Amoeba Forcing and inaccessible cardinals

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A set of reals A is said to be Lebesgue measurable iff for every Borel set B there is a Borel subset C, such that either $C \setminus A$ or $A \setminus C$ is of measure zero.

A set of reals on the other hand is said to have the Baire property iff for every Borel set B, there is a Borel subset C, such that either $C \setminus A$ or $A \setminus C$ is a meager set.

Notice that both the collection of meager sets and that of lebesgue measure zero sets form sigma ideals.

If I, is a σ ideal (which is $\mathbf{\Sigma}^1_2$) and $\mathbb{P}_I$ consists of Borel sets not in I, then we can define I regularity as follows: A set of reals A is I

regular iff for every Borel set B there is a subset C such that either $C \setminus A$ or $A \setminus C \in I$.

If B and C are elements of $\mathbb{P}_I$, then we say that $C \leq B$ iff $C \subseteq B$. This way $\mathbb{P}_I$ can be seen as a forcing notion.

Theorem

All analytic sets are regular

Proof.

Folklore □

Theorem

Regularity of $\mathbf{\Sigma}^1_2$ sets of reals is however independent of ZFC

Definition

A real x is said to be a quasi-generic over a model V iff it is not contained in any borel set $B \in I$, such that B is coded in V

Theorem (Ikegami)

If I is a σ ideal with $\mathbf{\Sigma}^1_2$ definition and $\mathbb{P}_I$ is a proper forcing notion, then

- all $\mathbf{\Delta}^1_2$ sets are I regular iff for every real r there is an I quasi-generic over $L[r]$.
- all $\mathbf{\Sigma}^1_2$ sets are I regular iff for every real r the complement of the set of quasi-generics over $L[r]$ is in I

Theorem

Forcing with $\mathbb{P}_I$ adds an I generic over the ground model and I generics are I quasi-generics.

This shows us a way to obtain $\mathbf{\Delta}^1_2$ regularity.

If $\mathbb{P}$ is a forcing notion as in Ikegami's theorem, one way to obtain $\mathbf{\Sigma}^1_2(\mathbb{P})$ is to iteratively add co-null sets of quasigeneric reals in an iteration of length ω_1. In order to do this, we would like to have natural forcing notions adding these large sets of quasigenerics, usually called *amoebas* of the original forcing.

Definition

Let $\mathbb{P}$ be an tree forcing notion and for $T \in \mathbb{P}$, let $[T]$ denote the set of branches through T and $\mathbb{Q}$ any other forcing notion. Then we say that $\mathbb{Q}$ is an amoeba for $\mathbb{P}$ iff for any $T \in \mathbb{P}$, any $\mathbb{Q}$-generic G, and any model $M \supseteq V[G]$, we have that

$$M \models \exists T' \leq T \forall x (x \in [T'] \rightarrow x \text{ is } \mathbb{P}\text{-generic over } V).$$

Some examples of amoeba forcing notions are as follows:
Amoeba forcing, denoted by $\mathbb{A}$, consists of the set of all pruned trees $T \subseteq 2^{<\omega}$ such that $\mu([T]) > \frac{1}{2}$, ordered by inclusion.

For Hechler forcing and eventually different forcing Brendle and Löwe [BL99, BL11] proved that one cannot achieve $\mathbf{\Sigma}^1_2$ regularity via forcing and that it infact implies that $\aleph_1$ is inaccessible by reals. The result about Hechler forcing will be used later to show that $\mathbf{\Sigma}^1_2$ regularity of amoeba forcing implies that $\aleph_1$ is inaccessible by reals.

Some regularity properties are not defined in terms of I-regularity or $\mathbb{P}$-measurability; e.g., the Baire property was originally defined in topological terms.
A general framework has been developed by Lucas in his PhD thesis with the help of weak category bases[Wan23]

Definition

Let X be a set and $C \subseteq \mathcal{P}(X)$. We call (X, C) is a *weak category base* if and only if

1. $X = \bigcup C$ and
2. for every $A, A' \in C$, $A \cap A'$ contains an element of C or for every $c \in C$, there is some $c' \in C$, such that $c' \subseteq c$ and $c' \cap (A \cap A') = \emptyset$.

- We refer to the elements of C as *regions*
- $A \subseteq X$ is C-*singular* if and only if for every region c there is a region $c' \subseteq c$ such that $c' \cap A = \emptyset$
- a subset $A \subseteq X$ is C-*small* if it is a countable union of C-singular sets
- $A \subseteq X$ is said to be C-*measurable* if and only if for every region c, there is a region c' such that $c' \subseteq c$ and either $c' \setminus A$ or $c' \cap A$ is C-small.

