Part III Michaelmas, 2022

COMBINATORICS

B. B.

Examples Sheet III.

If an exercise seems to make no sense, correct it and then solve it.

26. A family $\mathcal{A} \subset \mathcal{P}(n)$ is said to *shatter* a set $Y \subset X$ if for every set $Z \subset Y$ there is a set $A \in \mathcal{A}$ such that $Z = A \cap Y$. By 'down-compressing' the family $\mathcal{A}$, show that if $|\mathcal{A}| > \sum_0^{k-1} \binom{n}{i}$ then the family $\mathcal{A}$ shatters a k-set.
[Do not deduce this from the next exercise: use compressions.]

27. Show that every family $\mathcal{A} \subset \mathcal{P}(n)$ shatters at least $|\mathcal{A}|$ sets.

28. Let $\mathcal{A} \subset X^{(\leq n/2)}$ be an intersecting antichain. Show that

$$\sum_{A \in \mathcal{A}} \binom{n-1}{|A|-1}^{-1} \leq 1.$$

29. Let $A_1, A_2, \ldots, A_m; B_1, B_2, \ldots, B_m$ be subsets of $[n]$ such that the cardinality of $A_i \cap B_j$ is odd if and only if $i = j$. Show that $m \leq n$.

30. Let $K \subset \mathbb{R}^n$ be convex body of diameter 1 with smooth (differentiable) boundary. Prove that K can be partitioned into $n+1$ sets of diameter strictly less than 1.
[Hint. First partition the ball $B_n \subset \mathbb{R}^n$ of diameter 1.]

31. t Let $K \subset \mathbb{R}^2$ be a planar set of diameter 1. Show that K can be inscribed in a regular hexagon of width 1.

32. Show that every set of diameter 1 in the plane can be partitioned into three sets of diameter at most $\sqrt{3}/2$.

33. Let $S = \{s_1, \ldots, s_m\}$ be a set of m points in $\mathbb{R}^n$ such that any two points are at distance d_1 or d_2. Prove that $m \leq (n+1)(n+4)/2$.

[Hint. Write s_{ij} for the coordinates of our points: $s_j = (s_{1j}, s_{2j}, \ldots, s_{nj})$, and for $j = 1, \ldots, m$ define

$$f_j(X) = \left(\sum_{i=1}^{n}(X_i - s_{ij})^2 - d_1^2\right)\left(\sum_{i=1}^{n}(X_i - s_{ij})^2 - d_2^2\right) \in \mathbb{R}[X],$$

where $X = (X_1, \ldots, X_n)$. What next?]

34. Can you improve this bound to $(n+1)(n+2)/2$?

35. Construct a set $S \subset \mathbb{R}^n$ of cardinality $n(n+1)/2$ which determines only two distances.

36. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a family consisting of $2^{n-1}+1$ sets, where, as usual, $|X| = n$. At least how many disjoint pairs $\{A, B\}$ are in $\mathcal{A}$? And at most how many disjoint pairs $\{A, B\}$ are in $\mathcal{A}$ if $|\mathcal{A}| = 2^{n-1}$?

37. Prove the Four Functions Theorem in one dimension. Thus, let $\alpha, \beta, \gamma, \delta : Q^1 \to \mathbb{R}^+$ be such that

$$\alpha(a)\beta(b) \leq \gamma(a \vee b)\delta(a \wedge b)$$

for $a, b \in Q^1$. Show that for all $A, B \subset Q^1$ we have

$$\alpha(A)\beta(B) \leq \gamma(A \vee B)\delta(A \wedge B).$$

38. Let $a_1 < \cdots < a_n$ be $n \geq 2$ natural numbers. Show that $a_i/(a_i, a_j) \geq n$ for some a_i and a_j, $i \neq j$.

39. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ be such that every set $A \in \mathcal{A}$ contains a set $B \in \mathcal{B}$. Prove that

$$|\mathcal{A} - \mathcal{B}| \geq |\mathcal{A}|.$$

[Hint. Note that the pairs of subsets of $[n-1]$

$$(\mathcal{A}_0 \cap \mathcal{A}_1, \mathcal{B}_0) \quad \text{and} \quad (\mathcal{A}_0 \cup \mathcal{A}_1, \mathcal{B}_0 \cup \mathcal{B}_1)$$

satisfy the original conditions.]

40. Let $\mathcal{A}$ be a maximal intersecting family of subsets of $[n]$. Show that $|\mathcal{A}| = 2^{n-1}$.

41. Let $\mathcal{A} \subset \mathcal{P}(n)$ be a set system such that if $A, B, C \in \mathcal{A}$ are such that $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$ and $C \cap B \neq \emptyset$ then $A \cap B \cap C \neq \emptyset$. Show that $|\mathcal{A}| \leq 2^{n-1} + n$, and this bound can be attained for every n.

42. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ be pairwise incomparable families: if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $A \not\subset B$ and $B \not\subset A$. Show that

$$|\mathcal{A}|^{1/2} + |\mathcal{B}|^{1/2} \leq 2^{n/2}.$$

43. Let $(a_i)_1^n$ and $(b_i)_1^n$ be increasing sequences of reals. Show that

$$n \sum_1^n a_i b_{n+1-i} \leq \sum_1^n a_i \sum_1^n b_i \leq n \sum_1^n a_i b_i.$$

If, in addition, $\mu_1, \ldots, \mu_n$ are non-negative reals with $\sum_1^n \mu_i = 1$, then

$$\sum_1^n \mu_i a_i b_{n+1-i} \leq \left(\sum_1^n \mu_i a_i\right)\left(\sum_1^n \mu_i b_i\right) \leq \sum_1^n \mu_i a_i b_i.$$