Some results regarding the Predicate Modal Logic of Forcing STUCK 12, London

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The forcing interpretation of the modality \square

Interpret $\Box \varphi$ as

"in all forcing extensions arphi holds"

Question

What are the modal principles of forcing? That is, what are the modal statements for which all substitution instances of propositional/predicate variables for set theoretic formulas hold?

- $\Box p \rightarrow p$
- $\Box p \rightarrow \Box \Box p$
- $\Diamond \Box p \rightarrow \Box \Diamond p$
- A predicate example: $\Box \forall x P(x) \rightarrow \forall x \Box P(x)$

Propositional case

Definition

$$\mathsf{Force}^{\mathsf{ZFC}} = \{ \varphi \in \mathcal{L}^\square \mid \text{ all propositional substitution instances of } \varphi \\ \text{are provable from ZFC} \}$$

$$\mathsf{Force}^W = \{ \varphi \in \mathcal{L}^\square \mid \text{ all propositional substitution instances of } \varphi \\ \quad \mathsf{are true in } W \}$$

Theorem (Hamkins, Löwe [1])

If ZFC is consistent, then Force $^{ZFC} =$ **\$4.2**.

If $W \models \mathsf{ZFC}$, then $\mathsf{S4.2} \subseteq \mathsf{Force}^W \subseteq \mathsf{S5}$.

Predicate case so far...

We work in a modal language without equality.

Definition

$$\mathsf{Force}_\forall^{\mathsf{ZFC}} = \{\varphi \in \mathcal{L}^{\square,\forall} \mid \text{ all predicate substitution instances of } \varphi$$
 are provable from
$$\mathsf{ZFC}\}$$

$$\mathsf{Force}_\forall^{\ W} = \{\varphi \in \mathcal{L}^{\square,\forall} \mid \text{ all predicate substitution instances of } \varphi$$
 are true in $W\}$

Theorem (Hamkins, L.)

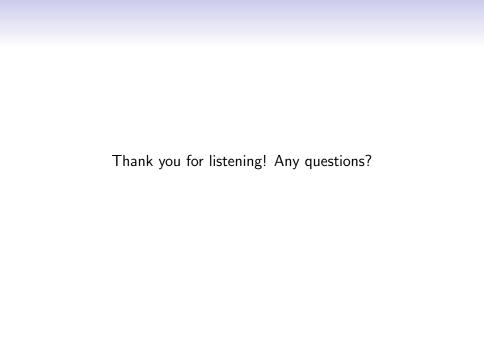
If ZFC is consistent, then $Force_{\forall}^{ZFC} = \mathbf{QS4.2}$. If $W \models \mathsf{ZFC}$, then $\mathbf{QS4.2} \subseteq \mathsf{Force}_{\forall}^{W}$.

Predicate case so far...

- We proved Force $_{\forall}^{ZFC} = \mathbf{QS4.2}$ by proving that Force $_{\forall}^{L} = \mathbf{QS4.2}$.
- To do this we introduced a new kind of system of control statements: a long ratchet system of buttons.

Theorem (Hamkins, L.)

If W satisfies the Ground Axiom and has a uniformly definable long ratchet system of buttons, then $Force_{\forall}^{W} = \mathbf{QS4.2}$.



References

- [1] J. D. Hamkins & B. Löwe, 'The modal logic of forcing', *Trans. Amer. Math. Soc.*, 360:4 (2008) 1793–1817.
- [2] J. D. Hamkins, G. Leibman & B. Löwe, 'Structural connections between a forcing class and its modal logic', Isr. J. Math., 207:2 (2015) 617–651.
- [3] U. Abraham, & S. Shelah, 'Forcing Closed Unbounded Sets', J. Symb. Log., 48:3 (1983) 643–657.