The Eight Types of Algebraic Surfaces

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Abstract: We proceed from the end of Part II Algebraic Geometry and first aim to prove Riemann-Roch. Sheaf theory is developed from scratch and we show there is a natural correspondence between divisors and invertible sheaves. We then define sheaf cohomology and use it to prove Riemann-Roch for curves. In the last section, we study cohomology on surfaces, leading to the Riemann-Roch theorem for surfaces. With the hard work done, we sit back and survey the classification of minimal complex algebraic surfaces.

Contents

1 Introduction 2
2 Preliminaries 3
  2.1 Categories 3
  2.2 Exact sequences of abelian groups 4
  2.3 Tensor products 5
  2.4 Varieties 6
3 Sheaf Theory 8
  3.1 Presheaves 8
  3.2 Sheaves 9
4 Sheaves of modules 14
  4.1 Quasi-coherent sheaves 16
  4.2 Tensor products of $\mathcal{O}_X$ modules 20
5 Divisors 21
  5.1 Divisor-Line bundle correspondence 21
  5.2 Divisors on curves 25
  5.3 The Cousin problem 29
6 Sheaf cohomology 31
  6.1 Definition 31
  6.2 Flabby sheaves 32
  6.3 Direct and inverse images 33
  6.4 Cohomology of curves 36

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6.5 Classification of curves 44
7 Surfaces 46
7.1 Overview 46
7.2 Riemann-Roch for surfaces 48
7.3 Birational invariants 55
7.4 Ruled surfaces 58
7.5 Surfaces with Kodaira dimension one 60
7.6 Surfaces with Kodaira dimension zero 61
7.7 Compact complex surfaces 64
References 66
Index 67

1. Introduction

We proceed from the end of the Part II Algebraic Geometry course and aim to prove the Riemann-Roch theorem for curves:

Riemann Roch: If X is a projective smooth curve over an algebraically closed field k, then for any divisor D

$$l(D) - l(K - D) = \text{deg}D + 1 - g$$

where $l(D) = \dim_k\{f : \text{div}(f) + D \text{ effective}\}$, K is the canonical divisor, and $g := l(K)$ is the genus.

The first step is to recast our current understanding of algebraic geometry in terms of 'sheaves', which are not specific to geometry and interesting in their own right. In Section 5, we see the true face of divisors swept under the rug in II Algebraic Geometry, which are the simplest class of sheaves acting on a variety. In Section 6, we see a new algebraic concept 'cohomology' which is exclusive to sheaves and apply it to prove Riemann-Roch for curves, which leads to the classification of curves.

Though Riemann-Roch for curves is an important result, it can be proven by elementary means, see Section 8 in [14]. In Section 7, we see Riemann-Roch for surfaces, which is a different beast altogether and proving it is widely considered intractable without the consideration of cohomology (discussed in [15]). Next, we change gears to a 'bedtime-read' style and survey the classification of complex algebraic surfaces, proving only the very easy parts. The relevant citations are provided for the purist, which are approachable given the tools developed in this article.

Sections 3 to 6 largely follow the text of Kempf [1] and this article serves as a 'reader's guide', including extra details in the proofs. The proof of Riemann-Roch for curves presented in [1] is Kempf's original proof (see [17]), which was of course not the first proof but still an innovation at the time. A memoriam to Kempf by David Mumford is found in [16]. Section 7 follows the texts of Beauville [3] and Barth, Peters, Van de ven [10].

We assume the reader is familiar with the material taught in the Part II Algebraic Geometry course at the University of Cambridge. The reader is referred to the notes of Scholl [20] or the condensed notes of Wilson [19]

Natural next steps after reading this article would be the study of schemes, which this article avoids, see the classical text of Hartshorne [2]. Many applications of cohomology are also discussed in [1]. More can be said about compact complex surfaces, which are the subject of [10].

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2. Preliminaries

2.1. Categories

A *category* is roughly a collection of *objects* with *arrows* between them. For our purposes, object means set and arrow means function (also called *morphisms* or *maps*). Key examples include:

- The category of sets **Set**. Every function between two sets is a morphism.
- The category of groups **Grp**. The morphisms are the group homomorphisms.
- The category of abelian groups **Ab**. The morphisms are the group homomorphisms.
- The category of topological spaces **Top**. The morphisms are continuous maps.
- The category of rings **Ring**. The morphisms are the ring homomorphisms.

The nice kind of map between categories is the *functor*. A functor F from $\mathbf{C}$ to $\mathbf{D}$ is a map that

(i) maps objects in **C** to objects in **D**

(ii) maps each morphism $f : X \to Y$ in **C** to a morphism $F(f) : F(X) \to F(Y)$ in **D**

such that:

(iii) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ (preserves identity morphisms)

(iv) $F(f \circ g) = F(f) \circ F(g)$ (respects composition)

Key examples include:

- The 'forgetful functor' **Ab** $\to$ **Set** which maps an abelian group to its underlying set and the morphisms map in the natural way. Similarly there is a forgetful functor **Ring** $\to$ **Ab**.
- The functor **Set** $\to$ **Grp** mapping a set to the free group on that set. A set functions maps to a group homomorphism by acting on each letter in a given string.

The next simplest type of map between categories is the *contravariant functor*, a map which reverses arrows. A contravariant functor F from C to D is a map that

(i) maps objects in **C** to objects in **D**

(ii) maps each morphism $f : X \to Y$ in C to a morphism $F(f) : F(Y) \to F(X)$ in **D**.

such that:

(iii) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ (preserves identity morphisms)

(iv) $F(f \circ g) = F(g) \circ F(f)$ (flips composition)

We will see later that 'sheaves' are an examples of a contravariant functors and the reversing of arrows is exactly what we want.

2.2. Exact sequences of abelian groups

An exact sequence is a compact form of notation for a common occurrence in algebra. Suppose B is an abelian group with subgroup A. We have the (injective) inclusion map $\iota : A \to B$ and the surjective map $q : B \to B/A$ with kernel A. This data is written compactly as,

$$0 \to A \xrightarrow{\iota} B \xrightarrow{q} B/A \to 0 \qquad (*)$$

where the first and final maps are trivial (one is inclusion, the other maps everything to the identity) and their purpose will be illuminated shortly. Thus we define:

Definition 2.1. A *short exact sequence of abelian groups* is a sequence of the form

$$0 \to A \to B \to C \to 0$$

where the map $A \to B$ is injective and the map $B \to C$ is surjective with kernel (the image of) A.

It is useful to think of any given short exact sequence as subgroup and a quotient, as in $(*)$. Another way to phrase this definition would be 'each map has kernel equal to the image of the previous map'. Thence we see how to generalise this definition:

Definition 2.2. An *exact sequence of abelian groups* is a countable collection of groups $(G_n)_n$ and group homomorphisms $(f_n)_n$ such that $f_n : G_n \to G_{n+1}$ and $\mathrm{im}(f_n) = \ker(f_{n+1})$ for all n. When the collection is finite, we write

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \dots \xrightarrow{f_{N-1}} G_N.$$

This definition generalises to categories other than abelian groups by replacing 'group homomorphism' with morphisms of the relevant category. However, it doesn't make sense for every category since we need $\ker f_{n+1}$ to make sense.

Example 2.3. Let A and B be abelian groups. Saying the sequence

$$0 \to A \to B$$

is exact is equivalent to saying A embeds into B. Saying the sequence

$$A \to B \to 0$$

is exact is equivalent to saying there is a surjective map from A to B. These are common shorthands.

2.3. Tensor products

Tensor products will be a useful construction for us later, but we won't use them in any difficult way. One interesting application will be turning a set of modules into a group with tensor product as the group law.

The definition should be reminiscent of usual multiplication:

Definition 2.4. Let R be a ring with R-modules M and N. The *tensor product over* R, denoted $M \otimes_R N$, is the R-module,

$$\{m \otimes_R n : m \in M, n \in N\}$$

with the relations

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$
$$(rm_1) \otimes n = r(m_1 \otimes n)$$

and vice versa in the other component, for $m_1, m_2 \in M$, $n \in N$, $r \in R$.

We drop the subscript when the context is clear.

Proposition 2.5. For R-modules M, N, and P:

- $M \otimes N \approx N \otimes M$ (symmetry)
- $M \otimes (N \oplus P) \approx (M \otimes P) \oplus (M \otimes N)$ (distributive law)

Proof. Exercise.

□

Definition 2.6. Let M be an R-module

- For k a positive integer, we define $M^{\otimes k} = \underbrace{M \otimes \cdots \otimes M}_{k \text{ times}}$.
- Define $\tau_{ij} : M^{\otimes k} \to M^{\otimes k}$ by swapping the ith and jth components.
- The *exterior product* $\wedge^k M$ is $M^{\otimes k}$ with the additional relations:

$$v = -\tau_{ij} v$$

for all $v \in M^{\otimes k}$ and $1 \leq i < j \leq k$.

Thus the exterior product contains the anti-symmetric tensors.

2.4. Varieties

The definitions of a 'variety' vary amongst II Algebraic Geometry lecturers. We briefly establish our convention here, with respect to a fixed algebraically closed field k.

Definition 2.7.

- An *affine variety* is the topological space given by any closed set of $\mathbb{A}^n$ equipped with the Zariski topology (for any n).
- A *quasi-affine variety* is the topological space given by any open subset of an affine variety (with the subspace topology)
- A *projective variety* is the topological space given by any closed set of $\mathbb{A}^n$ equipped with the Zariski topology (for any n).
- A *quasi-projective variety* is the topological space given by any open subset of a projective variety (with the subspace topology)
- For a polynomial f, define the *vanishing set* $V(f) := f^{-1}(0)$.

Remark 2.8.

- Zariski closed sets are vanishing sets of polynomials, thus we often do not distinguish between the topological space and the corresponding vanishing set embedded in $\mathbb{A}^n$.
- We don't require varieties to be irreducible
- Every type of variety defined above is a quasi-projective variety (exercise), thus we say *variety* to refer to a quasi-projective variety.

Definition 2.9. For X and Y varieties, a *morphism* is a map $\phi : X \to Y$ which is locally given by polynomials

With these morphisms, the varieties over the field k form a category.

Definition 2.10.

- A *rational function on* $\mathbb{A}^n$ is a map of the form $x \to p(x)/q(x)$ for $p, q \in k[X_1, \ldots, X_n]$.
- A *rational function on* $\mathbb{P}^n$ is a map $x \to p(x)/q(x)$ with $p, q \in k[X_1, \ldots, X_n]$ homogeneous and $\deg p = \deg q$.
- A *regular function* on a variety X is a map $X \to k$ which is locally given by rational functions. The set of all regular functions on X is denoted $k[X]$, or $\mathcal{O}_X$. Later, $\mathcal{O}_X$ will mean something related but different.

Fact 2.11. Let X be a smooth curve. Then for every $x \in X$, the ring $\mathcal{O}_{X,x}$ is local. A generator of the maximal ideal is called a *local parameter*.

3. Sheaf Theory

3.1. Presheaves

Let X be a topological space.

Definition 3.1. The category Open(X) consists of the open sets of X with inclusion morphisms (i.e. for every $U \subset V$ we have a morphism $\iota : U \to V$)

Note that this differs greatly from the category **Top** defined previously.

Definition 3.2. A *presheaf of abelian groups* F on a topological space X is a contravariant functor $F : \text{Open}(X) \to \mathbf{Ab}$.

In words, a presheaf assigns an abelian group to each open set of X with the extra condition that 'U is a subset of V' becomes '$F(V)$ subgroup of $F(U)$ ' in the image. This condition might seem backwards, which comes from the 'contravariant' part. We'll see why this is what we want in an example soon.

Definition 3.3. If $U \subset V$ are open with inclusion map $\iota : U \to V$ then

$$\text{res}_U^V := F(\iota)$$

is the *restriction map*, a group homomorphism $F(V) \to F(U)$.

The usual rules of a contravariant functor apply:

$$\text{res}_U^U \text{ is the identity map.} \tag{3.1}$$

$$\text{if } U \subset V \subset W \text{ then } \text{res}_U^W = \text{res}_U^V \circ \text{res}_V^W \tag{3.2}$$

These equations will be stated in much simpler notation shortly. Though the definition seems daunting, restriction maps are never an obstruction and always line up with restriction in the function sense.

Example 3.4. Let X a variety and consider the presheaf F given by the following data:

(i) For $U \subset X$ open, $F(U)$ is the set of regular functions on U (an abelian group under addition).

(ii) For $U \subset V$ open, the restriction map res_U^V is given by restricting functions in the usual sense.

Remark 3.5.

- Most presheaves arise in a similar way to Example 3.4 – for U open, $F(U)$ is the set of functions on U satisfying some property.

- This example also illuminates why presheaves were defined as 'contravariant' functors: for $U \subset V$ open sets and a presheaf F which maps open sets to functions, we can naturally map $F(V)$ into $F(U)$ by restriction (but it would be much less obvious how to map $F(U)$ into $F(V)$).
- In this article, the restriction map will always be obvious in any example and we often don't say what it is when defining a presheaf. Thus we will frequently define presheaves by saying 'F is the presheaf defined by $U \to \{f : f \text{ regular on } U\}$ for U open.'

Definition 3.6.

- If F is a presheaf and U open, a *section of F over U* is an element $f \in F(U)$. It is custom to drop "of F" or "over U" when the context is clear or irrelevant.
- If $U \subset V$ and f is a section over V, then $f|_U := \text{res}_U^V f$.

In this notation, equations 3.1 and 3.2 become

$$f|_U = f \text{ when } f \text{ is a section over } U \tag{3.3}$$

$$(f|_V)|_U = f|_U \text{ when } f \text{ is a section over } W\text{, where } U \subset V \subset W \tag{3.4}$$

respectively.

3.2. Sheaves

In Algebraic Geometry, it is often of interest to study local behaviour of functions. In this section we will define sheaves, which are presheaves with extra conditions to capture local behaviour.

Definition 3.7. Let F be a presheaf on X and $x \in X$. If f is a section over U and g is a section over V, we say f and g are *germ equivalent* at x, written $f \sim_x g$ if there exists a non-empty open $W \subset U \cap V$ such that $f|_W = g|_W$.

The *stalk* of F at x, written F_x, is the set of sections defined on a neighbourhood of x up to germ equivalence,

$$F_x = \{f : f \in F(U) \text{ for some open } U\}/ \sim_x$$

If f is a section of F over some neighbourhood of x, then the quotient projection naturally defines an element f_x in F_x.

A natural first question is whether a section is determined by its local behaviour. That is, can there be two different sections f and g over U such that $f_x = g_x$ for all $x \in U$? The broad definition of a presheaf does not prevent this.

Definition 3.8. A presheaf F is *decent* if any section of F is determined by its local behaviour, i.e. if f and g are sections over U such that $f_x = g_x$ for all $x \in U$, then $f = g$.

