ adds a κ-Cohen generic.

Proof. Partition $\{\alpha \in \kappa \mid \mathrm{cf}(\alpha) = \omega\}$ into stationary sets S_t labeled by $t \in {}^{<\kappa}2$. Let $\dot{m}_n$ name the n-th κ-Miller generic. Define sequences $\dot{x}, \dot{x}^*$ of elements in ${}^{<\kappa}2$:

$$\dot{x}(\alpha) = t \text{ if } \sup\{\dot{m}_n(\alpha) \mid n \in \omega\} \in S_t, \quad \dot{x}(\alpha) = \varnothing \text{ otherwise,}$$

$$\dot{x}^*(\alpha) = \dot{x}(0)^\frown \dot{x}(1)^\frown \cdots {}^\frown \dot{x}(\alpha)$$

Hence $\bigcup \dot{x}^*$ names an element of ${}^\kappa 2$. $\cdots$

$$\dot{x}(\alpha) = t \text{ if } \sup\{\dot{m}_n(\alpha) \mid n \in \omega\} \in S_t, \quad \dot{x}(\alpha) = \varnothing \text{ otherwise}$$

Let $p \in \overline{\mathbb{Mi}}$. Find $p_0 \leq p$ and $\delta \in \kappa$ and $t_0 \in \mathbf{V}$ s.t. for all $n \in \omega$ $p_0 \restriction n \Vdash$ "$\mathrm{ot}(\mathrm{stem}(p_0(n))) = \delta$" and $p_0 \Vdash$ "$\dot{x}^* \restriction \delta = \check{t}_0$".

Claim: For any $t \in {}^{<\kappa}2$, there is $p_\omega \leq p_0$ s.t. $p_\omega \Vdash$ "$\dot{x}(\delta) = t$".

Let $\langle \mathbf{M}_\alpha \mid \alpha \in \kappa \rangle$ be a continuous $\in$-chain with $\mathbf{M}_\alpha \prec \mathbf{H}_\chi$ and $|\mathbf{M}_\alpha| < \kappa$ and $\mathbb{P}, p_0, \delta \in \mathbf{M}_0$. Let $\kappa_\alpha = \mathbf{M}_\alpha \cap \kappa$ and fix some γ with $\kappa_\gamma \in S_t$, where $\langle \gamma_n \mid n \in \omega \rangle$ is cofinal in γ.

For each $n \in \omega$ find $p_{n+1}, \beta_n \in \mathbf{M}_{\gamma_{n+1}}$ with $p_{n+1} \leq p_n$, and $\kappa_{\gamma_n} \leq \beta_n$ s.t. $p_{n+1} \restriction n \Vdash$ "$p_{n+1}(n)(\delta) = \dot{m}_n(\delta) = \beta_n$" and $p_{n+1}(k) = p_0(k)$ for $k > n$. Then $p_\omega \Vdash$ "$\dot{x}(\delta) = t$". $\square_{\text{Claim}}$

Therefore $\dot{x}^*$ names a κ-Cohen generic. $\square$

Theorem *Brendle, Brooke-Taylor, Friedman, and Montoya (2018)*
The product $\mathbb{Mi}^{\mathcal{U}}_{\kappa} \times \mathbb{Mi}^{\mathcal{U}}_{\kappa}$ adds a κ-Cohen generic.

Note that classically the product $\mathbb{Mi} \times \mathbb{Mi}$ does not add a Cohen real, although $\mathbb{Mi} \times \mathbb{Mi} \times \mathbb{Mi}$ does.

Question
Is there a $<\kappa$-distributive forcing notion that adds many κ-Miller generics without adding a κ-Cohen generic?

$\mathbb{E}_\kappa$ has conditions $(s, F) \in {}^{<\kappa}\kappa \times [{}^{\kappa}\kappa]^{<\kappa}$ with $(t, G) \leq (s, F)$ if $s \subseteq t$, $F \subseteq G$ and $t(\alpha) \notin \{f(\alpha) \mid f \in F\}$ for $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$.

