Lent 2015 **INTRODUCTORY PROBLEMS – ANALYSIS 1** IBL

1. We say that a function f from $\mathbb{R}$ to $\mathbb{R}$ *crosses the axis* at a point x if $f(x) = 0$ but for any $\epsilon > 0$ there exist y and z in $(x-\epsilon, x+\epsilon)$ with $f(y) > 0$ and $f(z) < 0$. Can a continuous function cross the axis at uncountably many places?

2. Let $[a_1, b_1], [a_2, b_2], \ldots$ be a sequence of closed intervals whose union contains $[0,1]$. Does it follow that $\sum_{n=1}^{\infty}(b_n - a_n) \geq 1$?

3. Construct a continuous function from $[0,1]$ to $\mathbb{R}^2$ that is exactly $3-1$ onto its image.

4. Is there a continuous function from $[0,1]$ to $\mathbb{R}^2$ that is exactly $2-1$ onto its image?

5. Is every function from $[0,1]$ to $\mathbb{R}$ the pointwise limit of a sequence of continuous functions?

6. Let f be an infinitely-differentiable function from $\mathbb{R}$ to $\mathbb{R}$, such that for every x there is an n with all the derivatives $f^{(n)}(x), f^{(n+1)}(x), f^{(n+2)}(x), \ldots$ being zero. Must f be a polynomial?

7. Let f be a differentiable function from $\mathbb{Q}$ to $\mathbb{Q}$ such that $f' = f$. Must f be identically zero?

8. If f is a polynomial of one real variable that is bounded below (on $\mathbb{R}$), explain why f attains its minimum value. If f is a polynomial of two real variables that is bounded below (on $\mathbb{R}^2$), must f attain its minimum value?