Many results in Algebraic Geometry are described as 'local to global'; we're given local information, i.e. on stalks, and want to deduce global information, i.e. on open sets.

Proposition 3.9. If f and g are sections of F over U, then $f_x = g_x$ for all $x \in U$ if and only if there is an open cover $U = \bigcup U_\alpha$ such that $f|_{U_\alpha} = g|_{U_\alpha}$.

Proof. If $f_x = g_x$, then $f|_W = g|_W$ for some neighbourhood W of x, thus giving an open cover. The converse is 'global to local', so straightforward indeed. □

Given a section f over U and an open cover $U = \bigcup U_\alpha$, let $f_\alpha = f|_{U_\alpha}$. The set $\{f_\alpha : \alpha\}$ is not arbitrary. It satisfies the *patching condition*:

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta} \quad \forall \alpha, \beta \tag{3.5}$$

It is a common and desirable property that 'nice' functions that agree where they are both defined can be glued together to give another 'nice' function defined on both domains. Indeed, the functions we study often have this property e.g. morphisms of varieties. Thus we make the most important definition of today,

Definition 3.10. A *sheaf of abelian groups* F on X is a presheaf of abelian groups with the properties:

(3.6) Given any open U, open cover $U = \bigcup U_\alpha$, and sections f_α of F over U_α for all α such that the set $\{f_\alpha : \alpha\}$ has the patching condition (3.5), there exists a section f of F over U such that $f|_{U_\alpha} = f_\alpha$ for all α.

(3.7) F is decent, or equivalently (by Proposition 3.9) the section obtained in (i) is unique.

Key examples include:

Example 3.11.

(i) If X is a variety, *the sheaf of regular functions on* X defined by $U \to \{f : U \to k \mid f \text{ regular on } U\}$, is denoted $\mathcal{O}_X$. In this case, (3.6) and (3.7) are trivial as two regular functions which agree on an open set agree everywhere they are defined.

(ii) Let X and Y be varieties. Then $U \to \{f : U \to Y \mid f \text{ is a morphism}\}$ is a sheaf on X.

The functions we deal with often form a ring or module, not just an abelian group, thus we ought to extend our definition of (pre)sheaves to make use of this information. In fact, we can define a (pre)sheaf of any category:

Definition 3.12. Given a category $\mathbf{C}$, a *presheaf of* $\mathbf{C}$ *on* X is a contravariant functor $\mathrm{Open}(X) \to \mathbf{C}$. A presheaf F is a *sheaf of* $\mathbf{C}$ *on* X if it satisfies conditions (3.6) and (3.7).

Example 3.13. For a variety X, $\mathcal{O}_X$ is a sheaf of rings. Given $x \in X$, the stalk $\mathcal{O}_{X,x}$ is naturally a ring, identifying functions which agree near x. This is the sheaf theoretic definition of the local ring seen in II Algebraic Geometry.

Remark 3.14. Although sections of sheaves are almost always functions in practice, it is better to view functions as sections rather than view sections as functions. The key results we develop in this article come from abandoning the pointwise interpretation of functions and only comparing them on open sets. A stalk can be thought of as seeing the action of a function on a limit of smaller and smaller neighbourhoods.

Definition 3.15. A *morphism* α of (pre)sheaves F and G of the category $\mathbf{C}$ is a collection of morphisms of $\mathbf{C}$, $\alpha(U) : F(U) \to G(U)$ for each open U, such that $\alpha(U)$ commutes with the restriction maps: if $U \subset V$ open and f is a section of F over V, then

$$\alpha(U)(f|_U) = [\alpha(V)(f)]|_U. \tag{3.8}$$

Often we abbreviate $\alpha(V)(f)$ as $\alpha(f)$, thus (3.8) becomes

$$\alpha(f|_U) = \alpha(f)|_U. \tag{3.9}$$

With these morphisms, the sheaves of $\mathbf{C}$ on X form a category, denoted $\mathbf{Sh}(\mathbf{C})(X)$.

Definition 3.16. Let F and G be sheaves and $\alpha : F \to G$ a morphism

- α is *surjective* if for all U open, the map $\alpha(U)$ is surjective.
- α is *injective* if for all U open, the map $\alpha(U)$ is injective.
- α is an *isomorphism* if it is injective and surjective.

We establish our first local to global result on sheaves:

Proposition 3.17. (Local to global for isomorphisms) Let F and G be sheaves. A morphism $\alpha : F \to G$ is an isomorphism if and only if the induced map on stalks $\alpha_x : F \to G$ is an isomorphism for all $x \in X$

Proof. The forwards direction is clear, so assume α_x is an isomorphism for all x.

α is injective If $\alpha(f) = \alpha(g)$, then $\alpha_x(f) = \alpha_x(g)$ for all x, so by injectivity, for every x there is some neighbourhood U_x with $f|_{U_x} = g|_{U_x}$. Since sections of a sheaf are determined by their local behaviour, we have $f = g$.

α is surjective Let g be a section of $G(U)$. For all $x \in U$, there are sections $f_x \in F_x$ such that $\alpha_x(f_x) = g_x$. Choose a section $f(x)$ of some open set $V(x)$ such that $f(x)_x = f_x$. Then $\alpha(f(x))_x = g_x$, so by shrinking $V(x)$ we may assume $\alpha(f(x))|_{V(x)} = g|_{V(x)}$. Thus for all $x, y \in U$,

$$\alpha(f(x)|_{V(x)\cap V(y)}) = g|_{V(x)\cap V(y)} = \alpha(f(y)|_{V(x)\cap V(y)})$$

so by injectivity, $f(x)|_{V(x)\cap V(y)} = f(y)|_{V(x)\cap V(y)}$. Therefore since F is a sheaf, $\{f(x) : x \in U\}$ can be patched to give a section $f \in F(U)$ and we have $\alpha(f) = g$. $\square$

Remark 3.18.

- The proof only required G to be a presheaf, but we will not concern ourselves with presheaves much longer.
- This proof does not say that two sheaves are isomorphic if all their stalks are isomorphic. It tells us that we don't need to check that the inverse isomorphisms of stalks glue together to give an inverse morphism of sheaves.

Definition 3.19. We say F *is a subsheaf of* G or write $F \subset G$ if there is an injective morphism from F to G, referred to as the *inclusion map*.

Example 3.20. If F is a sheaf of category $\mathbf{C}$ on X and $U \subset X$ open then the *restriction of* F *to* U is the sheaf denoted $F|_U$ and defined by

$$V \to F(U \cap V)$$

for $V \subset X$ open. There is an injective morphism $\iota : F \to F|_U$ given by $\iota(V)(f) = f|_U$, thus F is a subsheaf of $F|_U$.

Example 3.21. If $\alpha : F \to G$ is a morphism of sheaves of abelian groups, we define the *kernel of* α, denoted $\ker(\alpha)$, as the sheaf defined by

$$U \to \alpha(U)^{-1}(0)$$

In words, U maps to the sections of U which map to the identity under α. The reader should convince themselves that $\ker(\alpha)$ is indeed a sheaf of abelian groups and moreover a subsheaf of F. When F is a ring, $\ker(\alpha)$ is a sheaf of ideals.

Definition 3.22. A sequence of sheaves of abelian groups on X

$$F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} \dots \xrightarrow{\zeta} F_n$$

is *exact* if the corresponding sequence of abelian groups with the induced stalk maps

$$(F_1)_x \xrightarrow{\alpha_x} (F_2)_x \xrightarrow{\beta_x} \dots \xrightarrow{\zeta_x} (F_n)_x$$

is exact for all $x \in X$.

Remark 3.23.

- The stalk of a sheaf of abelian groups is naturally an abelian group, to which this definitions refers. Similarly, a stalk of a sheaf of rings is naturally a ring. However, this wouldn't make sense for an arbitrary category, but it does generalise to so-called 'abelian categories'.
- In light of the previous remark, an *exact sequence of sheaves of rings* is defined by replacing 'abelian group' with 'ring' in Definition 3.22.
- These definitions extend in the obvious way if the sequence is countably infinite.

Unfortunately, the world is not perfect and not every presheaf we are interested in is a sheaf. There is a natural construction to get around this, which involves identifying sections which agree locally and dding in glued-together sections – though we won't get into the details here. Moreover, whenever we are required to apply this procedure in this article, the reader is encouraged to ignore it and pretend we already had a sheaf.

Definition 3.24. For a presheaf F, a *sheafification* of F, is a sheaf of the same category $F^{\#}$ and a presheaf morphism $\phi : F \to F^{\#}$ such that

(i) any sheaf morphism $\alpha : F \to G$ can be written in the form

$$\alpha = \alpha^{\#} \circ \phi$$

for some unique sheaf morphism $\alpha^{\#} : F^{\#} \to G$.

(ii) for all $x \in X$, the induced map on stalks $\phi_x : F_x \to F_x^{\#}$ is an isomorphism

Remark 3.25. The reader should reconcile (ii) with Proposition 3.17, also considering the case when F is a sheaf.

Example 3.26. Let F and G be sheaves.

- The *tensor product sheaf* over the ring R, denoted $F \otimes_R G$, is the sheafification of the presheaf defined by

$$U \to F(U) \otimes_R G(U)$$

- Let $\alpha : F \to G$. The *image sheaf* $\mathrm{im}(\alpha)$ is the sheafification of the presheaf defined by

$$U \to \alpha(F(U))$$

- Let G be a subsheaf of F over a suitable category. The *quotient sheaf* is defined by the sheafification of the presheaf defined by

$$U \to F(U)/G(U)$$

4. Sheaves of modules

For a ring R, the category R-modules exists and Definition 3.12 applies. However, this would not be a very useful notion of a 'sheaf of modules' because R is fixed over the entire domain. We would like R to represent functions, thus should depend on the open set in question.

Definition 4.1. Given a sheaf of rings $\mathcal{R}$, a *sheaf of* $\mathcal{R}$*-modules* $\mathcal{M}$ is a sheaf of abelian groups on X such that

- $\mathcal{M}(U)$ is a $\mathcal{R}(U)$-module for all U open
- $(a \cdot m)|_V = a|_V \cdot m|_V$ for all $V \subset U$ open, $a \in \mathcal{R}(U)$, $m \in \mathcal{M}(U)$. I.e. restriction is a module homomorphism.

Remark 4.2. Definition 3.22 generalises in the obvious way to sheaves of $\mathcal{R}$-modules.

The simplest example is given by the following:

Example 4.3. Let $\mathcal{R}$ be a sheaf of rings.

- Let I be a set, the *direct sum of* $\mathcal{R}$ *by* I, denoted $\mathcal{R}^I$, is the sheaf of $\mathcal{R}$-modules, defined by
$$U \to \bigoplus_I \mathcal{R}(U)$$
for $U \subset X$ open. We will only ever concern ourselves with direct sums and not Cartesian products, so there will be no ambiguity in notation.

- Let $\alpha : \mathcal{R} \to \mathcal{S}$ be a sheaf morphism then $\ker \alpha$ (see Example 3.21) is a sheaf of $\mathcal{R}$-modules (naturally, as ideals are $\mathcal{R}$-modules).

The following definition provides further examples, which will be useful later:

Definition 4.4. Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves of $\mathcal{R}$-modules.

- The *sheaf tensor product* $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$ is the sheafification of the presheaf defined by $U \to \mathcal{M}(U) \otimes_{\mathcal{R}(U)} \mathcal{N}(U)$.

- For $n \in \mathbb{N}$, the *sheaf exterior product* $\wedge^n \mathcal{M}$ is the sheafification of the presheaf defined by $U \to \wedge^n \mathcal{M}(U)$.

Recall that if R is a ring, an R-module M is free if $M = \bigoplus_I R$ for some set I and $|I|$ is the rank of M. These notions generalise to sheaves of modules.

Definition 4.5. Let $\mathcal{R}$ be a sheaf of rings and $\mathcal{M}$ a sheaf of $\mathcal{R}$-modules.

- $\mathcal{M}$ is *locally free* if every $x \in X$ has an open neighbourhood U such that $\mathcal{M}(U)$ is free.

- $\mathcal{M}$ is *rank* n if every $x \in X$ has an open neighbourhood U such that $\mathcal{M}(U)$ is free with finite rank n. In the case $n = 1$ we say $\mathcal{M}$ is *invertible* or a *line bundle*.

Remark 4.6. The most interesting sheaves of modules are sheaves of $\mathcal{O}_X$-modules. In fact, for an arbitrary topological space X and an *arbitrary* sheaf of rings $\mathcal{O}_X$ called the 'structure sheaf of X', one can study the so-called 'ringed space' $(X, \mathcal{O}_X)$. The remainder of this article could be summarised as the study of ringed spaces when X is a variety and the structure sheaf is the sheaf of regular functions. The question of what happens when X is not a variety leads to the study of 'schemes'.

Example 4.7. Let π be the projection from $\mathbb{A}^{n+1}\backslash\{0\}$ to $\mathbb{P}^n$. Define $\mathcal{O}_{\mathbb{P}^n}(m)$, a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules, by

$$U \to \{f \in \mathcal{O}_{\pi^{-1}U} : f \text{ homogeneous with degree} \geq m\}$$

Thus,

- $\mathcal{O}_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(0)$
- For $m_1, m_2 \in \mathbb{Z}$, there is a natural isomorphism

$$\mathcal{O}_{\mathbb{P}^n}(m_1) \otimes \mathcal{O}_{\mathbb{P}^n}(m_2) \to \mathcal{O}_{\mathbb{P}^n}(m_1 + m_2) \tag{4.1}$$

- Let X_i be a coordinate variable. Then

$$\mathcal{O}_{\mathbb{P}^n}(m)|_{X_i \neq 0} = X_i^m \cdot \mathcal{O}_{\mathbb{P}^n}$$

  Hence $\mathcal{O}_{\mathbb{P}^n}(m)$ is invertible. Later we will see all invertible sheaves of $\mathcal{O}_X$-modules on $\mathbb{P}^n$ are isomorphic to $\mathcal{O}_{\mathbb{P}^n}(m)$ for some m.

Inspired by (4.1), we define

Definition 4.8. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules and $m \in \mathbb{Z}$, then

$$\mathcal{M}(m) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$$

is a sheaf of $\mathcal{O}_X$ modules.

Definition 4.9. If $\mathcal{M}$ is locally free of rank n, we define the *determinant sheaf* $\det \mathcal{M} := \wedge^n \mathcal{M}$.

4.1. Quasi-coherent sheaves

Locally free is usually too much to ask for. Instead, we often have a local generating set with linear relations between them.