Let $\mathcal{F}$ be a filter, then a set $Q \subseteq \mathbb{P}$ is $(\kappa, \mathcal{F})$**-linked** if every $\langle p_\alpha \mid \alpha \in \kappa\rangle \subseteq Q$ has a $q \in \mathbb{P}$ s.t. $q \Vdash$ " $\{\alpha \in \kappa \mid p_\alpha \in \dot{G}\} \in \mathcal{F}^+$ ". The union of μ-many $(\kappa, \mathcal{F})$-linked sets is $(\mu, \kappa, \mathcal{F})$**-linked**.

Proposition *Goldstern, Mejía, and Shelah (2016)*
If $\mathbb{P}$ is $(\kappa, \kappa, \mathcal{F})$-linked for a $<\kappa$-complete nonprincipal filter $\mathcal{F}$, then $\mathbb{P}$ does not add dominating κ-reals.

Proposition *Goldstern, Mejía, and Shelah (2016), Mejía (2019)*
$\mathbb{E}$ is $(\aleph_0, \aleph_0, \mathcal{F})$-linked for $\mathcal{F}$ the Fréchet filter or any nonprincipal ultrafilter on ω. Hence $\mathbb{E}$ does not add dominating reals.

Lemma

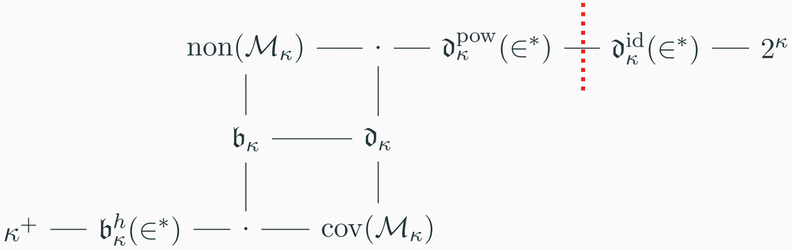
If κ is measurable, $\mathbb{E}_\kappa$ is $(\kappa, \kappa, \mathcal{F})$-linked for $\mathcal{F}$ the κ-Fréchet filter or any nonprincipal $<\kappa$-complete ultrafilter on κ.

Question

Let κ be Laver indestructibly supercompact. Does $<\kappa$-support iteration preserve $(\kappa, \kappa, \mathcal{F})$-linkedness?

The problem

The ξ-th iterate $(\dot{\mathbb{E}}_\kappa)_\xi$ must be $(\kappa, \kappa, \dot{\mathcal{U}}_\xi)$-linked for a $<\kappa$-complete ultrafilter $\dot{\mathcal{U}}_\xi \in \mathbf{V}^{\mathbb{P}_\xi}$ extending all previous ultrafilters $\dot{\mathcal{U}}_\eta$ for $\eta < \xi$. At limit stages ξ, how can we be sure that $\bigcup_{\eta<\xi} \dot{\mathcal{U}}_\eta$ can be extended to a $<\kappa$-complete ultrafilter?



Theorem *Bartoszyński (1987)*
(Classically) $\operatorname{cof}(\mathcal{N}) = \mathfrak{d}^h(\in^*)$ and $\operatorname{add}(\mathcal{N}) = \mathfrak{b}^h(\in^*)$ for **any** cofinally increasing $h \in {}^\omega\omega$.

Theorem *Brendle, Brooke-Taylor, Friedman, and Montoya (2018)*
Let $\mathrm{id} : \alpha \mapsto |\alpha|^+$ and $\mathrm{pow} : \alpha \mapsto (2^{|\alpha|})^+$, then $\mathfrak{d}_\kappa^{\mathrm{pow}}(\in^*) < \mathfrak{d}_\kappa^{\mathrm{id}}(\in^*)$ holds in the κ-Sacks model.

Let $\mathbb{P}$ be a forcing notion and $h \in {}^{\kappa}\kappa$ be cofinally increasing.

