Some results regarding the Predicate Modal Logic of Forcing

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The forcing interpretation of the modality $\Box$

Interpret $\Box\varphi$ as

"in all forcing extensions φ holds"

Question

What are the modal principles of forcing? That is, what are the modal statements for which all substitution instances of propositional/predicate variables for set theoretic formulas hold?

- $\Box p \to p$
- $\Box p \to \Box\Box p$
- $\Diamond\Box p \to \Box\Diamond p$
- A predicate example: $\Box\forall x P(x) \to \forall x \Box P(x)$

Propositional case

Definition

$$\text{Force}^{\text{ZFC}} = \{\varphi \in \mathcal{L}^{\Box} \mid \text{ all propositional substitution instances of } \varphi \text{ are provable from ZFC}\}$$

$$\text{Force}^{W} = \{\varphi \in \mathcal{L}^{\Box} \mid \text{ all propositional substitution instances of } \varphi \text{ are true in } W\}$$

Theorem (Hamkins, Löwe [1])

If ZFC is consistent, then $\text{Force}^{\text{ZFC}} = \mathbf{S4.2}$.
If $W \models \text{ZFC}$, then $\mathbf{S4.2} \subseteq \text{Force}^{W} \subseteq \mathbf{S5}$.

Predicate case so far...

We work in a modal language without equality.

Definition

$$\mathrm{Force}_{\forall}{}^{\mathrm{ZFC}} = \{\varphi \in \mathcal{L}^{\Box,\forall} \mid \text{ all predicate substitution instances of } \varphi \text{ are provable from ZFC}\}$$

$$\mathrm{Force}_{\forall}{}^{W} = \{\varphi \in \mathcal{L}^{\Box,\forall} \mid \text{ all predicate substitution instances of } \varphi \text{ are true in } W\}$$

Theorem (Hamkins, L.)

If ZFC is consistent, then $\mathrm{Force}_{\forall}^{\mathrm{ZFC}} = \mathbf{QS4.2}$.
If $W \models \mathrm{ZFC}$, then $\mathbf{QS4.2} \subseteq \mathrm{Force}_{\forall}^{W}$.

Predicate case so far...

- We proved $\text{Force}_{\forall}^{\text{ZFC}} = \textbf{QS4.2}$ by proving that $\text{Force}_{\forall}^{L} = \textbf{QS4.2}$.
- To do this we introduced a new kind of system of control statements: *a long ratchet system of buttons*.

Theorem (Hamkins, L.)

If W satisfies the Ground Axiom and has a uniformly definable long ratchet system of buttons, then $\text{Force}_{\forall}^{W} = \textbf{QS4.2}$.

Thank you for listening! Any questions?

References

[1] J. D. Hamkins & B. Löwe, 'The modal logic of forcing', *Trans. Amer. Math. Soc.*, 360:4 (2008) 1793–1817.

[2] J. D. Hamkins, G. Leibman & B. Löwe, 'Structural connections between a forcing class and its modal logic', *Isr. J. Math.*, 207:2 (2015) 617–651.

[3] U. Abraham, & S. Shelah, 'Forcing Closed Unbounded Sets', *J. Symb. Log.*, 48:3 (1983) 643–657.