Definition 4.10. Let $\mathcal{R}$ be a sheaf of rings and $\mathcal{M}$ a sheaf of $\mathcal{R}$-modules. A *presentation* of $\mathcal{M}$ is an exact sequence of sheaves

$$\mathcal{R}^I \to \mathcal{R}^J \to \mathcal{M} \to 0$$

If J is finite, we say $\mathcal{M}$ is *finitely generated*. If both I and J are finite, we say $\mathcal{M}$ is *finitely presented*

For the reader who is new to exact sequences, the following example gives an illustration of Definition 4.10:

Example 4.11. We consider the simplest case of Definition 4.10 when the sheaves are constant: we have a ring R, an R-module M, and an exact sequence of R-modules:

$$R^I \to R^J \to M \to 0 \tag{4.2}$$

Suppose $|J| = n$ is finite and let $(e_j)_1^n$ be a basis of R^J. Equation 4.2 tells us there is a surjective map from R^J to M, thus we can identify the $(e_j)_1^n$ with a spanning set for M.

Let $(f_i)_I$ be a basis of R^I with images under the first map

$$a_{1,i}e_1 + \cdots + a_{n,i}e_n, \quad i \in I.$$

Equation 4.2 tells us these expressions generate the kernel of the second map. We can view this statement as telling us precisely what the relations between the elements of the spanning set $(e_j)_1^n$ are:

$$a_{1,i}e_1 + \cdots + a_{n,i}e_n = 0, \quad \forall i \in I$$

Altogether, J is the number of generators and I is the number of relations between them.

Definition 4.12. A sheaf of $\mathcal{R}$-modules $\mathcal{M}$ is *quasi-coherent* if any $x \in X$ has an open neighbourhood U such that there is an exact sequence of sheaves of $R|_U$-modules

$$\mathcal{R}^I|_U \to \mathcal{R}^J|_U \to \mathcal{M}|_U \to 0$$

for some sets I and J. We say $\mathcal{M}$ is *coherent* if I and J can be taken to be finite for all x.

In light of Example 4.11, we informally say that $\mathcal{M}$ is quasi-coherent if it is locally given by generators with linear relations. While this definition is easy to picture, it is surprisingly difficult to prove things with. For the remainder of this section, we work towards a more productive characterisation of quasi-coherent sheaves.

Definition 4.13. For a sheaf F, we define $\Gamma(X, F) := F(X)$. The map $F \to \Gamma(X, F)$ is called the *global sections functor on* X.

Example 4.14. For a variety X, we have $\Gamma(X, \mathcal{O}_X) = k[X]$, the space of regular functions on X (aka the 'coordinate ring').

Definition 4.15. Let M be a $\mathcal{R}(X)$-module. Define $M \otimes_{\mathcal{R}(X)} \mathcal{R}$ to be the sheafification of the presheaf of $\mathcal{R}$-modules

$$U \to M \otimes_{\mathcal{R}(X)} \mathcal{R}(U)$$

Our next fact says that quasi-coherent modules are precisely the sheaves of modules which are locally of this form.

Fact 4.16. Let $\mathcal{M}$ be a sheaf of $\mathcal{R}$-modules on a variety X. The following are equivalent:

(i) $\mathcal{M}$ is quasi-coherent

(ii) for all x, there is a neighbourhood U and a $\mathcal{R}(U)$-module M such that

$$\mathcal{M}|_U = M \otimes_{\mathcal{R}(U)} (\mathcal{R}|_U)$$

Proof. Kempf [1] Section 5.

□

We will not prove or use this. It is used to prove the following fact, which we will use:

Fact 4.17. Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves of $\mathcal{O}_X$-modules on a variety X

(i) If $\mathcal{M}$ is locally free of rank n, then every stalk $\mathcal{M}_x$ is free of rank n.

(ii) If $\mathcal{M}$ is coherent and every stalk $\mathcal{M}_x$ is free of rank n, then F is locally free of rank n.

(iii) If $\mathcal{M}$ is (quasi-)coherent and $\alpha : \mathcal{M} \to \mathcal{N}$, then $\ker \alpha$ and im α are (quasi-)coherent.

Proof. Also Kempf [1] Section 5.

□

Example 4.18. Let Y be a variety and X a subvariety with inclusion map $\phi : X \to Y$. There is a morphism of sheaves of rings $\phi^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ given by precomposition:

$$\phi^\sharp(U)(f) = f \circ \phi$$

for U open and f a section. Define the *ideal sheaf of regular functions on Y vanishing on X*, denoted I_X,

$$I_X := \ker(\phi^\sharp)$$

Let f be a real polynomial in n variables. Given a point $a \in \mathbb{R}^n$, f is equal to its (finite) Taylor series:

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_a (x_i - a_i) + O\left((x_i - a_i)^2\right), \quad \forall x \in \mathbb{R}^n$$

Consider the differential map $df : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$df(x, a) = f(x) - f(a) \mod (x_i - a_i)^2.$$

Let

$$\Delta = V_{\mathbb{R}^n \times \mathbb{R}^n} (x_1 - a_1, \ldots, x_n - a_n)$$

and $I(\Delta)$ be the ideal sheaf of regular functions on $\mathbb{R}^n \times \mathbb{R}^n$ vanishing on Δ. Then df is an element of the 'sheaf of differentials on $\mathbb{R}^n$': $\Omega_{\mathbb{R}^n} := I(\Delta)/I(\Delta)^2$, where the quotient and square is taken on ideals. This motivates the next definition.

Definition 4.19. Let $X \subset \mathbb{A}^n$ be a quasi-affine variety and write $X \times X = \{(x, a) : x, a \in X\}$ with projections π_1 and π_2. Define

$$\Delta = V_{X \times X}(x_1 = a_1, \ldots, x_n = a_n)$$

then the *sheaf of differentials* Ω_X is the sheaf of $\mathcal{O}_X$-modules obtained by taking the sheafification of the presheaf defined by

$$U \to I(\Delta)(U)/I(\Delta)(U)^2$$

for U open with $\mathcal{O}_X$-module structure given by acting on the first component: for $f \in \mathcal{O}_X, \omega \in \Omega_X$

$$f \cdot \omega = (f \circ \pi_1)\omega.$$

By Fact 4.17, $I(\Delta)$ and Ω_X are coherent. The *differential map* from $\mathcal{O}_X$ to Ω_X is given by

$$f \to df := f \circ \pi_1 - f \circ \pi_2$$

Remark 4.20. Definition 4.19 extends to quasi-projective varieties by taking affine patches.

4.2. Tensor products of $\mathcal{O}_X$ modules

Tensor products will be essential in the following sections and we establish some preliminary results here. In this section, and following sections, we will almost always be considering sheaves of $\mathcal{O}_X$-modules rather than any other sheaf of rings.

Proposition 4.21. If $\mathcal{M}$ and $\mathcal{N}$ are locally free sheaves of $\mathcal{R}$ modules of ranks m and n respectively, then $\mathcal{M} \oplus_{\mathcal{R}} \mathcal{N}$ is locally free of rank mn.

Proof. Let U be an open set such that $\mathcal{M}(U) = R^m$ and $\mathcal{N}(U) = R^n$, where $R = R(U)$. It suffices to show the purely algebraic fact:

$$R^m \otimes_R R^n \approx R^{nm}$$

The proof is thus not relevant to us, nor difficult, and we briefly outline it below and suppress subscripts.

$$\begin{aligned} R^m \otimes R^n &\approx R^m \otimes (R^{n-1} \oplus R) \\ &\approx (R^m \otimes R^{n-1}) \oplus (R^m \otimes R) \\ &\approx R^{m(n-1)} \oplus R^m \\ &\approx R^{mn} \end{aligned}$$

□

Corollary 4.22. If F and G are invertible sheaves of $\mathcal{R}$-modules, then $F \otimes_{\mathcal{R}} G$ is invertible.

This motivates the following definition:

Definition 4.23. The *Picard group* $\mathrm{Pic}(X)$, is the set of isomorphism classes of invertible sheaves of $\mathcal{O}_X$-modules on X with the tensor product as the group law.

It is not yet clear that the Picard group is a group. Commutativity, associativity, and now closure are clear. $\mathcal{O}_X$ acts as the identity. We have not mentioned inverses.

Definition 4.24. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules. Then the *dual sheaf* $\mathcal{M}^*$ is defined

$$\mathcal{M}^* := \mathrm{Hom}(\mathcal{M}, \mathcal{O}_X)$$

which denotes the set of sheaf morphisms $\mathcal{M}$ to $\mathcal{O}_X$. This is a sheaf of $\mathcal{O}_X$-modules, mapping

$$U \to \mathrm{Hom}(\mathcal{M}|_U, \mathcal{O}_X|_U).$$

Remark 4.25.

- Unlike previous examples of sheaves, $\mathcal{M}^*(U)$ does not contain functions which act on U
- $\mathcal{M}^*$ is indeed a sheaf of $\mathcal{O}_X$-modules since any $\mathcal{O}_X|_U$-linear combination of sheaf morphisms defines another sheaf morphism: for $f \in \mathcal{O}_X|_U$ and $\alpha, \beta \in \mathrm{Hom}(\mathcal{M}|_U, \mathcal{O}_X|_U)$

$$(\alpha + f\beta)(t) := \alpha(t) + f \cdot \beta(t), \quad t \in \mathcal{M}(U)$$

- In this definition we could replace $\mathcal{O}_X$ with any other sheaf of rings, but this wont be of use to us.

Proposition 4.26. If $\mathcal{M}$ is invertible, then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}^* \approx \mathcal{O}_X$. Thus, $\mathrm{Pic}(X)$ is an abelian group with $\mathcal{M}^{-1} = \mathcal{M}^*$.

Proof. Consider $\varphi : \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}^* \to \mathcal{O}_X$ defined by $f \otimes \alpha \to \alpha(f)$. By Proposition 3.17, it suffices to show the induced stalk maps φ_x are isomorphisms for all $x \in X$.

Invertible sheaves have rank 1 stalks (as $\mathcal{O}_{X,x}$-modules), so we can let $\mathcal{M}_x = (f_x)$ and $\mathcal{M}_x^* = (\alpha_x)$. Then $\alpha_x(f_x)$ is a non-zero element of $\mathcal{O}_{X,x}$, so by rescaling we can assume $\alpha_x(f_x) = 1$. Thus the space $(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}^*)_x = \mathcal{M}_x \otimes_{\mathcal{O}_X} \mathcal{M}_x^*$ is generated by $f_x \otimes \alpha_x$ and $\varphi(f_x \otimes \alpha_x) = 1$.

Moreover the inverse morphisms ψ_x are defined by $\psi_x(1) = f_x \otimes \alpha_x$ □

5. Divisors

5.1. Divisor-Line bundle correspondence

In this section, we assume that X is an irreducible variety. Divisors on curves naturally generalise to arbitrary varieties:

Definition 5.1.

- An *irreducible divisor* D is a closed subvariety $D \subset X$ with $\dim D = \dim X - 1$. For each D, we have the ideal sheaf I_D taken with respect to X.
- The *(Weil) divisor group* Div(X) is the free abelian group of finite support on the set of irreducible divisors. Thus every $D \in$Div(X) is of the form $\sum_i n_i D_i$ with D_i irreducible and finitely many non-zero n_i.
- D is effective if $n_i \geq 0$ for all i.

Remark 5.2. This is the right generalisation of divisors on curves. Codimension one subvarieties coincide with hypersurface cuts of X.

We also assume the following fact:

Fact 5.3. If X is a smooth variety, then all local rings $\mathcal{O}_{X,x}$ are UFDs.

Proof. The Auslander-Buchsbaum theorem. See Appendix 7 in [5].

□

Remark 5.4. A variety is called *factorial* if all its local rings are UFDs. There exist non-smooth varieties which are factorial, e.g. $x^2 + y^3 + z^5 = 0$, but this is hard to prove, see [4].

Lemma 5.5. Let D be an irreducible divisor. Then I_D is an invertible sheaf.

Proof. Let $f \in I_{D,x}$ irreducible. Taking X to be smaller, we may assume X is quasi-affine and f extends to a regular function f^* on X. Write $f^{*-1}(0) = D \sqcup C$. D and C are closed distinct varieties, so from II Algebraic Geometry we can find a polynomial g^* that vanishes on C but not D.

Let $h^* \neq 0$ be arbitrary but vanishing on D. Then $h^* g^*$ vanishes on $f^{*-1}(0)$, so by Nullstellensatz f^* divides $(h^* g^*)^n$ for some n. Taking germs at x we have f divides $(hg)^n$, but f does not divide g since g doesn't vanish on D. Thus f divides h as $\mathcal{O}_{X,x}$ is a UFD. Therefore $(f) = I_{D,x}$, so I_D is an invertible sheaf.

□

We will study the following new concepts:

Definition 5.6.

- Define the *field of rational functions on X* by

$$k(X) := \bigcup_{U \subset X \text{ open}} \mathcal{O}_U$$

  Every element of $k(X)$ is defined on all but a closed subset of X.

- Let $\mathrm{Rat}(X)$ denote the sheaf of rings defined by $U \to k(X)|_U$ for U open, where the restriction is in the function sense.

- A *sheaf of fractional ideals I* is any coherent subsheaf of $\mathrm{Rat}(X)$.

- $\mathrm{IFI}(X)$ denotes the *group of invertible fractional ideal sheaves*, with ideal multiplication as the group law.

The key theorem we want to prove is:

Theorem 5.7. (Divisor-Line bundle correspondence) There is an isomorphism $\psi : \mathrm{Div}(X) \to \mathrm{IFI}(X)$

Given this theorem, we can make the following definition

Definition 5.8. For any divisor D, there is a corresponding invertible subsheaf of $\mathrm{Rat}(X)$, denoted $\mathcal{O}_X(D)$. Moreover, every invertible subsheaf of $\mathrm{Rat}(X)$ arises in this way.

Remark 5.9.

- Since ψ is a homomorphism, we have $\mathcal{O}_X(D+E) = \mathcal{O}_X(D)\mathcal{O}_X(E)$.

- In the proof, we'll see $\mathcal{O}_X(D)(U) = \{f \in O_U : D + (f) \text{ is effective}\}$, where (f) is the divisor of zeroes and poles as seen in II Algebraic Geometry.

In the proof, we will assume the following facts:

Fact 5.10. Let g be a non-zero regular function on an irreducible variety X. Then each irreducible component of the closed subset $g^{-1}(0)$ has dimension $\dim X - 1$.

Proof. Kempf [1] Section 2.6.

□

Proof of Theorem 5.7. A homomorphism ψ is determined by its action on the irreducible divisors. For D irreducible we define

$$\psi(D) = I_D^{-1}$$

where we recall the inverse of an invertible sheaf is its dual. Then ϕ defines an injective homomorphism.

Let I be an invertible ideal. We prove the following:

Claim: If $I \subset \mathcal{O}_X$, then $I = \psi(-D)$ for some effective D

Note $\mathcal{O}_X = \mathcal{O}_X(0)$. By induction we may assume the claim is true for all ideals strictly containing I. Define $V = \{x \in X : I_x = 0\}$, which is a closed subvariety since it can be written as an intersection of the sections of I.

