Part III Michaelmas, 2022

COMBINATORICS

B. B.

Examples Sheet II.

If an exercise seems to make no sense, correct it and then solve it.

11. Let $2 \le 2k < n$, and let $\mathcal{A} \subset [n]^{(k)} \cup [n]^{(n-k)}$ be an antichain (Sperner system). Set $\mathcal{A}_i = \mathcal{A} \cap [n]^{(i)}$. At most how large is

$$\min \{|\mathcal{A}_k|, |\mathcal{A}_{n-k}|\}?$$

12. Two partitions of $\mathcal{P}(2n)$ are *orthogonal* if no two sets belong to the same chain in both partitions. Show that for $n \ge 1$ there are two orthogonal partitions into $\binom{2n}{n}$ chains each.

13. Let $\mathcal{A} \subset \mathcal{P}(n)$ be an antichain with

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} = 1.$$

Show that $\mathcal{A} = [n]^{(r)}$ for some r, $0 \le r \le n$.

14. Let $\mathcal{A} \subset \mathcal{P}(n)$ be an antichain. Show that $\sum_{A \in \mathcal{A}} |A| \le m\binom{n}{m}$, where $m = \lfloor n/2 \rfloor + 1$.

15. Show that if $\mathcal{F} \subset [n]^{(\le k)}$ is an intersecting antichain then $|\mathcal{F}| \le$
$2k \le n$ $\binom{n-1}{k-1}$.

16. Let X be the disjoint union of sets Y and Z with $|Y|$ and $|Z|$ even. What is the maximal cardinality of a set system $\mathcal{A} \subset \mathcal{P}(X)$ if $A, B \in \mathcal{A}$, $A \ne B$ and $A \subset B$ imply that

$$A \cap Y \ne B \cap Y \quad \text{and} \quad A \cap Z \ne B \cap Z?$$

17. Let $1 < k < n$. What is the maximal cardinality of a family $\mathcal{M}$ of $k \times k$ submatrices of an $n \times n$ matrix, such that any two matrices in $\mathcal{M}$ have a common entry?

18. An $s \times s$ matrix (a_{ij}) whose rows and columns are permutations of a set S with s elements is called a *Latin square of order* s, *based on* S. Two $n \times n$ Latin squares, (a_{ij}) and (b_{ij}), say, are *orthogonal* if the n^2 ordered pairs (a_{ij}, b_{ij}) are all distinct.

(i) Show that for $n \ge 3$ there are at most $n - 1$ orthogonal Latin squares of order n.

(ii) Show that for $n \geq 3$ there is a projective plane of order n if and only if there is a complete set of $n-1$ orthogonal Latin squares of order n.

19. Let $\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathcal{P}(n)$ be a set system such that if $A_i, A_j, A_k \in \mathcal{A}$ satisfy $A_i \cap A_j \neq \emptyset$, $A_j \cap A_k \neq \emptyset$ and $A_k \cap A_i \neq \emptyset$ then $A_i \cap A_j \cap A_k \neq \emptyset$. Show that $|\mathcal{A}| \leq 2^{n-1} + n$, and this bound can be attained for every n.

20. Let $Z_1, \ldots, Z_n$ be i.i.d. Bernoulli random variables with $\mathbb{P}(Z_i = 1) = p \geq 1/2$ and $\mathbb{P}(Z_i = 0) = 1 - p$, and let Z be a convex linear combination of the Z_i. Thus $\mathbb{P}(Z_i = 1) = p$ and $Z = \sum_{i=1}^n c_i Z_i$ with $c_1, \ldots, c_n$ non-negative reals summing to 1. Make use of the Erdős–Ko–Rado theorem to prove that

$$\mathbb{P}(Z \geq 1/2) \geq p.$$

21. Let $\mathcal{F} \subset X^{(r)}$ be such that if A, B and C are three sets in $\mathcal{F}$ then $A \cap B \not\subset C$. Show that $|\mathcal{F}| \leq \binom{r}{\lfloor r/2 \rfloor} + 1$. What bound can you give for the cardinality of $\mathcal{F}$ if we do not assume that it is uniform, i.e. we assume only that $\mathcal{F} \subset \mathcal{P}(n)$?

22. A family $\mathcal{A} \subset \mathcal{P}(n)$ is convex if $A \subset B \subset C$ and $A, C \in \mathcal{A}$ imply that $B \in \mathcal{A}$. Show that if $\mathcal{A} \subset \mathcal{P}(n)$ is convex then

$$\sum_{A \in \mathcal{A}} (-1)^{|A|} \leq \binom{n}{\lfloor n/2 \rfloor}.$$

23. Let $x, x_1, \ldots, x_n$ be positive real numbers. Without any reference to the Littlewood–Offord section, show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the sums $\sum_{i \in A} x_i$ are equal to x.

24. What is the maximal cardinality of a family $\mathcal{A} \subset \mathcal{P}(n)$ without a chain of length five? [Thus $\mathcal{A}$ does not contain five nested sets, $A_1 \subset \cdots \subset A_5$.]

For what values of $n \geq 5$ is the extremal family unique?

25. Let $x_1, \ldots, x_n$ be real number of modulus at least 1, and let $I \subset \mathbb{R}$ be an interval of length ℓ. For at most how many sequences $\varepsilon = (\varepsilon_i)_1^n$, $\varepsilon_i = \pm 1$, do the sums $\sum_1^n \varepsilon_i x_i$ belong to I?

Prove a bound that can be attained whenever $\ell < 2n$.