$\mathbb{P}$ has the h**-Laver property** if for all $p \in \mathbb{P}$ and $\dot{f}$ with $p \Vdash$ "$\dot{f} \in {}^{\kappa}\kappa$" and $p \Vdash$ "$\dot{f} \leq^* g$" for some $g \in {}^{\kappa}\kappa$, there exists $q \leq p$ and $\varphi \in \mathrm{Sl}_h$ such that $q \Vdash$ "$\dot{f} \in^* \varphi$".

$\mathbb{P}$ has the h**-Sacks property** if for all $p \in \mathbb{P}$ and $\dot{f}$ with $p \Vdash$ "$\dot{f} \in {}^{\kappa}\kappa$", there exists $q \leq p$ and $\varphi \in \mathrm{Sl}_h$ such that $q \Vdash$ "$\dot{f} \in^* \varphi$".

If $\mathbb{P}$ has the h-Sacks property, then $(\mathrm{Sl}_h)^{\mathbf{V}}$ witnesses $\mathfrak{d}^h_\kappa(\in^*)$.

A tree $T \subseteq {}^{<\kappa}\kappa$ is called **perfect** if every node $s \in T$ has $t \in T$ with $s \subseteq t$ that is splitting. Let **κ-Sacks forcing** $\mathbb{S}_\kappa$ consist of perfect splitting-closed trees on ${}^{<\kappa}2$, ordered by inclusion.

Theorem *Brendle, Brooke-Taylor, Friedman, and Montoya (2018)*
$\mathbb{S}_\kappa$ has the pow-Sacks property, and $\mathbb{S}_\kappa$ adds a κ-real f such that $f \not\in^* \varphi$ for all id-slaloms φ from the ground model.

Proof sketch. Let $T \in \mathbb{S}_\kappa$ and $\dot{f}$ be a name. Using fusion, we may assume that all nodes in the α-th splitting level decide $\dot{f}(\alpha)$. Furthermore, since $\mathbb{S}_\kappa$ is splitting-closed, there is a club set of α's for which $T \cap {}^\alpha 2$ is exactly the set of splitting nodes at the α-th splitting level.

Since $|T \cap {}^\alpha 2| = 2^{|\alpha|}$, we can collect the possible values of $\dot{f}(\alpha)$ with a pow-slalom, but (possibly) not with an id-slalom. $\square$

How to separate **more** localisation cardinals?

Let $h \in {}^{\kappa}\kappa$ be cofinally increasing. We define κ**-Miller-Lite forcing** $\mathbb{ML}^h_\kappa$ to have perfect splitting-closed trees $T \subseteq {}^{<\kappa}\kappa$ as conditions such that if $s \in T$ is an α-th level splitting node of T, then s has exactly $h(\alpha)$-many successors in T.

We order $T' \leq T$ iff $T' \subseteq T$ and for any $s \in T'$ with $\mathrm{suc}_{T'}(s) \neq \mathrm{suc}_T(s)$ we also require that $|\mathrm{suc}_{T'}(s)| < |\mathrm{suc}_T(s)|$. This guarantees that $\mathbb{ML}^h_\kappa$ is $<\kappa$-closed.

Let $\mathrm{id}_h = \mathrm{id} \circ h$ and $\mathrm{pow}_h = \mathrm{pow} \circ h$.

Theorem *vdV. (2024)*

$\mathbb{ML}^h_\kappa$ has the pow_h-Sacks property, and $\mathbb{ML}^h_\kappa$ adds a κ-real f such that $f \not\in^* \varphi$ for all id_h-slaloms φ from the ground model.

Proof sketch. Same as with $\mathbb{S}_\kappa$. Note that if $T \cap {}^\alpha\kappa$ is the set of α-th level splitting nodes, then $|T \cap {}^\alpha\kappa| = 2^{h(\alpha)}$. □

Suppose that $g \in {}^\kappa\kappa$ is a cardinal-valued function satisfying:

- $g(\alpha)^{|\alpha|} = g(\alpha)$, and
- $2^{g(\alpha)} \leq g(\beta)$ for all $\alpha < \beta$.