Subclaim: $\dim V = \dim X - 1$

Recall that for any variety, $\dim X = \dim U$ for any non-empty open $U \subset X$. Applying this fact to V, it suffices to show $V \cap U$ has dimension $\dim X - 1$ for some U open. By invertibility of I, choose U such that $V \cap U \neq \emptyset$ and $I(U) = (f)$ for some $f \in \mathcal{O}_X$. Then $V \cap U = V(f) \cap U$ and by Lemma 5.10, $V(f)$ has dimension $\dim X - 1$, hence $V \cap U$ does too. □

Let W be an irreducible component of V such that $\dim V = \dim W$. Then W is an irreducible divisor. The ideal $I' = I\mathcal{O}_X(W)$ is an invertible ideal strictly larger than I, so by induction $I' = \psi(-D')$ for some D' effective. Hence $I = \psi(-D' - E)$ □.

Now let I be an invertible fractional ideal. Let $J = I \cap \mathcal{O}_X$.

Subclaim: J is invertible

Let $x \in X$. Locally, I is generated by some f/g with $f, g \in \mathcal{O}_X(U)$. Since $\mathcal{O}_{X,x}$ is a UFD, if f_x and g_x are not coprime in $\mathcal{O}_{X,x}$, then they have a gcd given by a germ h_x, which after shrinking U we may assume is an element of $\mathcal{O}_X(U)$. Thus we may assume f_x and g_x are coprime. But then g_x is a unit in $\mathcal{O}_{X,x}$ so non-vanishing on a neighbourhood $V \subset U$ of x. Thus locally, $J|_V$ is free generated by f and the subclaim is proven.

To conclude, we observe $I = J(JI^{-1})^{-1}$ thus I has the form $\mathcal{O}_X(D)$ for some D.

□

We now see some applications of Theorem 5.7.

Fact 5.11. Every dimension $n-1$ irreducible subvariety of $\mathbb{A}^n$ is of the form $V(f)$ for some irreducible polynomial f. The same is true for $\mathbb{P}^n$ but additionally f is homogeneous.

This can be proven using the methods of II Algebraic Geometry. For example, if $X \subset \mathbb{A}^n$ is irreducible and dimension $n-1$, then we can choose an irreducible $f \in I(X)$. Then $V(f) \subset I(X)$ and $V(f)$ is dimension $n-1$. One can conclude by showing an irreducible variety cannot have a proper subvariety of the same dimension.

Corollary 5.12. $\text{Pic}(\mathbb{A}^n) = \{\mathcal{O}_{\mathbb{A}^n}\}$

Proof. By Theorem 5.7, it suffices to show I_D is principal whenever D is an irreducible divisor. By Fact 5.11, $D = V(f)$ for some irreducible f, therefore $I_D = f\mathcal{O}_{\mathbb{A}^n}$. □

Fact 5.13. The only regular functions defined on all of $\mathbb{P}^n$ are the constants

This fact is also an exercise in II Algebraic Geometry.

Corollary 5.14. $\text{Pic}(\mathbb{P}^n) = \{\mathcal{O}_{\mathbb{P}^n}(m) : m \in \mathbb{Z}\}$, as defined in Example 4.7. In other words, $\text{Pic}(\mathbb{P}^n) \approx \mathbb{Z}$.

Proof. By Fact 5.11, if D is an irreducible divisor then $D = V(f)$ for some irreducible and homogeneous f of degree d. Any section of I_D is a multiple of f and is a ratio of two homogeneous polynomials of the same degree, thus $I_D = f\mathcal{O}_{\mathbb{P}^n}(-d) \approx \mathcal{O}_{\mathbb{P}^n}(-d)$.

Lastly, suppose $\mathcal{O}_{\mathbb{P}^n}(a+b) \approx \mathcal{O}_{\mathbb{P}^n}(a)$. Then by Example 4.7, $\mathcal{O}_{\mathbb{P}^n}(b) \approx \mathcal{O}_{\mathbb{P}^n}(0)$. By Fact 5.13, $\mathcal{O}_{\mathbb{P}^n}(b)$ has no global sections for $b > 0$. However, $\mathcal{O}_{\mathbb{P}^n}$ has global sections (the constants).

□

5.2. Divisors on curves

In this subsection we fix X to be a projective non-singular curve. We have seen there is a correspondence between divisors and invertible subsheaves of $\text{Rat}(X)$. For curves, every coherent subsheaf of $\text{Rat}(X)$ is invertible:

Proposition 5.15. Let I be a non-zero coherent subsheaf of $\text{Rat}(X)$. Then I is invertible.

Proof. Let $x \in X$. Then I_x is an ideal of $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is a DVR, $I_x = t^n O_{X,x}$ for some generator t. Since I is coherent with every stalk being a rank 1 $\mathcal{O}_{X,x}$-module, it follows that I is locally free of rank 1. □

Definition 5.16. Let F be a sheaf of $\mathcal{O}_X$-modules on X. For sections f over U and g over V, we say f and g are *rationally equivalent* and write $f \sim g$ if there is a non-empty open $W \subset U \cap V$ such that $f|_W = g|_W$.

The rational sections of F are defined,

$$\Gamma_{\text{rat}}(F) := \bigsqcup_{U \subset X \text{ open}} F(U)/\sim$$

Let $\text{Rat}(F)$ denote the sheaf on X defined by $U \to \Gamma_{\text{rat}}(F)|_U$ with restriction in the functional sense.

Remark 5.17.

- $\sim$ is indeed an equivalence relation, which uses the fact X is an irreducible variety.
- Previously we defined $k(X)$, which coincides with $\Gamma_{\text{rat}}(\mathcal{O}_X)$.
- We have seen a similar notion of 'rational maps' in II Algebraic Geometry.

Proposition 5.18. If F is coherent, then $\text{Rat}(F)$ is a finite dimensional $k(X)$-vector space

Proof. Since F is coherent, we can find a non-empty open set U with $F|_U$ finitely generated, i.e. there are sections $(f_i)_1^n$ over U such that for every $V \subset U$, $F(V)$ is generated by $(f_i|_V)_1^n$ as an $\mathcal{O}_V$-module, and thus as a $k(X)$-module.

Let g be a section over $W \subset X$. By density, $W \cap U$ is non-empty, so we can take $V = W \cap U$, then $g|_V \in \text{span}_{k(X)}(f_i)_1^n$. Thus $(f_i)_1^n$ span $\text{Rat}(F)$. $\square$

Note that in the proof of the previous proposition, we see if F is locally free of rank n, then the dimension of $\text{Rat}(F)$ is n. This motivates the following definition:

Definition 5.19. If F is a coherent sheaf, then $\text{rank}(F) := \dim_{k(X)}\text{Rat}(F)$.

Recall that an R-module M is torsion if for all $m \in M$ there exists $r \in R\backslash 0$ such that $rm = 0$. This notion generalises to sheaves:

Definition 5.20. Let F be a sheaf of $\mathcal{O}_X$-modules.

- F is *torsion* if F_x is a torsion module for all x.

- F is *torsion-free* if F_x is a torsion-free module for all x.
- The *support* of F is the set $\text{Supp}(F) = \{x \in X : F_x \neq 0\}$.

Proposition 5.21. Let F be a coherent torsion sheaf on X. Then

(i) F has finite support.

(ii) For all $x \in \text{Supp}(F)$, F_x is a finite dimensional k-vector space.

Proof. (i) Let U be a non-empty open subset such that $F|_U$ is generated by sections $f_1, \dots, f_n$ over U. Pick $x \in U$, then $(g_1)(f_1)_x = 0$ for some $g_1 \in \mathcal{O}_{X,x} \backslash 0$. Pick g_i for $i \leq n$ similarly and let $g = g_1 ... g_n \neq 0$. Then $gf_i = 0$ for all i thus $gf = 0$ for all $f \in F(U)$. Since U is a neighbourhood of x, we have $F_x = 0$ and this holds for all $x \in U$. As X is an irreducible curve, non-empty open sets are co-finite.

(ii) Recall since X is a non-singular curve, $\mathcal{O}_{X,x}$ is a DVR. Let t generate the maximal ideal, then all we know all ideals of $\mathcal{O}_{X,x}$ are of the from (t^n) for some n. In particular, $\mathcal{O}_{X,x}$ is a PID. Applying Structure Theorem to the finitely generated $\mathcal{O}_{X,x}$-module F_x, we have

$$F_x \approx \frac{\mathcal{O}_{X,x}}{(t^{n_1})} \oplus \cdots \oplus \frac{\mathcal{O}_{X,x}}{(t^{n_k})}$$

as $\mathcal{O}_{X,x}$-modules with $n_i \neq 0$ since otherwise F_x would not be torsion. Recall that $\dim_k \mathcal{O}_{X,x}/(t^m) = m$ for all $m > 0$ (from II Algebraic Geometry, otherwise an exercise). Thus F_x is finite dimensional. □

We can thus define:

Definition 5.22. Let F be coherent and torsion. Then we define its divisor

$$\text{Div}(F) := \sum_i (\dim_k F_{x_i}) x_i$$

where (x_i) is the finite set of points where F is supported.

Proposition 5.23. Let F be coherent and torsion. Then, for U open,

$$F(U) \approx \bigoplus_{x_i \in U} F_{x_i}$$

as $\mathcal{O}_X(U)$-modules and in particular, $\dim_k \Gamma(X, F) = \deg \text{Div}(F)$.

Proof. We have an $\mathcal{O}_X(U)$-linear evaluation map $\text{ev} : F(U) \to \oplus_{x_i \in U} F_{x_i}$ by taking germs. Without loss of generality, assume $x_i \in U$ for all i.

ev injective

If f is a section of U and $f_{x_i} = 0$ for all i, then since this the entire support of F in U, we have $f_x = 0$ for all $x \in U$. Thus $f = 0$.

ev surjective

Let f_i be germs in F_{x_i} for each i and lift them to sections of open sets U_i such that $x_j \notin U_i$ for all $j \neq i$. By taking stalks, we see for all $i \neq j$, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} = 0$, thus the set $\{f_i : i\}$ satisfies the patching condition (3.5), so they can be patched to give a section in $F(U)$ as desired. This inverse map is clearly $\mathcal{O}_X(U)$-linear.

□

Next we consider the torsion-free case:

Proposition 5.24. Let F be coherent and torsion-free. Then there is a sequence of sheaves

$$0 \subset F_1 \subset \cdots \subset F_{\text{rank}(F)}$$

such that

(i) F is locally free of rank = rank F. Moreover, F_i is coherent for each i and locally free of rank i.

(ii) F_i/F_{i-1} is an invertible sheaf

Proof. We proceed by induction, noting the case rank $F = 0$ is trivial. Let $f \in \Gamma_{\text{rat}}(F)$ be a non-zero rational section and let L be the constant subsheaf of $\text{Rat}(F)$ defined by $U \to f|_U$. Define $F_1 = L \cap F$, which is an invertible sheaf. Then the quotient sheaf $F/F_1 \subset \text{Rat}(F)/L$ is torsion-free of one less rank, so induction applies. □

Definition 5.25. For F coherent and D a divisor, define

$$\begin{aligned} F(D) &:= F \cdot \mathcal{O}_X(D) \\ &\approx F \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \end{aligned}$$

which is the subsheaf of $\mathrm{Rat}(F)$ defined by $U \to F(U) \cdot \mathcal{O}_U(D)$ with element-wise multiplication. Moreover if D is effective we define,

$$F|_D := F/F(-D)$$
$$\mathcal{O}_D := \mathcal{O}_X|_D.$$

Proposition 5.26. Let F be coherent and torsion free. Then,

(i) $F|_D$ is torsion

(ii) $\mathrm{Div}(F|_D) = (\mathrm{rank}\ F)D$

(iii) $\dim_k \Gamma(X, F|_D) = (\mathrm{rank}\ F)(\deg D)$

Proof. Recall $\mathcal{O}_X(D) = \{g \in \mathcal{O}_X : (g) + D \text{ is effective}\}$. Since X is a curve, $D = n_1 p_1 + \cdots + n_k p_k$ for some $p_i \in X$ and positive integers n_i

(i) Let t_i be a local parameter at x_i, then t_i is a regular function on an open subset of X. Since X is an non-singular curve, t_i extends to an element of $\Gamma(X, \mathcal{O}_X)$. Thus take $t = t_1^{n_1} \ldots t_k^{n_k} \in \Gamma(X, \mathcal{O}_X) \backslash 0$ and we have $tf \in F(-D)$ for any f a section of F. Thus $F|_D$ is torsion.

(ii, iii) By Proposition 5.23, it suffices to show (a) $F|_D$ is supported on the p_i and (b) $\dim_k F_{p_i} = (\mathrm{rank}\ F)n_i$ for each i. We recall the general fact that quotients commute with stalks: $(F/F(-D))_x = F_x/F(-D)_x$.

(a) For any $q \notin \{p_i : i\}$, $F(-D)_q = F_q$, thus $(F|_D)_q = 0$.

(b) Fix $p = p_i$, $n = n_i$ and $t = t_i$. Then

$$(F|_D)_x = F_x/(F_x(t^n))$$

as a $\mathcal{O}_{X,p}$-module. This is naturally isomorphic to the k-vector space $k^n \otimes_k F_x$, as we recall $\dim_k \mathcal{O}_{X,p}/(t^n) = n$. Thus $\dim_k (F|_D)_x = n\mathrm{rank}\ F$. □

5.3. The Cousin problem

In this subsection, we let F be a locally free sheaf on a smooth curve X

Definition 5.27.

- For $x \in X$, the *group of principal parts at x* is

$$\mathrm{Prin}_x(F) := \Gamma_{\mathrm{rat}}(F)/F_x$$

- The *sheaf of principal parts*, denoted $\mathrm{Prin}(F)$, is the sheaf defined by

$$U \to \bigoplus_{u \in U} \mathrm{Prin}_u(F)$$

for U open and restriction to $V \subset U$ forgets coordinates outside of V.

- The *principal sections* of F are

$$\begin{aligned}\Gamma_{\mathrm{prin}}(F) &:= \Gamma(X, \mathrm{Prin}(F)) \\ &= \bigoplus_{x \in X} \mathrm{Prin}_x(F)\end{aligned}$$

Proposition 5.28. There is an exact sequence of sheaves

$$0 \to F \to \mathrm{Rat}(F) \xrightarrow{\alpha} \mathrm{Prin}(F) \to 0 \tag{5.1}$$

We need to define a homomorphism $\alpha(U) : \mathrm{Rat}(F)(U) \to \mathrm{Prin}(F)(U)$. Let f be a section of $\mathrm{Rat}(F)$ over U. Then $f \in \Gamma_{\mathrm{rat}}(F)$ so there is an open $V \subset U$ with $f \in F(V)$. Define

$$\alpha(U)(f) = \sum_{u \in U} f_u \bmod F_u$$

This is a finite sum because $f_u \in F_u$ if $u \in V$. This definition commutes with restriction, thus α is well defined. Note also $\ker(\alpha) = F$. It remains to show α is surjective, so suffices to show α_x is surjective for all $x \in X$. But $\alpha_x : \Gamma_{\mathrm{rat}}(F) \to \Gamma_{\mathrm{rat}}(F)/F_x$ is just the quotient projection, thus surjective.