For a region c, we say that

- $A \subseteq X$ is *C-not small in c* if $c \cap A$ is not C-small
- it is *C-not small everywhere in c* if $c' \cap A$ is not C-small for any region $c' \subseteq c$
- The ideal I_C^* is defined to be the collection of all subsets of X such that there is no region c for which A is C-not small everywhere in c.

- The set C is a partial order with the ordering $\subseteq$; we say that the weak category base (X, C) has the *countable chain condition* if the partial order $(C, \subseteq)$ does.
- A weak category base (X, C) is called *proper* if $(C, \subseteq)$ is a proper forcing notion, $\mathcal{I}_C$ is a proper σ-ideal, and every region is not C-small.

If X is a Polish space and (X, C) is a weak category base, we say that C *is Borel compatible with* X if every region is Borel and every Borel set is C-measurable.

Definition

Amoeba infinity forcing, denoted by $\mathbb{A}_\infty$ consists of the set of all pairs $(O, \epsilon) \in \boldsymbol{O} \times \mathbb{R}^+_\infty$ such that $\mu(O) < \epsilon\}$ ordered by $(O', \epsilon') \leq (O, \epsilon)$ iff $O \subseteq O'$ and $\epsilon' \leq \epsilon$.

This forcing notion lives on the Polish space of all open subsets of $\mathbb{R}$ denoted by $\boldsymbol{O}$.

$$[O, \epsilon] := \{U \in \boldsymbol{O}\,;\; O \subseteq U \text{ and } \mu(U) \leq \epsilon\}$$

for $(O, \epsilon) \in \mathbb{A}_\infty$. The collection of sets $[O, \epsilon]$ is denoted by $C_{\mathbb{A}_\infty}$.

The pair $(\boldsymbol{O}, C_{\mathbb{A}_\infty})$ is a c.c.c category base which is Borel compatible with $\boldsymbol{O}$ and the ideal of $C_{\mathbb{A}_\infty}$-small sets is Borel generated.

Definition

Amoeba forcing for category, also known as *universally meagre forcing* and denoted by $\mathbb{UM}$, consists of the set of all (σ, E) such that $\sigma = (\sigma(0), ..., \sigma(n-1))$ is a finite sequence of elements of $2^{<\omega}$ and E is an open dense subset of $2^{<\omega}$. This set is partially ordered as follows:

$$(\sigma', E') \leq (\sigma, E) \text{ iff } \sigma \subseteq \sigma' \text{ and } \forall n \in \mathrm{dom}(\sigma' \setminus \sigma)(\sigma'(n)) \in E.$$

Amoeba forcing for category is an Amoeba for $\mathbb{C}$. It lives on the Polish space $\boldsymbol{U}$ which is the collection of all $x \in (2^{<\omega})^{\omega}$ such that for all $s \in 2^{<\omega}$ there are infinitely many m with $s \subseteq x(m)$, via the the collection U of sets $[\sigma, E]$ for $(\sigma, E) \in \mathbb{UM}$.

The pair $(\boldsymbol{U}, U)$ is a proper weak category base which is Borel compatible with $\boldsymbol{U}$.

Definition

Localisation forcing, denoted by $\mathbb{LOC}$, consists of the set of all pairs (σ, F) such that $\sigma = (\sigma(0), ..., \sigma(n-1))$ is a finite sequence of elements of $[\omega]^{<\omega}$ and F is a finite set of elements of ω^ω such that for every $k < n$, $|\sigma(k)| = k+1$ and $|F| \leq n+1$. The set is partially ordered as follows:

$$(\sigma', F') \leq (\sigma, F) \text{ iff } \sigma \subseteq \sigma' \text{ and } F \subseteq F' \text{ and } \forall x \in F$$

$\forall n \in \mathrm{dom}(\sigma' \setminus \sigma)(x(i) \in \sigma'(i))$.

We define **Loc** to be the Polish space of all slaloms that is of functions $f : \omega \to [\omega]^{<\omega}$ such that for all $n \in \omega$, $|f(n)| \leq n + 1$.

The pair $(\mathbf{Loc}, L)$ forms a proper weak category base such that it is Borel compatible with **Loc**.

As mentioned earlier the $\mathbf{\Sigma}^1_2$ regularity of Hechler and Eventually different Forcings both imply the inaccessibility of $\aleph_1$ by reals. Up next we shall show that the $\mathbf{\Sigma}^1_2$ regularity of Amoeba Forcing, Amoeba for Category and the **Loc** all imply the inaccessibility of $\aleph_1$ by reals.

The general strategy for proving it in case of the two amoeba forcing notions involves an indirect approach in which we show that $\mathbf{\Sigma}^1_2(\mathbb{P})$ implies $\mathbf{\Sigma}^1_2(\mathbb{D})$, where $\mathbb{P} \in \{\mathbb{A}, \mathbb{UM}\}$.

