Part III Michaelmas, 2022

COMBINATORICS

B. B.

Examples Sheet I.

If an exercise seems to make no sense, correct it and then solve it.

1. For $U, V \subset X$ with $|U| = |V|$ and $U \cap V = \emptyset$, define the *UV-compression* of a set $A \subset X$ as follows:

$$C_{UV}(A) = \begin{cases} (A \cup U) \setminus V & \text{if } V \subset A,\ A \cap U = \emptyset, \\ A & \text{otherwise.} \end{cases}$$

Furthermore, for $\mathcal{A} \subset \mathcal{P}(n)$, set

$$C_{UV}(\mathcal{A}) = \{C_{UV}(A) : A \in \mathcal{A}\} \bigcup \{A \in \mathcal{A} : C_{UV}(A) \in \mathcal{A}\}.$$

A family $\mathcal{A} \subset \mathcal{P}(X)$ is *UV-compressed* if $C_{UV}(\mathcal{A}) = \mathcal{A}$.

Let $U, V \subset X$ with $|U| = |V|$ and $\max U < \max V$. Suppose that $\mathcal{A}$ is $U'V'$-compressed for all $|U'| = |V'| < |U| = |V|$ with $\max U' < \max V'$. Show that $\partial C_{UV}(\mathcal{A}) \subset C_{UV}(\partial \mathcal{A})$.

2. Deduce the Kruskal–Katona theorem from the previous exercise.

3. What is the 111^{th} element of $\mathbb{N}^{(4)}$ in the colex order? And the 112^{th} element of the cube Q_{10} in the simplicial order?

4. Is there an order on the reals in which every number has a predecessor and a successor? [In the standard order, a real number has neither a predecessor nor a successor.]
   Hint. Learn from the colex order on $\mathbb{N}^{(<\omega)}$.

5. Let $r \geq 1$ and $\mathcal{A} \subset X^{(r)}$ with $|\mathcal{A}| = \binom{y}{r} > 1$ and $|\partial \mathcal{A}| = \binom{y}{r-1}$. Is it true that y is an integer, and $\mathcal{A} = Y^{(r)}$ for some set $Y \in X^{(y)}$?

6. Let $\mathcal{L} \subset X^{(r)}$ be a family of r-sets such that any two of them intersect in exactly one element, but there is no element in all. Show that $|\mathcal{L}| \leq r^2 - r + 1$. For what families do we have equality?

7. Let $L_1, \ldots, L_m \subset X$, $|X| = n$, be such that $|L_i \cap L_j| = 1$ for all i, j with $1 \leq i < j \leq m$. Is there a set $Y \subset X$ such that $1 \leq |L_i \cap Y| \leq 2$ for every i?

8. Let $L_1, L_2, \ldots$ be an infinite sequence of infinite sets such that any two intersect in exactly one element. Is there a set Y such that $1 \leq |L_i \cap Y| \leq 2$ for every i?

9. Let $1 < k < r < n$ and let $A_1, \ldots, A_m \in S^{(r)}$ be such that $|A_i \cap A_j| \leq k$ for all i, j, $i \neq j$. Show that $|S| \geq r^2m/(k(r-1)+r)$.

Show also that equality holds for infinitely many systems with k tending to infinity.

10. A family of sets is a Δ-system, if any two elements of it intersect in the same set. Suppose $\mathcal{A} \subset X^{(r)}$ does not contain a Δ-system of size m. Show that $|\mathcal{A}| \leq (m-1)^r \times r!$

Classical Isoperimetric Inequalities

Do not get bogged down in approximations: in your measure theory arguments you are allowed to be cavalier.

1. The (Minkowski) sum of two sets $K, L \subset \mathbb{R}^n$ is

$$K + L = \{x + y : \; x \in K, \; y \in L\}.$$

Show that if K and L are finite unions of unit cubes with edges parallel with the axes, then

$$\left(\mathrm{vol}_n(K + L)\right)^{1/n} \geq \left(\mathrm{vol}_n(K)\right)^{1/n} + \left(\mathrm{vol}_n(L)\right)^{1/n}. \tag{1}$$

[By trivial approximations, (4) holds for compact sets, bounded open sets, bounded half-closed sets, if the Lebesgue measure is replaced by the outer measure, etc.]