By taking global sections of (5.1), we obtain:

Corollary 5.29. The sequence

$$0 \to \Gamma(X, F) \to \Gamma_{\mathrm{rat}}(F) \xrightarrow{\alpha} \Gamma_{\mathrm{prin}}(F) \tag{5.2}$$

is exact.

The *Cousin problem* asks whether α is surjective. Cokernel(α) is an important global invariant of F, we will see it is the 'cohomology group' $H^1(X, F)$.

Definition 5.30. F is *ordinary* if α is surjective.

Remark 5.31. One can show that if X is affine, then α is always surjective. On the other hand, if X is projective and $F = \mathcal{O}_X$, then α is surjective if and only if $X \approx \mathbb{P}^1$.

Lemma 5.32. The following are equivalent,

(i) F is ordinary

(ii) For all $x \in X$ and D an effective divisor,

$$\dim_k \left(\Gamma(X, F(D+x))/\Gamma(X, F(D))\right) = \text{rank } F.$$

Proof. Let D be effective. By Proposition 5.26,

$$\dim_k \Gamma(X, F(D)/F)) = (\deg D)(\text{rank } F)$$

and embeds naturally to a subspace of $\Gamma_{\text{prin}}(F)$ denoted V_D. Note that $\Gamma_{\text{prin}}(F) = \bigcup_{D \text{ effective}} V_D$, thus α is surjective if and only if its image contains V_D for all D. Observe that the intersection of the image of α with V_D is isomorphic to $\Gamma(X, F(D))/\Gamma(X, F)$, thus α is surjective iff

$$\dim_k \Gamma(X, F(D))/\Gamma(X, F) = (\deg D)(\text{rank } F)$$

which is equivalent to (ii). □

6. Sheaf cohomology

6.1. Definition

In Algebraic Topology, 'cohomology' refers to 'number of holes' amongst other exciting things. To us, cohomology will be a powerful proof device with no interpretation. In this section, we let $\mathbf{C}$ be the category of sheaves of $\mathcal{O}_X$-modules.

In the next section we will study the following class of sheaves, which are also used in the definition of cohomology:

Definition 6.1. A sheaf F is *flabby* if every section over U extends to a section over X, i.e. the restriction map $\text{res}_U^X : F(X) \to F(U)$ is surjective.

Example 6.2. $\text{Rat}(X)$ is a flabby sheaf. If X is a curve, then any torsion sheaf is flabby (Hint: Proposition 5.23) and $\text{Prin}(F)$ is flabby.

We are now ready to define cohomology:

Fact 6.3. There exists a sequence of functors $H^i(X, \cdot\ ) : \mathbf{Sh}(\mathbf{C})(X) \to \mathbf{C}$ such that the following hold:

(i) $H^0(X, F) = \Gamma(X, F)$

(ii) Given a short exact sequence of sheaves

$$0 \to E \to F \to G \to 0$$

there exists a long exact sequence of sheaves

$$0 \to H^0(X, E) \to H^0(X, F) \to H^0(X, G) \to H^1(X, E) \to \dots$$

(iii) There is a class of sheaves called *injective* satisfying the following:

(a) any injective sheaf is in the kernel of $H^i(X, \cdot)$ for all $i > 0$

(b) any sheaf embeds into some injective sheaf

(c) injective sheaves are flabby

Definition 6.4. The *i-th cohomology functor* $H^i(X, \cdot\ )$ is the functor described in Fact 6.3. For a sheaf F, $H^i(X, F)$ is an $\mathcal{O}_X$ module called the *i-th cohomology group*. We drop the 'X' and write $H^i(F)$ when the context is clear.

Surprisingly, this definition and Fact 6.3 have little to do with sheaves. A general construction in homological algebra called a 'derived functor' allows one to go from a short exact sequence to a long exact sequence in an 'abelian category'. An insightful exposition without proofs is provided in Hartshorne [2] page 207.

The properties listed in Fact 6.3 *almost* uniquely determine the functor $H^i(X, \cdot)$. We will not concern ourselves with such issues and only use these properties of cohomology. $H^i(F)$ is naturally a k-vector space, which will be important later sections.

6.2. Flabby sheaves

In this section we prove the following theorem,

Theorem 6.5. (Flabby sheaves have no cohomology) If F is a flabby sheaf, then $H^i(X, F) = 0$ for all $i > 0$.

We will use this theorem in the proof of Riemann Roch and it also serves as a simple demonstration of cohomology.

We will use the following fundamental fact about flabby sheaves:

Fact 6.6. Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence of sheaves of abelian groups with F_1 flabby. Then

1. the sequence of groups

$$0 \to F_1(U) \to F_2(U) \to F_3(U) \to 0$$

   is exact for all U open.

2. if F_2 is flabby, then F_3 is flabby.

Proof. Exercise. Only the basic definitions from Section 3 are required.

□

Proof of Theorem 6.5. By Fact 6.3 there is an injective sheaf I such that the sequence

$$0 \to F \xrightarrow{\alpha} I \xrightarrow{\beta} Q \to 0 \tag{6.1}$$

is exact, for $Q = I/\alpha(F)$ the quotient sheaf. Thus we have a long exact sequence

$$0 \to \Gamma(X, F) \to \Gamma(X, I) \to \Gamma(X, Q) \to H^1(F) \to H^1(I) \to \dots \tag{6.2}$$

Since I is injective, $H^1(X, I) = 0$. Hence the following is exact,

$$\Gamma(X, I) \to \Gamma(X, Q) \xrightarrow{\gamma} H^1(F) \to 0$$

We can take the first map to be induced by β from (6.1), thus by Lemma 6.6 is surjective. Therefore $\ker \gamma = \Gamma(X, Q)$ but the last map tells us γ is surjective. Hence, $H^1(X, F) = 0$.

Looking at the next part of (6.2), we have

$$\cdots \to H^1(Q) \to H^2(F) \to H^2(I) \to \dots \tag{6.3}$$

By Lemma 6.6, Q is flabby. Thus from what we have already established, $H^1(Q) = 0$. We also know $H^2(I) = 0$ as it is injective. Hence, (6.3) gives $H^2(F) = 0$. This continues for all i. □

6.3. Direct and inverse images

In this section, we develop some more machinery required for Riemann-Roch. Given varieties X and Y and a continuous (in the topological sense) map between them, we wish to 'push forward' a sheaf on X to a new sheaf on Y.

Definition 6.7. Let $f : X \to Y$ continuous. The *direct image functor* $f_\star$ is a map from $\mathbf{Sh(C)}(X)$ to $\mathbf{Sh(C)}(Y)$ defined by

$$(f_\star F)(U) = F(f^{-1}(U))$$

for $F \in \mathbf{Sh}(\mathbf{C})(X)$ and U open. The sheaf $(f_\star F)$ is called the *push-forward sheaf*.

Remark 6.8. Continuity of f is precisely what is needed for $f^{-1}(U)$ to be open, thus sheaves act on it. We leave it as an exercise to argue $(f_\star F)$ is indeed a sheaf.

Example 6.9. Let $X = V \times V$, $Y = V$, and $f = \pi$, the projection onto the second factor. Then

$$(\pi_\star F)(U) = F(\{y \mid \exists x : (x, y) \in U\})$$

This example will re-emerge in the proof of Riemann-Roch.

The next thing to do is go the other way around: we still have $f : X \to Y$, can we 'pull back' sheaves from Y to sheaves on X? The answer is less straightforward. Let G be a sheaf on Y, and we naively attempt to define a sheaf on X by $U \to G(f(U))$. This falls at the first hurdle, because $f(U)$ is not necessarily open. The next best thing we can do is approximate $f(U)$ by open sets. To do this, we use a similar construction to stalks:

Definition 6.10. Let (V_α) be a collection of open subsets of Y and G a sheaf on Y. We say $G(V_\beta) \sim G(V_\gamma)$ if there exists a non-empty open set $W \subset V_\beta \cap V_\gamma$ such that $G(V_\beta)|_W = G(V_\gamma)|_W$. If $\sim$ is an equivalence relation, then we define the *direct limit*

$$\lim_\alpha G(V_\alpha) := \bigsqcup_\alpha G(V_\alpha)/\sim .$$

Whether $\sim$ is an equivalence relation or not depends on the collection (V_α) and G.

Example 6.11. Let $y \in Y$. Then

$$G_y = \lim_{V \ni y} G(V)$$

Under the hood, we have been using direct limits all along.

Definition 6.12. Let $f : X \to Y$ and G a sheaf on Y of category **C**. The *inverse image* $f^{-1}G$ is the sheafification of the presheaf defined by

$$U \to \lim_{V \supset f(U)} G(V)$$

and also a sheaf of category **C**.

This is still not the perfect sheaf. Often we have a sheaf of $\mathcal{O}_Y$-modules on Y and the inverse sheaf would also be a sheaf of $\mathcal{O}_Y$-modules on X. In the abstract world of sheaves this is not an issue but this is morally wrong in the philosophy of ringed spaces – the 'natural' sheaves on X are sheaves of $\mathcal{O}_X$-modules.

Definition 6.13. Let X and Y be varieties and $f : X \to Y$ a morphism of varieties. For a sheaf of $\mathcal{O}_Y$-modules G on Y, the *pull-back sheaf* is the sheaf of $\mathcal{O}_X$-modules,

$$f^\star G := f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Fortunately, this concept is not used too much in algebraic geometry.

Remark 6.14. A morphism of varieties, which has not made an appearance in our exposition so far, is a special case of a 'morphism of ringed spaces'.

The last interesting part of maps between varieties is what happens to the cohomology. The following result which we state but do not prove gives an answer for affine varieties

Fact 6.15. Let $f : X \to Y$ be a morphism of affine varieties and F a quasi-coherent $\mathcal{O}_X$-module. Then for all i we have a natural isomorphism

$$H^i(X, F) \approx H^i(Y, f_\star F)$$

Proof. Kempf [1], Corollary 8.2.3, page 102

□

In general, life is not this simple. The same construction from homological algebra, the 'derived functor' which when applied to the global sections functor $\Gamma(X, \cdot)$ gives the cohomology functors $H^i(X, \cdot)$, can be applied to the direct image functor.

Definition 6.16. The *i-th higher direct image functor* $R^i f_\star : \mathbf{Sh}(C)(X) \to \mathbf{Sh}(C)(Y)$, is the i-th derived functor of $f_\star$.

Of course, we do not know what direct functors are. Instead, we use the analogous properties described in Fact 6.3.

Fact 6.17. The functors $R^i f_\star : \mathbf{Sh}(\mathbf{C})(X) \to \mathbf{Sh}(\mathbf{C})(Y)$ satisfy

(i) $R^0(f_\star F) \to f_\star F$

(ii) Given a short exact sequence of sheaves

$$0 \to E \to F \to G \to 0$$

there exists a long exact sequence of sheaves

$$0 \to R^0 f_\star E \to R^0 f_\star F \to R^0 f_\star G \to R^1 f_\star E \to \dots$$

Remark 6.18. $R^i f_\star F$ is the sheaf associated to the presheaf defined by $U \to H^i(f^{-1}U, F)$ for U open in Y.

Example 6.19. If $Y = \{y\}$ is a point, then $R^i f_\star F(Y) = H^i(X, F)$.

We briefly recall that given a k-vector space V, we can define the sheaf of k-vector spaces $\mathcal{O}_X \otimes_k V$ by the sheafification of $U \to O_U \otimes_k V$. The following result is what we use in the proof of Riemann-Roch.

Lemma 6.20. Let X and Y be varieties, F a quasi-coherent sheaf on Y, and π_X and π_Y the usual projections on $X \times Y$. Then for all i,

$$(R^i \pi_{X\star})\,(\pi_Y^\star F) \approx \mathcal{O}_X \otimes_k H^i(Y, F)$$

Proof. Kempf [1], Lemma 8.3.3, page 103 □

We understand this result as saying, if we have a sheaf on $X \times Y$ which is constant in X (i.e. the push-forward of a sheaf on Y), then the higher direct image only depends on the cohomology of Y.

6.4. Cohomology of curves

In this subsection, F is a locally free coherent sheaf of $\mathbf{C}$ on a smooth projective curve X. Our goal is to understand the cohomology groups $H^i(X, F)$.

Proposition 6.21. $H^i(F) = 0$ if $i > 1$

Proof. Recall (5.1), the exact sequence

$$0 \to F \to \mathrm{Rat}(F) \to \mathrm{Prin}(F) \to 0$$

From the corresponding long exact sequence, we have

$$\cdots \to H^1(\mathrm{Prin}(F)) \to H^2(F) \to H^2(\mathrm{Rat}(F)) \to \dots$$

$\mathrm{Rat}(F)$ and $\mathrm{Prin}(F)$ are flabby, so have no cohomology. Thus $H^2(F) = 0$. This argument repeats for all higher cohomology groups. ☐

Remark 6.22. The first part of the long exact sequence gives

$$0 \to H^0(F) \to \mathrm{Rat}(F) \to \mathrm{Prin}(F) \to H^1(F) \to 0$$

Thus the prophecy in Section 5.3 is fulfilled: $H^1(F)$ is the cokernel of α and hence measures the obstruction to solving the Cousin problem.

Corollary 6.23. F is ordinary if and only if $H^1(F) = 0$.

We recall that cohomology groups (of $\mathbf{C}$) are naturally k-vector spaces and define:

Definition 6.24.

- $h^i(F) := \dim_k H^i(X, F)$.
- $\chi(F) := h^0(F) - h^1(F)$ is the *Euler characteristic* of F.
- $g := h^1(\mathcal{O}_X)$ is the *genus* of X.

The numbers $h^i(F)$ are not specific to curves. In general, they are an incredibly important invariant of varieties. In the definition, we already see two main characters of geometry appear, though proving equivalence of the various definitions is always difficult. With that in mind, our next theorem should seem intimidating:

Theorem 6.25. Let F be a locally free coherent sheaf of $\mathcal{O}_X$-modules. Then

(i) $h^0(F), h^1(F) < \infty$

(ii) $\chi(F) = \deg(\det\ F) + (\mathrm{rank}\ F)(1 - g)$

We require some tools before we can prove this theorem.