On the other hand for $\mathbb{LOC}$, we develop a Brendle-Labdezki style lemma to prove it.

Let us first outline what a Brendle-Labdezki style lemma is and also the mechanism of the indirect proofs and we shall proceed to the specifics in the next section.

Definition

An *assignment* is a pair of formulas (Ψ, Φ) such that in every transitive model of set theory M, we have that if $A \in M$ and $M \models \Psi(A)$, then, in M, Φ defines an injection $a \mapsto c_a^A$ with domain A.

For an ideal I on a Polish space X,

Definition

An assignment (Ψ, Φ) is called *I-canonical* if for each transitive model M of set theory, we have that

$$M \models \exists A(\Psi(A) \wedge |A| = 2^{\aleph_0})$$

and if $M \models \Psi(A)$ and $a \in A$, then c_a^A is a Borel code in M such that the Borel set B_a^A coded by c_a^A is in I.

Definition

A forcing notion defined as $\mathbb{P}_I$ on a Polish space X is said to satisfy a Brendle-Labdezki lemma if, there is an I-canonical assignment (Φ, Ψ) such that for each $Z \in I$ and any set A we have that

$$\{B_a^A ; a \in A \wedge B_a^A \subseteq Z\}$$

is countable.

Theorem

Let $\mathbb{P}$ be a forcing notion satisfying the conditions of Ikegami's theorem that in addition also satisfies a Brendle-Łabędzki lemma, then $\mathbf{\Sigma}^1_2(\mathbb{P})$ implies that $\aleph_1$ is inaccessible by reals.

Proof.

Fix any $x \in \mathbb{R}$. Due to Ikegami's theorem, the assumption gives us that the set

$$N_x := \{z \in X\,;\, z \text{ is not } I\text{-quasigeneric over } \mathrm{L}[x]\}$$

is small. By definition of quasigenericity, every Borel set B with Borel code in $\mathrm{L}[x]$ satisfies $B \subseteq N_x$. By the canonicity assumption, we find a particular A in $\mathrm{L}[x]$ such that $\mathrm{L}[x] \models \Psi(A)$ and $|A| = \aleph_1^{\mathrm{L}[x]}$ and for all $a \in A$, we have $B_a^A \subseteq N_x$. But by the Brendle-Łabędzki lemma, the set $\{B_a^A\,;\, a \in A \wedge B_a^A \subseteq N_x\}$ is countable, so $\aleph_1^{\mathrm{L}[x]}$ is countable. $\square$

Now, in case of $\mathbb{LOC}$, we look at the assignment defined as follows:

$$x \in \omega^\omega \mapsto X_x = \{f \in \mathbb{LOC} : \exists^\infty n \in \omega(x(n) \notin f(n))\}$$

We of course mean the Borel code corresponding to X_x. This

assignment is made canonical for the $\mathbb{LOC}$ ideal by an eventually different family E (that is for $x, y \in E$, $\forall^\infty n \in \omega x(n) \neq y(n)$).

Theorem

$\mathbb{LOC}$ satisfies Brendle-Labedzki lemma for the above canonical embedding.

Proof.

[Wan23, Lemma 2.5.9] □

We now move on to develop the frame work which will enable us to show that $\mathbf{\Sigma}^1_2(\mathbb{P})$ implies $\mathbf{\Sigma}^1_2(\mathbb{D})$ for $\mathbb{P} \in \{\mathbb{A}, \mathbb{UM}\}$.

Let (X, C) and (Y, D) be weak category bases. A function $f : C \to D$ is called a *projection* if

(i) whenever $c \subseteq c'$, then $f(c) \subseteq f(c')$ and
(ii) for every $c \in C$ and $d \leq f(c)$ there is a $c' \subseteq c$ such that $f(c') \leq d$.

Lemma (Wansner's Implication Lemma)

Let X and Y be uncountable Polish spaces and (X, C) and (Y, D) be proper weak category bases such that (X, C) and (Y, D) are Borel compatible with X and Y, respectively. Assume that I_D^* is Borel generated, let $\alpha > 0$ be an ordinal, that $\langle h_\beta : \beta < \alpha \rangle$ is a sequence of Borel functions from X to Y, and that $\langle \bar{h}_\beta : \beta < \alpha \rangle$ be a sequence of projections from C to D such that

a. *for every $\beta < \alpha$ and every $c \in C$, there is some region $c' \subseteq c$ such that $h_\beta[c'] \subseteq \bar{h}_\beta(c)$ and*
b. *for every $d \in D$, there are $\beta < \alpha$ and $c \in C$ such that $\bar{h}_\beta(c) \subseteq D$.*

Then for every projective pointclass $\mathbf{\Gamma}$, we have that if every $\mathbf{\Gamma}(X)$ set is C-measurable, then every $\mathbf{\Gamma}(Y)$ set is D-measurable.