If $S_0, S_1 \subseteq \kappa$ are almost disjoint stationary sets, and h_0, h_1 are such that $h_i \restriction S_i = g(\alpha)$ and $h_i \restriction (\kappa \setminus S_i) = 2^{g(\alpha)}$, then it is consistent that $\mathfrak{d}^{h_0}_\kappa(\in^*) < \mathfrak{d}^{h_1}_\kappa(\in^*)$ and $\mathfrak{d}^{h_1}_\kappa(\in^*) < \mathfrak{d}^{h_0}_\kappa(\in^*)$.

Under assumption of $\diamondsuit_\kappa$ there exists a family $\langle S_\xi \mid \xi \in \kappa^+ \rangle$ of mutually almost disjoint stationary sets.

Theorem *vdV. (2024)*
If:

- $2^\kappa = \kappa^+$, and
- $\langle S_\xi \mid \xi \in \kappa^+ \rangle$ is a family of mutually almost disjoint stationary sets, and
- $\iota : \kappa^+ \to \kappa^+$ is bijective, and
- we define $h_\xi \restriction S_\xi = g(\alpha)$ and $h_\xi \restriction (\kappa \setminus S_\xi) = 2^{g(\alpha)}$,

then there is a forcing extension in which for all $\xi, \xi' \in \kappa^+$ with $\xi < \xi'$ we have $\mathfrak{d}_\kappa^{h_{\iota(\xi)}}(\in^*) < \mathfrak{d}_\kappa^{h_{\iota(\xi')}}(\in^*)$.

References

- Bartoszyński, T. (1987). **“Combinatorial aspects of measure and category”.** In: **Fundamenta Mathematicae** 127.3, pp. 225–239.
- Blass, A., Hyttinen, T., and Zhang, Y. (n.d.). **Mad families and their neighbors.** Preprint. URL: `https://dept.math.lsa.umich.edu/~ablass/madfam.pdf`.
- Brendle, J. (2022). **“The higher Cichoń diagram in the degenerate case”.** In: **Tsukuba Journal of Mathematics** 42.2, pp. 255–269.
- Brendle, J., Brooke-Taylor, A., Friedman, S.-D., and Montoya, D. C. (2018). **“Cichoń’s diagram for uncountable cardinals”.** In: **Israel Journal of Mathematics** 225.2, pp. 959–1010.
- Goldstern, M., Mejía, D. A., and Shelah, S. (2016). **“The left side of Cichoń’s diagram”.** In: **Proceedings of the American Mathematical Society** 144.9, pp. 4025–4042.
- Khomskii, Y., Koelbing, M., Laguzzi, G., and Wohofsky, W. (2022). **“Laver trees in the generalized Baire space”.** In: **Israel Journal of Mathematics** 255.2, pp. 599–620.
- Mejía, D. A. (2019). **“Matrix Iterations with Vertical Support Restrictions”.** In: **Proceedings of the 14th and 15th Asian Logic Conferences**, pp. 213–248.
- Mildenberger, H. and Shelah, S. (2021). **“Higher Miller forcing may collapse cardinals”.** In: **The Journal of Symbolic Logic** 86.4, pp. 1721–1744.
- Miller, A. W. (1981). **“Some Properties of Measure and Category”.** In: **Transactions of the American Mathematical Society** 266.1, pp. 93–114.
- Shelah, S. (2017). **“A parallel to the null ideal for inaccessible λ: Part I”.** In: **Archive for Mathematical Logic** 56.3-4, pp. 319–383.
- Truss, J. (1977). **“Sets Having Calibre $\aleph_1$”.** In: **Logic Colloquium 76**. Ed. by R. Gandy and J. Hyland. Vol. 87. Studies in Logic and the Foundations of Mathematics. Elsevier, pp. 595–612.
- Vlugt, T. van der (2024). **“Separating many localisation cardinals on the generalised Baire space”.** In: **Journal of Symbolic Logic** 89.3, pp. 1212–1231.