Lemma 6.26. (Long sequence trick) Suppose we have a long exact sequence of vector spaces:

$$0 \to V_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} V_n \to 0$$

with f_i of rank r_i and $v_i = \dim V_i$. Then

(i) $v_i \le v_{i-1} + v_{i-2}$

(ii) If all the v_i's are finite, then

$$\sum_{i \text{ odd}} v_i = \sum_{j \text{ even}} v_j,$$

Proof. We have

(exactness)	$\text{rank } f_i = \text{null } f_i$
(rank-nullity theorem)	$\text{rank } f_i + \text{null } f_i = v_i$

Thus $r_i + r_{i-1} = v_i$ for all i, so (i) follows and

$$\sum_{i \ge 2} (-1)^i (r_i + r_{i-1}) = r_1 + (-1)^{n-1} r_{n-1}$$

$$\therefore \quad \sum_{i \ge 2} (-1)^i v_i = v_1 + (-1)^{n-1} v_n$$

$$\therefore \quad \sum_{i \text{ even}} v_i = \sum_{i \text{ odd}} v_i$$

□

Lemma 6.27. (High degrees have no cohomology) Let F be an invertible sheaf.

(i) If $\Gamma(X, \Omega_X \otimes F^*) = 0$, then $h^1(F) = 0$.

(ii) If $\deg F > \deg \Omega_X$, then $h^1(F) = 0$.

Proof. We first note (i) implies (ii) since $\deg(\Omega_X \otimes F^*) = \deg \Omega_X - \deg F < 0$ thus has no global sections. For (i), we aim to show F is ordinary and then we're done by Corollary 6.23. To this end, we use the criterion in Lemma 5.32. Let $G = F(D)$ for some effective D, Then

$$\begin{aligned}\Gamma(X, \Omega_X \otimes G^*) &= \Gamma(X, (\Omega_X \otimes F^*)(-D)) \\ &\subset \Gamma(X, \Omega_X \otimes F^*) \\ &= 0\end{aligned}$$

Let $x \in X$. We have the exact sequence $0 \to F \to F(x) \to F(x)|_x \to 0$. The corresponding long exact sequence is

(6.4)

$$0 \to \Gamma(X, F) \to \Gamma(X, F(x)) \to \Gamma(X, F(x)|_x) \xrightarrow{\delta_x} H^1(F) \to H^1(F(x)) \to 0$$

<u>Claim: δ_x is zero</u>

The idea of the proof is to see how δ_x varies with x globally in X in order to upgrade (6.4) to a global statement. To do this, we look at the diagonal Δ in the product $X \times X$ and recall π_1 and π_2 are the natural projections. We have the exact sequence of sheaves,

$$0 \to \mathcal{O}_{X\times X} \to \mathcal{O}_{X\times X}(\Delta) \to \Omega_X^* \to 0$$

where we recall $\Omega_X = \mathcal{O}_{X\times X}(-\Delta)/\mathcal{O}_{X\times X}$ is the sheaf of differentials with dual $\Omega_X^* = \mathcal{O}_{X\times X}(\Delta)/\mathcal{O}_{X\times X}$. Tensoring with $\pi_1^\star F$, this becomes the exact sequence:

$$0 \to \pi_1^\star F \to \pi_1^\star F(\Delta) \to \pi_1^\star F \otimes_{\mathcal{O}_X} \Omega_X^* \to 0$$

Taking the long exact sequence corresponding to the higher direct image of $\pi_{2\star}$, we obtain the exact sequence

$$0 \to \pi_{2\star}\pi_1^\star F \to \pi_{2\star}\pi_1^\star F(\Delta) \to \pi_{2\star}(\pi_1^\star F \otimes_{\mathcal{O}_X} \Omega_X^*) \to (R^1\pi_{2\star})(\pi_1^\star F) \tag{6.5}$$

There is a natural isomorphism from $\pi_{2\star}(\pi_1^\star F \otimes_{\mathcal{O}_X} \Omega_X^*)$ to $F \otimes_{\mathcal{O}_X} \Omega_X^*$. Using this and Lemma 6.20, we can rewrite (6.5) as

$$0 \to \Gamma(X, F) \otimes_k \mathcal{O}_X \to \pi_{2\star}(\pi_1^\star F(\Delta)) \to F \otimes_{\mathcal{O}_X} \Omega_X^* \xrightarrow{\delta} H^1(X, F) \otimes_k \mathcal{O}_X$$

We know that $F^* \otimes_{\mathcal{O}_X} \Omega_X$ has no non-zero global sections, but given a sheaf morphism $\alpha : A \to B$ we can define its dual $\alpha^* : B^* \to A^*$ by precomposition: $\alpha^*(f) = f \circ \alpha$. Applying this to δ, we obtain a sheaf morphism δ^* from $H^1(X, F)^* \otimes_k \mathcal{O}_X$ to $F^* \otimes_{\mathcal{O}_X} \Omega_X$. The domain consists entirely of global sections and the image has none, so δ^* must be the zero map. It follows that

δ must also be the zero map. Moreover, any $\delta_x : \Gamma(X, F(x)|_x) \to H^1(F)$ being non-zero would lift to a map $\delta : F \otimes_{\mathcal{O}_X} \Omega^*_X \to H^1(F) \otimes_k \mathcal{O}_X$ which was not identically zero, and the claim is proven.

We conclude as follows. The exact sequence (6.4) becomes

$$0 \to \Gamma(X, F(D)(x)) \to \Gamma(X, F(D)(x)) \to \Gamma(X, F(D)(x)|_x) \to 0$$

where D is any effective divisor. This completes the proof as

$$\dim_k \Gamma(X, F(D)(x)|_x) = \text{rank } F$$

by Proposition 5.26.

□

Proof of Theorem 6.25. We induct on rank(F). The base case $i = 1$ is difficult and we do it later. Let F have rank greater than 1, then we have an exact sequence

$$0 \to L \to F \to Q \to 0$$

where L is invertible and Q is locally free of rank $=$ (rank F) $- 1$. Consider the corresponding long exact sequence of k-vector spaces:

$$0 \to H^0(L) \to H^0(F) \to H^0(Q) \to H^1(L) \to H^1(F) \to H^1(Q) \to 0$$

with the last 0 coming from Proposition 6.21. By the long sequence trick, we have that $h^0(F)$ and $h^1(F)$ are finite and

$$h^0(L) + h^0(Q) + h^1(F) = h^0(F) + h^0(Q) + h^1(L) + h^1(Q)$$

Thus,

$$\begin{aligned}
\chi(F) &= \chi(Q) + \chi(L) \\
&= \deg(\det L) + (1-g) + \deg(\det Q) + (\text{rank } F - 1)(1-g) \\
&= \deg(\det(L \otimes Q)) + (\text{rank } F)(1-g) \\
&= \deg(\det F) + (\text{rank } F)(1-g)
\end{aligned}$$

and we're done. We now turn to the base case where F is invertible. Recall that $F = \mathcal{O}_X(D) := \{f : D + (f) \text{ is effective}\}$ for some divisor D. We saw in II Algebraic Geometry that $\dim_k \Gamma(X, \mathcal{O}_X(D)) \leq \deg D + 1$. Thus $h^0(F) < \infty$.

By Lemma 6.27, we can find an invertible sheaf with $h^1(F) = 0$. Suppose L is such a sheaf and $x \in X$. Recall that invertible sheaves of the form $\mathcal{O}_X(D)$ for some divisor D. In particular, they are torsion free and $F(x)$ is invertible. Thus we have an exact sequence

$$0 \to F \to F(x) \to F(x)|_x \to 0$$

The corresponding long exact sequence is

$$0 \to H^0(F) \to H^0(F(x)) \to H^0(F(x)|_x) \to H^1(F) \to H^1(F(x)) \to 0$$

By Proposition 5.26, we have $h^0(F(x)|_x) = 1$. By the long sequence trick, we have $h^1(F) < \infty$ if and only if $h^1(F(x)) < \infty$ and

$$\chi(F(x)) = \chi(F) + \chi(F(x)|_x)$$

$F(x)|_x$ is torsion, so flabby, thus $h^1(F(x)|_x) = 0$ by Theorem 6.5. Altogether,

$$\chi(F(x)) = \chi(F) + 1$$

We can write $F = L + D$ where $D = \sum x_i - \sum y_i$ for some $x_i, y_i \in X$. Applying the previous argument repeatedly, we have:

$$h^1(F) < \infty \iff h^1(F + \sum y_i) < \infty$$
$$\chi(F + \sum y_i) = \chi(F) + \deg \sum y_i$$

and

$$h^1(L) < \infty \iff h^1(L + \sum x_I) < \infty$$
$$\chi(L + \sum x_i) = \chi(L) + \deg \sum x_i$$

thus $h^1(F) < \infty$ and $\chi(F) = \chi(L) + \deg D$. In the case $F = \mathcal{O}_X$, we have $\chi(\mathcal{O}_X) = \chi(L) + \deg(-E)$ where $L = \mathcal{O}_X(E)$. In this notation, $F = \mathcal{O}_X(E + D)$, thus

$$\begin{aligned}\chi(F) &= \chi(\mathcal{O}_X) + \deg(E + D)\\ &= (\dim_k \Gamma(X, \mathcal{O}_X) - h^1(\mathcal{O}_X)) + \deg F\\ &= 1 - g + \deg F \qquad \square\end{aligned}$$

Corollary 6.28.

(i) $h^0(F) \geq \deg \det F + (\text{rank } F)(1-g)$

(ii) $h^1(F) \geq -\deg \det F - (\text{rank } F)(1-g)$

Remark 6.29. Before this point, we didn't know there were sheaves F with $h^1(F) \neq 0$. Corollary 6.28 tells us if F is invertible and $\deg F$ is sufficiently negative, then $h^1(F)$ can be arbitrarily large.

Lemma 6.30. There is a canonical vector space isomorphism from $H^1(\Omega_X)$ to k.

Proof. By Lemma 6.27 and Corollary 6.28, we can find an invertible sheaf L with $h^1(L) \neq 0$ but $h^1(L(x)) = 0$ for all $x \in X$. Recall from the proof of Lemma 6.27 we have the exact sequences

$$\Gamma(X, L(x)|_x) \xrightarrow{\delta_x} H^1(M) \to \underbrace{H^1(M(x))}_{0} \tag{6.6}$$

Thus δ_x is surjective, and since $h^0(L(x)|_x) = 1$ it must also be injective. Hence $h^1(L) = 1$. From the same proof, we have the global map

$$\delta : F \otimes_{\mathcal{O}_X} \Omega_X^* \to H^1(F) \otimes_k \mathcal{O}_X$$

which is therefore an isomorphism of sheaves. Since $h^1(F) = 1$, we have $H^1(F) \otimes \mathcal{O}_X \approx \mathcal{O}_X$ as sheaves. Hence, by uniqueness of inverses in the Picard group, $F \approx \Omega_X$ as sheaves. Thus,

$$\delta : \mathcal{O}_X \to H^1(\Omega_X) \otimes \mathcal{O}_X$$

is an isomorphism. Altogether, $\delta(1) = e \otimes 1$ where e is the canonical generator of $H^1(\Omega_X)$, thence we obtain a vector space isomorphism $k \to H^1(\Omega_X)$. $\square$

The last piece of the puzzle is to show the first cohomology groups are isomorphic to zeroth cohomology groups (of some other sheaf). Results of this kind are known as 'duality'. With the results to follow, we will completely understand cohomology and can prove Riemann-Roch.

For an arbitrary locally free sheaf F on X, if $\alpha \in \text{Hom}(F, \Omega_X)$ then we have an exact sequence

$$0 \to F \xrightarrow{\alpha} \Omega_X \to Q \to 0$$

so by taking the long exact sequence, and recalling $H^1(\Omega_X) = k$, we have a linear map

$$H^1(\alpha) : H^1(F) \to k$$

Thus we have a map

$$H^1 : \mathrm{Hom}(F, \Omega_X) \to H^1(F)^*$$

Theorem 6.31. (Serre duality) H^1 is an isomorphism between $\mathrm{Hom}(F, \Omega_X)$ and $H^1(F)^*$ as k-vector spaces.

The bulk of the theorem is in the following lemma. We recall a commutative diagram has arrows which represent functions and the any composition of arrows from the same starting point and to the same end point gives the same function. We distinguish two copies of the global map δ (with respect to different sheaves) with a tilde.

Lemma 6.32. Assume we have a commutative diagram

$$\begin{array}{ccc} \tilde{\delta} : \Omega_X \otimes_{\mathcal{O}_X} \Omega_X^* & \xrightarrow{\approx} & H^1(\Omega_X) \otimes_k \mathcal{O}_X \\ {\scriptstyle \alpha\otimes 1}\uparrow & & \uparrow{\scriptstyle \lambda\otimes 1} \\ \delta : F \otimes_{\mathcal{O}_X} \Omega_X^* & \longrightarrow & H^1(F) \otimes_k \mathcal{O}_X \end{array}$$

where $\alpha \in \mathrm{Hom}(F, \Omega_X)$ and $\lambda : H^1(F) \to H^1(\Omega_X)$ is a linear map. Then α determines λ and vice versa.

Proof.
Claim: λ determines α

The top arrow is an isomorphism so we can run the arrow backwards, i.e. $\alpha \otimes 1 = (\tilde{\delta})^{-1}(\lambda \otimes 1)(\delta)$

Claim: α determines λ

From the diagram we have

$$\hat{\delta} \circ (\alpha \otimes 1) = (\lambda \otimes 1) \circ \delta$$

Thus λ is determined on the image of δ. Recall $\mathrm{im}(\delta)$ is the image sheaf contained in $H^1(F) \otimes_k \mathcal{O}_X$, which projects to give a subspace of $H^1(F)$.

Choose distinct points $(x_i)_1^N$ such that $h^1(F(D)) = 0$ where $D = \sum_i x_i$ (by Lemma 6.27). We have an exact sequence

$$0 \to \Gamma(X, F) \to \Gamma(X, F(D)) \to \Gamma(X, F(D)|_D) \xrightarrow{\delta_D} H^1(F) \to 0$$

and for each i, thus $H^1(F)$ is spanned by the images of a basis of the (finite dimensional) k-vector space $\Gamma(X, F(D)|_D)$. Thus is spanned by $\mathrm{im}(\delta)|_{x_i}$ for $i = 1, \ldots, n$. $\square$

Proof of Theorem 6.31. In the language of Lemma 6.32, we have the commutative diagram and the map $\alpha \to \lambda$ is H^1 and the inverse is given by $\lambda \to \alpha$, thus an isomorphism.

Now we come to the gem.

Corollary 6.33. (Riemann-Roch) Let F be a locally free sheaf of $\mathcal{O}_X$-modules on a smooth projective curve X. Then,

$$h^0(F) - \dim_k \operatorname{Hom}(F, \Omega_X) = \deg \det F + (\text{rank } F)(1 - g)$$

Proof. Combine Theorem 6.31 with Theorem 6.25

□

6.5. Classification of curves

In this section we use Riemann-Roch to give a classification of smooth projective curves up to isomorphism over an algebraically closed field k. The ideas in this section will be similar to those in the classification of surfaces.

First, we use deduce the form of Riemann-Roch seen in II Algebraic Geometry from Corollary 6.33.

Definition 6.34.