Proof.

[Wan23, Theorem 2.2.46] □

So, clearly the wansner implication lemma when at the $\boldsymbol{\Sigma}^1_2$ level gives us the necessary technique to prove the intended results.

We first of all intend to show that $\mathbf{\Sigma}^1_2(\mathbb{A}_\infty) \implies \mathbf{\Sigma}^1_2(\mathbb{D})$.

To that end, we define a borel function h as follows:

Let $\{I^n_k \subseteq \mathbb{R} : n, k \in \omega\}$ be a recursive family of pairwise disjoint open intervals with rational endpoints such that for every $n, k \in \omega$ $\mu(I^n_k) = 2^{-2n}$. Then

$$h(U)(n) = \begin{cases} 0 & \text{if } \mu(U) = \infty, \\ \min\{k \in \omega : \forall \ell \geq k(I^n_\ell)\} & \text{otherwise.} \end{cases}$$

After this we wish to define a projection $\bar{h}$ on a dense subset of $\boldsymbol{O}$.

$$D = \{(O, \epsilon) \in \mathbb{A}_\infty : \exists n \in \omega(\sum_{m \geq n} 2^{-2m} < \epsilon - \mu(O) < 2^{-2(n-1)})\}.$$

which is defined as:

$$D' = \{(O, \epsilon) \in D : \forall U \in [O, \epsilon](h(O) \restriction n_{(O,\epsilon)} = h(U)) \restriction n_{(O,\epsilon)}\}$$

where $n_{(O,\epsilon)} \in \omega$ such that

$$\epsilon - \mu(O) \geq 2^{-2(n-1)} > 2^{-2(n-1)}/3 = \sum_{m \geq n} 2^{-2m}.$$

It being dense actually suffices since $C_{D'} = \{[O, \epsilon] : (O, \epsilon) \in D'\}$ will render $\boldsymbol{O}$ to be as category base equivalent to the original one.

It can be checked that the above satisfy the conditions of the Wansner lemma [Wan23, Proposition 2.3.24]

Next up we show that $\mathbf{\Sigma}^1_2(\mathbb{A}) \implies \mathbf{\Sigma}^1_2(\mathbb{A}_\infty)$.

To that end we fix a recursive family $\{A^n_k : n, k \in \omega\}$ of open independent subsets of 2^ω with $\mu(A^n_k) = 2^{(-n+1)}$.

Moreover let $I^{\ell,n}$ be the set of finite unions of intervals with rational endpoints with measure $\leq 4^{\ell-n}$ and let $\{U^{\ell,n}_k : k, \ell, n \in \omega\}$ be a recursive family of open sets such that for $\ell, n \in \omega$, $\{U^{\ell,n}_k : k \in \omega\}$ enumerates $\mathcal{I}^{\ell,n}$ and each element of $I^{\ell,n}$ occurs infinitely often in this enumeration.

For every $\ell \in \omega$, we define functions $h_\ell : \boldsymbol{R} \cup \mathbb{A} \to \boldsymbol{O}$ and $\bar{h}_\ell : \mathbb{A} \to \mathbb{A}_\infty$ by

$$h_\ell(S) = \bigcup\{U_k^{\ell,n} : \mu(A_k^n \cap [S]) = 0\} \text{ and}$$

$$\bar{h}_\ell(T) = (h_\ell(T), \sup\{\mu(h_\ell(S)) : S \leq T\})$$

This completes the proof of $\boldsymbol{\Sigma}_2^1(\mathbb{A})$ implies the inaccessibility of $\aleph_1$ by reals [Wan23, Proposition 2.3.27].

Now, we prove that $\mathbf{\Sigma}^1_2(\mathbb{UM}) \implies \mathbf{\Sigma}^1_2(\mathbb{D})$.

For this we define a Borel function $h : \boldsymbol{U} \to \omega^\omega$ and a projection $\bar{h} : \mathbb{UM} \to \mathbb{D}$ as follows:

$$h(x)(n) := \min\{|s| : s \in \operatorname{ran}(s) \text{ and } t_n \subseteq s\} \text{ and}$$
$$\bar{h}(\sigma, E) := (n_{(\sigma,E)}, f_{(\sigma,E)})$$

where $n_{(\sigma,E)}$ is maximal such that for every $x, x' \in [\sigma, E]$, $h(x) \restriction n = h(x') \restriction n$ and $f_{(\sigma,E)}(n) = \min\{h(x)(n) : x \in [\sigma, E]\}$.

To check that the above satisfies the Wansner lemma we refer to [Wan23, Theorem 2.4.10]

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Thank you!