2. Let A be a non-empty subset of $\mathbb{R}^n$. Show that for $\varepsilon > 0$ the ε-neighbourhood of A,

$$A_{(\varepsilon)} = \{x \in A : \; d(x, A) < \varepsilon\},$$

is Lebesgue measurable.

3. Let A be a bounded open set in $\mathbb{R}^n$. Given a hyperplane (i.e. 1-codimensional subspace) H in $\mathbb{R}^n$ and a line ℓ perpendicular to H, let $\ell(A)$ be the open segment on ℓ that is symmetric about H and whose length is the measure ('length') of $A \cap \ell$. Define

$$H(A) = \bigcup \{\ell(A) : \; \ell \; \text{ is perpendicular to } \; H\},$$

so that $H(A)$ is an open set in $\mathbb{R}^n$ symmetric about H, with $\mathrm{vol}_n(H(A) = \mathrm{vol}_n A$. Show that for $\varepsilon > 0$ we have

$$\mathrm{vol}_n(H(A)_{(\varepsilon)}) \leq \mathrm{vol}_n(A_{(\varepsilon)}).$$

4. Let $A \subset \mathbb{R}^n$ be an open set of measure m and let $B \subset \mathbb{R}^n$ be a Euclidean ball of volume m. Then for $\varepsilon > 0$ we have

$$\mathrm{vol}_n(A_{(\varepsilon)}) \geq \mathrm{vol}_n(B_{(\varepsilon)}). \tag{2}$$

5. Let $K \subset \mathbb{R}^n$ be a bounded convex set with surface measure $s(K) = \mathrm{vol}_{n-1}(\partial K)$. Show that for $\varepsilon > 0$ we have

$$\mathrm{vol}_n(K_{(\varepsilon)}) - \mathrm{vol}_n(K) = s(K)\varepsilon + O(\varepsilon^2)$$

as $\varepsilon \to 0$.

Be more precise for $n = 2$.

6. Note that the previous two exercises imply the usual form of the isoperimetric inequality in $\mathbb{R}^n$ in terms of the volume of a body and the surface measure of its boundary.

7. Given a bounded convex set $K \subset \mathbb{R}^n$ with non-empty interior, write $K_h = \{(x_i)_1^n \in K : x_n = h\}$, and set $f(h) = \left(\mathrm{vol}_{n-1}(K_h)\right)^{1/(n-1)}$. Show that $f(h)$ is concave on its support.

8. Fix $\mathrm{v} \in S^2$. Given $\mathrm{s} \in S^2$ not orthogonal to v, define $\mathrm{s}_-, \mathrm{s}_+ \in S^2$ such that $\mathrm{s}_+ - \mathrm{s}_- = \lambda\mathrm{v}$ for some $\lambda > 0$ and $\mathrm{s} \in \{\mathrm{s}_+, \mathrm{s}_-\}$. [Thus either $\mathrm{s}_- = s$ and $\mathrm{s}_+ = \mathrm{s} + \lambda\mathrm{v}$ or $\mathrm{s}_+ = \mathrm{s}$ and $\mathrm{s}_- = s - \lambda\mathrm{v}$.] For $\mathrm{s} \in S^2$, $\mathrm{s} \perp \mathrm{v}$ set $\mathrm{s}_+ = \mathrm{s}_- = \mathrm{s}$. Finally, define the compression of s as $C_{\mathrm{v}}(\mathrm{s}) = \mathrm{s}_+$, and the compression of a set A, $\emptyset \neq A \subset S^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$ be open in S^2, as

$$C_{\mathrm{v}}(A) = \{C_{\mathrm{v}}(a) : a \in A\} \cup \{a \in A : C_{\mathrm{v}}(a) \in A\}.$$

Show that, as subsets of S^2,

$$\mathrm{vol}_2\left(C_{\mathrm{v}}(A)\right) = \mathrm{vol}_2(A) \text{ and } \mathrm{vol}_2\left((C_{\mathrm{v}}(A))_{(\varepsilon)}\right) \leq \mathrm{vol}_2(A_{(\varepsilon)}).$$

9. Deduce the isoperimetric inequality on the sphere S^2 from the previous exercise, namely that spherical caps are best.