- A *canonical divisor* K is a divisor such that the corresponding line bundle $\mathcal{O}_X(K)$ is isomorphic to Ω_X.
- For a divisor D, $l(D) := h^0(X, \mathcal{O}_X(D))$

The bulk of the proof of Corollary 6.33 was proving the case of invertible sheaves. Indeed, we retain a lot of information by specialising to this case, which we can state in terms of divisors:

Theorem 6.35. (Divisor Riemann-Roch) Let X be a smooth projective curve of genus g and D a divisor. Then

$$l(D) - l(K - D) = \deg D + 1 - g$$

Proof. To arrive at this from Corollary 6.33, all we have to do is show $l(K - D) = \dim_k \operatorname{Hom}(\mathcal{O}_X(D), \Omega_X)$. We use the fact from algebra:

$$\operatorname{Hom}(Y \otimes X, Z) \approx \operatorname{Hom}(Y, \operatorname{Hom}(X, Z))$$

which holds for arbitrary modules X, Y, Z (proof: stare at it). We deduce this holds for sheaves, because it holds on stalks. Thus

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_X(D), \Omega_X) &\approx \mathrm{Hom}(\mathcal{O}_X(D)\otimes\Omega_X^*, \mathcal{O}_X)\\ &= (\mathcal{O}_X(D)\otimes\Omega_X^*)^*\\ &= \mathcal{O}_X(-D)\otimes\mathcal{O}_X(K)\\ &= \mathcal{O}_X(K-D)\end{aligned}$$

Thus since X is a curve, $\dim_k \mathrm{Hom}(\mathcal{O}_X(D), \Omega_X) = \dim_k \Gamma(X, \mathcal{O}_X(K-D))$ as required. □

Corollary 6.36. Let X be a curve with canonical divisor K. Then

(i) $\deg K = 2g-2$

(ii) $l(nK) \leq (2n-1)g+1$ for $n > 1$.

Proof. (i) is from taking $D = K$ in Theorem 6.35, noting also $l(0) = 1$. (ii) is from taking $D = nK$. □

The following definitions are motivated by (ii):

Definition 6.37. Let X be any smooth variety (not necessarily a curve)

- The *canonical bundle* or *sheaf of differential forms*, written ω_X is $\wedge^n\Omega_X$, where $n = \dim X$.
- A *canonical divisor* is any divisor K such that $\mathcal{O}_X(K) \approx \omega_X$.
- The *n-th plurigenus* of X is $P_n(X) := h^0(X, nK)$ where K is a canonical divisor.
- The *Kodaira dimension* $\kappa(X)$, is the smallest $k \in \mathbb{N}\cup\{-\infty\}$ such that $\sup_n P_n(X)/n^k < \infty$

Remark 6.38. On a smooth surface X, Ω_X is locally free of rank 2. Suppose the generators at $p \in X$ are dx and dy. Then ω_X is generated by $dx \wedge dy$ (since $dx \wedge dy = -dy \wedge dx$, we get everything), thus ω_X is invertible and canonical divisors exist (by the line bundle correspondence).

By Corollary 6.36, if X is a curve then κ is either 1, 0, or $-\infty$ (when $l(K) = 0$)

Corollary 6.39. For a curve X of genus g,

$$\kappa(X) = \begin{cases} -\infty & g = 0 \\ 0 & g = 1 \\ 1 & g \geq 2 \end{cases}$$

Proof. Apply Riemann-Roch to the cases $g = 0, 1$ and $g \geq 2$ □

We conclude by recalling the classification obtained in II Algebraic Geometry (phrased instead in terms of Kodaira dimension)

κ	Type
$-\infty$	conics (isomorphic to $\mathbb{P}^1$)
0	elliptic curves (isomorphic to a degree 3 curve in $\mathbb{P}^1$)
1	general type

FIG 1. *The classification of smooth curves*

Classification of curves beyond genus 1 becomes very difficult, hence 'general type'. We'll have similar disappointment in classifying surfaces, but there is much more to say about those.

Theorem 6.40. Let X be a variety. Then $\kappa(X) \leq \dim X$.

Proof. Difficult. Theorem 8.1 in Ueno [7] □

Definition 6.41. A variety X is of *general type* if $\kappa(X) = \dim X$.

There is often not much we can say about varieties of general type.

Example 6.42.

- $\kappa(\mathbb{P}^n) = -\infty$
- $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$. Thus arbitrary Kodaira dimensions are possible.

7. Surfaces

7.1. Overview

In this chapter, a *surface* refers to a smooth variety of dimension two over $\mathbb{C}$, except in Section 7.2 where the results (notably, Riemann-Roch for surfaces) hold over any algebraically closed field k.

Our goal is to classify all algebraic complex surfaces. By the end of this section, we will understand the following figure:

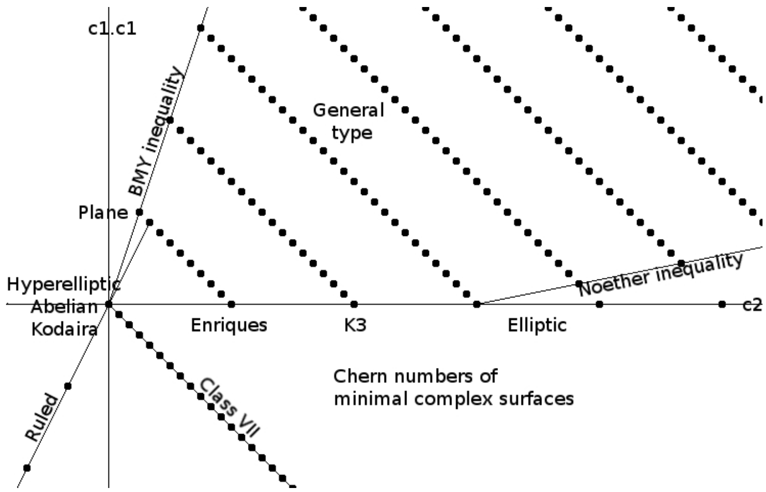

FIG 2. *Plot of the 'Chern numbers' c_1^2 against c_2, which we will define later*

Unlike with curves, classifying up to isomorphism is too much to ask. Instead we classify up to 'birational equivalence', which we recall here:

Definition 7.1. Let X and Y be varieties.

- A *rational map* f, written $f : X \dashrightarrow Y$ is a morphism from a non-empty open $U \subset X$ to Y.
- A *birational map* is a rational map with rational inverse.
- If a birational map is defined everywhere (i.e. a morphism), we call it a *birational morphism*.
- We say X and Y are *birationally equivalent*, written $X \sim Y$, if there is a rational map with rational inverse from X to Y.

Remark 7.2. Recall open sets in the Zariski topology are huge, so this is not a weak condition. From II Algebraic Geometry, we recall a birational map on curves is in fact an isomorphism.

There is a very neat description of birational morphisms between surfaces.

Definition 7.3. Let X be a surface and $x \in X$. Then there exists a surface $\hat{X}$ called the *blow-up of X at x* and a morphism $\epsilon : \hat{S} \to S$, which are unique up to isomorphism, such that

(i) the restriction of ϵ to $\epsilon^{-1}(S - \{p\})$ is an isomorphism onto $S - \{p\}$.

(ii) $\epsilon^{-1}(p) = E$ (the 'exceptional curve') is isomorphic to $\mathbb{P}^1$.

Remark 7.4. We have seen the construction of ϵ in II Algebraic Geometry.

Theorem 7.5. Let $f : X \to Y$ be a birational morphism of surfaces. Then there are a sequence of blow-ups $\epsilon_k : S_k \to S_{k-1}$, $k = 1, \dots, n$ and an isomorphism $u : S \to S_n$ such that

$$f = \epsilon_1 \circ \cdots \circ \epsilon_n \circ u$$

Proof. Theorem II.11 in Beauville [3]. □

Definition 7.6. Let X be a surface.

- $B(X)$ denotes the set of isomorphism classes of surfaces birationally equivalent to X
- If $S_1, S_2 \in B(X)$, then S_1 *dominates* S_2 if there is a birational morphism $S_1 \to S_2$.
- X is *minimal* if for every surface Y, every birational morphism $X \to Y$ is an isomorphism.

The following should not be surprising.

Proposition 7.7. Every surfaces dominates a minimal surface

Proof. Proposition II.16 in Beauville [3]

□

Thus, every surface is obtained by a finite sequence of blow-ups of a minimal surface.

7.2. Riemann-Roch for surfaces

In this section, we develop machinery required to begin the classification of surfaces. We allow k to be an arbitrary algebraically closed field, but we will switch back to $\mathbb{C}$ in the next section. Recall that irreducible divisors are co-dimension 1 subvarieties, thus divisors are now curves. Our plan is to extract information about X from curves on the surface.

Lemma 7.8. Let C be a curve on the surface X. Then locally, C is cut out by one polynomial, i.e. for all $x \in X$ and $I = I_C$

$$I_x = (f)$$

for some polynomial f dependent on x.

Proof. This is not obvious and uses Fact 5.3. If $g \in I_x$, then $V(g)$ is a finite union of irreducible components, one of which is C. This corresponds with (g) an intersection of prime ideals. Since g factors into irreducibles, one of those factors f satisfies $V(f) = C$. □

Definition 7.9. Let C and C' be two distinct irreducible curves on a surface X. The *intersection multiplicity at x* is defined,

$$m_x(C \cap C') = \dim_k \mathcal{O}_{X,x}/(f, g)$$

where C and C' are locally cut out by f and g respectively.

Remark 7.10. We know that in a local ring, the only ideals are multiples of the maximal ideal.

Example 7.11. Let $X = \mathbb{A}^2$ with coordinates x and y.

- Consider $C = V(x)$ and $C' = V(y)$. Then $\mathcal{O}_{X,0}/(x, y) \approx k$ so the intersection multiplicity is 1.
- Consider $C = V(y - x^2)$ and $C' = V(y)$. Then $\mathcal{O}_{X,0}/(y - x^2, y) \approx \mathcal{O}_{X,0}/(x)$ is 2 dimensional.

Proposition 7.12. $m_x(C \cap C')$ is finite

Proof. Assume $x \in C \cap C'$. In the notation of Definition 7.9, we choose a local affine neighbourhood such that f and g belong to a polynomial ring $k[x_1, \dots, x_n]/I$ with each x_i vanishing at x. Let m_x be the maximal ideal of $\mathcal{O}_{X,x}$ and $I(\cdot)$ be the ideal generated by a variety.

$$\begin{aligned} & V(m_x) \subset V(f \cap g) \\ (\forall i) \quad & \therefore x_i \in I(V(f \cap g)) \\ (\text{Nullstellensatz}) \quad & \therefore x_i \in \sqrt{(f \cap g)} \\ & \therefore x_i^{r_i} \in (f \cap g) \end{aligned}$$

for some integers r_i. Choosing $r = \max_i r_i$, we have $m_x^r \subset (f \cap g)$, and we know (else exercise) that $\mathcal{O}_{X,x}/(m_x)^r$ is a finite dimensional k-vector space. Thus $\mathcal{O}_{X,x}/(f, g)$ is finite dimensional. □

Definition 7.13. Let C and C' be distinct irreducible curves on X. Their *intersection number* is

$$(C, C') = \sum_{x \in C \cap C'} m_x(C \cap C')$$

Remark 7.14. The intersection of two distinct irreducible curves is a closed subset of each curve, thus a finite number of points, so this sum is finite.

Example 7.15. Recall from the proof of the Divisor-Line bundle correspondence that the ideal sheaf I_C is given by the invertible sheaf $\mathcal{O}_X(-C)$. We can define the sheaf $I_C + I_{C'}$ in the obvious way: $U \to I_C(U) + I_{C'}(U)$ for U open. We define the *intersection sheaf*

$$\mathcal{O}_{C \cap C'} = \mathcal{O}_X/(I_C + I_{C'})$$

The stalks behave, i.e.

$$(\mathcal{O}_{C \cap C'})_x = \mathcal{O}_{X,x}/(I_{C,x} + I_{C',x})$$

and we note $\mathcal{I}_{C,x} = \mathcal{O}_{X,x}$ for $x \notin C$. Thus $\mathcal{O}_{C \cap C'}$ is supported on the finite set $C \cap C'$ and at each of those points $(\mathcal{O}_{C \cap C'})_x = \mathcal{O}_{X,x}/(f, g)$, for the respective local cut-outs f and g. The conclusion is,

$$\begin{aligned} (C, C') &= \dim \Gamma(X, \mathcal{O}_{C \cap C'}) \\ &= h^0(X, \mathcal{O}_{C \cap C'}) \end{aligned}$$

Definition 7.16. For a sheaf F on any variety Y, the *Euler characteristic of F* is

$$\chi(F) := \sum_i (-1)^i h^i(Y, F)$$

which generalises the same notion on curves.

Alarm bells should be ringing as we don't even know $h^i(Y, F)$ is finite, never mind whether the series converges (i.e. the $h^i(Y, F)$ are eventually zero). The following theorems address this, and we will certainly not attempt to prove them:

Theorem 7.17. (Grothendieck's Vanishing Theorem) Let X be a variety of dimension d. Then for any F, $h^i(F) = 0$ for $i > d$.

Proof. See Grothendieck's famous 'Tohoku paper' [6]. The Wikipedia page will likely be just as inspiring to the reader as the paper itself.

□

Theorem 7.18. (Cartan-Serre Finiteness Theorem) Let X be a proper projective variety. Then $h^i(X, F)$ is finite for all i if F is coherent.

Proof. See Serre [8]. □

We also take note of the following result which we already used in the proof of Theorem 6.25:

Lemma 7.19. (Additivity of Euler) Let

$$0 \to F_1 \to F_2 \to \cdots \to F_n \to 0$$

be a long exact sequence of sheaves on a variety X. Then

$$\sum_{i \text{ odd}} \chi(F_i) = \sum_{i \text{ even}} \chi(F_n)$$

Proof. Long sequences trick with Theorem 7.17.

□

Our next result shows, intersection numbers can be entirely recovered from Euler characteristics.

Theorem 7.20. For F and G in $\mathrm{Pic}(X)$, define

$$(F, G) := \chi(\mathcal{O}_X) - \chi(F^{-1}) - \chi(G^{-1}) + \chi(F^{-1} \otimes G^{-1}) \tag{7.1}$$

Then

(i) this coincides with our previous definition in the sense that

$$(\mathcal{O}_X(C), \mathcal{O}_X(C')) = (C, C')$$

(ii) and $(\cdot, \cdot)$ is a symmetric bilinear form on $\mathrm{Pic}(X)$, i.e. $(L, G) = (G, F)$ and $(L \otimes G, H) = (L, H) + (G, H)$.

Proof. One should remind themselves that divisors correspond with $\mathrm{Pic}(X)$, so the fact this definition itself extends to $\mathrm{Pic}(X)$ is not exactly surprising. We prove (i) here, and the proof of (ii), which is non-trivial and uses Riemann-Roch for curves, is Theorem 1.4 in Beauville [3].

Consider the sequence of sheaves

$$0 \to \mathcal{O}_X(-C-C') \xrightarrow{\alpha} \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C') \xrightarrow{\beta} \mathcal{O}_X \to \mathcal{O}_{C\cap C'} \to 0 \tag{7.2}$$

where α and β are given by the following: fix sections $s \in \Gamma(X, \mathcal{O}_X(C))$ and $s' \in \Gamma(X, \mathcal{O}_X(C'))$ which are non-zero and vanish on C and C' respectively. Then

$$\alpha(t) = (t/s, -t/s')$$
$$\beta(r,t) = r/s' + t/s$$

Claim (7.2) is exact.

Let $f, g \in \mathcal{O}_{X,x}$ be local equations for C, C' at x. Taking stalks of (7.2) we have

$$0 \to \mathcal{O}_{X,x} \xrightarrow{\alpha_x} \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x} \xrightarrow{\beta_x} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/(f,g) \to 0$$

where the local maps are $\alpha_x(t) = (t/f, -t/g)$ and $\beta_x(t) = r/g + t/f$.

Clearly α_x is injective. For β_x, if $r/g + t/f = 0$ then $rf = -tg$ so since $\mathcal{O}_{X,x}$ is a UFD and f, g are irreducible we have $r = kg$ and $t = kf$ for some k and $\beta_x(k) = (r,t)$, so the kernel is the image of α_x.

With the claim proven, by the additivity of Euler characteristic on (7.2) we obtain

$$\chi(\mathcal{O}_X(-C-C')) - \chi(\mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C')) + \chi(\mathcal{O}_X) - \chi(\mathcal{O}_{C\cap C'}) = 0$$

and we have a long exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C') \to O_X(-C') \to 0$$

so applying additivity of Euler again,

$$\chi(\mathcal{O}_X(-C)) - \chi(O_X(-C) \oplus O_X(-C')) + \chi(\mathcal{O}_X(-C')) = 0. \tag{7.3}$$

Combining these and noting $\chi(\mathcal{O}_{C\cap C'}) = h^0(\mathcal{O}_{C\cap C'})$ as it is flabby, we get the required equality. $\square$

Remark 7.21.

- Although (i) provides interpretation of this symbol, (ii) allows effective computation which we will see in the remainder of this section.
- Note the RHS of (7.1) depends only on their Euler characteristic, thus if $F \approx F'$ and $G \approx G'$ then $(F, G) = (F', G')$.

Definition 7.22. Let D and D' be any divisors on X. Then

$$D.D' := (\mathcal{O}_X(D), \mathcal{O}_X(D'))$$

Corollary 7.23. (Bezout) If $C, C' \subset \mathbb{P}^2$ are curves of degree d and d' respectively, then the number of intersections counted with multiplicity is dd'.

Proof. Let C and C' be degree d and d' curves. Recall $\text{Pic}(\mathbb{P}^2) = \{\mathcal{O}_X(m) : m \in \mathbb{Z}\}$. The proof of this (Corollary 5.14) shows that any curves of the same degree have their corresponding invertible sheaves isomorphic. Let L and L' be any pair of distinct lines in $\mathbb{P}^2$, then $C \approx dL$ and $C' \approx d'L$ thus

$$\begin{aligned} C.C' &= dL.dL' \\ \text{(bilinear)} \qquad &= dd'(L.L') \\ \text{(lines intersect at a point)} \qquad &= dd' \end{aligned}$$

□

We now turn to Riemann-Roch for surfaces, quoting the following difficult result:

Theorem 7.24. (Serre duality for surfaces) Let X be a smooth projective surface and F an invertible sheaf. Then $h^2(\Omega_X) = 1$ and for $i = 0, 1, 2$ we have an isomorphism of k-vector spaces

$$H^{2-i}(\omega_X \otimes F^{-1}) \approx H^i(F)^*$$

In particular, $\chi(F) = \chi(\omega_X \otimes F^{-1})$.

Proof. Hartshorne [2] III 7.7 □

Theorem 7.25. (Riemann-Roch for surfaces) Let F be an invertible sheaf and X a smooth projective surface, then

$$\chi(F) = \chi(\mathcal{O}_X) + \frac{1}{2}(F.F - F.\Omega_X)$$

Proof. By Serre duality for surfaces, we have $\chi(\omega_X) = \chi(\mathcal{O}_X)$ and $\chi(\omega_X \otimes F^{-1}) = \chi(F)$. To make these terms appear in the definition of the intersection product, we compute

$$\begin{aligned}(F^{-1}.F \otimes \omega_X^{-1}) &= \chi(\mathcal{O}_X) - \chi(F) - \chi(\omega_X \otimes F^{-1}) + \chi(\omega_X) \\ &= 2(\chi(\mathcal{O}_X) - \chi(F))\end{aligned}$$

Also

$$(F^{-1}, F \otimes \omega_X^{-1}) = -F.F + F.\omega_X$$

and we're done. □

Corollary 7.26. (The genus formula) Let C be an smooth curve on a smooth projective surface X. The genius $g_C := h^1(C, \mathcal{O}_C)$ is given by

$$g_C = 1 + \frac{1}{2}(C.C + C.K)$$

where K is a canonical divisor on X.

Proof. We have an exact sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$. By additivity of the Euler characteristic, we have $\chi(\mathcal{O}_C) = 1 - g_C = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C))$. Applying Riemann-Roch for surfaces to $\mathcal{O}_X(-C)$, we are done.

□

Remark 7.27. We can again interpret Riemann-Roch in terms of divisors. Recall for D a divisor, $h^i(D) = h^i(X, \mathcal{O}_X(D))$. Then Serre duality says, for any canonical divisor K and $i = 0, 1, 2$, we have $h^i(D) = h^{2-i}(K - D)$ and Riemann-Roch says

$$h^0(D) + h^0(K - D) - h^1(D) = \chi(\mathcal{O}_X) + \frac{1}{2}(D.D - D.K)$$

Usually, we do not have any information about $h^1(D)$ and we use the so-called Riemann-Roch inequality

$$h^0(D) + h^0(K - D) \geq \chi(\mathcal{O}_X) + \frac{1}{2}(D.D - D.K)$$

7.3. Birational invariants

We now specialise to $k = \mathbb{C}$.

Definition 7.28. Let X be a surface.

- The *geometric genus* is $p_g(X) := h^2(X, \mathcal{O}_X)$
- The *irregularity* is $q(X) := h^1(X, \mathcal{O}_X)$.
- The *arithmetic genus* is $p_a(X) := 1 + \chi(\mathcal{O}_X)$

Remark 7.29. Suppose X were instead a curve and we defined $p_a = 1 - \chi(\mathcal{O}_X)$ and $p_g = h^1(\mathcal{O}_X)$ (i.e. the usual definition). Then $p_a = p_g$. So for surfaces, the irregularity q measures the departure from the nice situation where both types of genera coincide.

Fact 7.30. The integers q, p_g, P_n, p_a, κ are birational invariants. I.e. if $X \sim Y$ then $q(X) = q(Y)$ as with the others.

Proof. Proposition III.20 in Beauville [3]. The proof is short.

□

These invariants are far from independent from each other. Much can be said about the relations between them.

Proposition 7.31. $p_g = P_1$

Proof. Serre duality □

Definition 7.32. Given the complex structure on X, we can also define the following

- The *de-Rham cohomology* are $H^i(X, \mathbb{R})$, where $\mathbb{R}$ is the constant sheaf.
- The *Betti numbers* are $b_i(X) = \dim_{\mathbb{R}} H^i(X, \mathbb{R})$
- The *topological Euler characteristic* is defined by $\chi_{\text{top}}(X) = \sum_i (-1)^i b_i(X)$.

Remark 7.33. The quantities in Definition 7.32 are those studied by the algebraic topologists. For example, the i-th Betti number is interpreted as the number of i-dimensional holes in X.

Example 7.34. Let $X = \mathbb{P}^2$. Using the techniques of algebraic topology, one can show $(b_0, b_1, b_2, b_3, b_4) = (1, 0, 1, 0, 1)$. A *fake projective plane* is a surface with the same Betti numbers as $\mathbb{P}^2$ but is not isomorphic to it. There are exactly 50 (non-isomorphic) fake projective planes, see [11]. One can show a fake projective plane is always of 'general type' – to be discussed later.

Fact 7.35. $q(X) = \frac{1}{2} b_1(X)$

Proof. This is a result from Hodge theory, see Weil [9].

□

The relationship between the usual Euler characteristic $\chi(X) := \chi(\mathcal{O}_X)$ and the topological genus $\chi_{\text{top}}(X)$ is given by the following, which we will not attempt to prove:

Theorem 7.36. (Noether) For X a smooth complex surface and K a canonical divisor,

$$\chi(X) = \frac{1}{12}(K.K + \chi_{\text{top}}(X))$$

Remark 7.37. $K.K$ and b_2 are not birational invariants, though their images under birational maps is understood, see Proposition II.3 in [3].

Definition 7.38. Let X be a surface. The *Chern numbers* c_1^2 and c_2 are $K.K$ and $\chi_{\text{top}}(X)$ respectively.

Remark 7.39.

- There is much more to this definition in higher dimensions which we have swept under the rug.
- Chern numbers are not birational invariants, but for most of the classes in our classification we can determine them.

Proposition 7.40. (Noether's inequality Version 0) Let X be a minimal surface. Then

$$p_g \leq \frac{1}{2} c_1^2 + 2.$$

Proof. See Theorem 3.1 in [10]. □

Corollary 7.41. (Noether's inequality) $5c_1^2 - c_2 + 36 \geq 0$

$$\frac{1}{12}(c_1^2 + c_2) = p_g - q + 1 \quad \text{(Theorem 7.36)}$$

$$\therefore \frac{1}{12}(c_1^2 + c_2) \leq \frac{1}{2}c_1^2 + 2 \quad \text{(Proposition 7.40)}$$

and the conclusion follows.

Remark 7.42. This corresponds to 'Noether's inequality' as written in Figure 7.1. A discussion of the surfaces that satisfy Noether's inequality can be found in VII.10 in [10].

We also have:

Theorem 7.43. (Bogomolov-Miyaoka-Yau) $c_1^2 \leq 3c_2$

Proof. Theorem 4.1 in [10] □

Remark 7.44. This is the 'BMY inequality' as written in Figure 7.1. It was shown in [11] that there surfaces with $c_1^2 = 3c_2 = 9n$ for every n. Mumford found a fake projective plane with $c_1^2 = 3c_2$, see [12].

We now define more numbers:

Definition 7.45. Let X be a surface.

- The *sheaf of differential i-forms* is the sheaf $\wedge^i \Omega_X$.
- The *Hodge numbers* $h^{i,j}$ are $h^j(X, \Omega_X^i)$ for $0 \leq i, j \leq 2$

Remark 7.46. We often write the Hodge numbers in a diamond

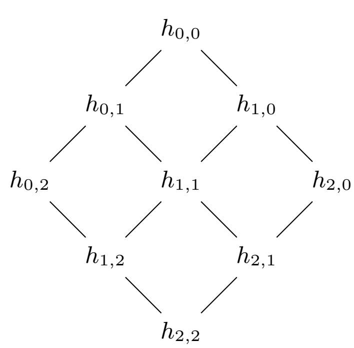

Fig 3. *A Hodge diamond*

Proposition 7.47. We have:

(i) $h^{i,j} = h^{2-i,2-j}$.

(ii) $h^{0,0} = h^{2,2} = 1$.

(iii) All 'outer' Hodge numbers are birationally invariant (i.e. all but $h^{1,1}$).

(iv) Blowing up a point increases $h^{1,1}$ by one.

Proof. (i) is Serre duality. (ii) is clear given (i). (iii) uses the same proof as Proposition III.20 in Beauville [3]. (iv) is tricky. □

Most of the numbers we have seen so far can be written in terms of Hodge numbers:

Fact 7.48.

(i) $q = h^{0,1}$, $p_g = h^{0,2}$, $p_a = h^{0,2} - h^{0,1}$.

(ii) $\chi(X) = h^{0,2} - h^{0,1} + 1$.

(iii) $b_1 = h^{2,1} + h^{1,2}$, $b_2 = h^{2,0} + h^{1,1} + h^{0,2}$.

7.4. Ruled surfaces

Definition 7.49. A surface is *ruled* if it is birational equivalent to $C \times \mathbb{P}^1$ for a smooth curve C. If $C = \mathbb{P}^1$ then the surface is *rational*.

Ruled surfaces will give a large class of surfaces in our classification.

Proposition 7.50. Let X be ruled over C. Then

$$\begin{aligned} q(X) &= g(C) \\ p_g(X) &= 0 \\ P_n(X) &= 0, \qquad n \geq 2 \end{aligned}$$

Proof. Proposition III.21 in Beauville [3]. □

This proposition has an incredible converse!

Theorem 7.51. (Enriques) Let X be a surface with (a) $P_4 = P_6 = 0$ or (b) $P_{12} = 0$. Then X is ruled.

Proof. Theorem VI.17 in Beauville [3].

□

Corollary 7.52. Let X be a surface. Then $\kappa(X) = 0$ if and only if X is ruled.

Another fantastic criterion exists for rational surfaces:

Theorem 7.53. (Castelnuovo's Criterion) If X is a surface with $q = P_2 = 0$ then X is rational.

Proof. Theorem V.1 in Beauville [3].

□

Corollary 7.54. A ruled surface with $p_g = 0$ is rational.

Remark 7.55. In the proof, one shows that $P_2 = 0$ implies $p_g = 0$. However, there are many non-rational surfaces with $q = p_g = 0$, as we will see later. A classification of all surfaces with $p_g = 0$ and $q \geq 1$ is in Chapter VI of Beauville.

Proposition 7.56. Let $X \sim C \times \mathbb{P}$ be a ruled surface with $g = g_C$. Then,

(i) $c_2 = 4 - 4g$ and $c_1^2 = 2c_2$

(ii) X has Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ 0 & & 2 & & 0 \\ & g & & g & \\ & & 1 & & \end{array}$$

Proof. We prove (i). A useful fact from topology states $\chi_{\text{top}}(A \times B) = \chi_{\text{top}}(A)\chi_{\text{top}}(B)$. Thus, $c_2 = \chi_{\text{top}}(C)\chi_{\text{top}}(\mathbb{P})$. One can also show that for curves, $\chi_{\text{top}}(C) = 2\chi(C)$, thus $c_2 = 2(2 - 2g)$. By Proposition 7.50 we have $p_g = 0$ and $q = g$ thus

$$
\begin{aligned}
\text{(Noether)} \qquad \frac{1}{12}(c_1^2 + c_2) &= 1 + p_g - q \\
&= 1 - g \\
\therefore c_1^2 &= 2c_2
\end{aligned}
$$

□

Remark 7.57. This proves the line on Figure 7.1 for ruled surfaces.

7.5. Surfaces with Kodaira dimension one

Definition 7.58. A surface X is *elliptic* if there is a smooth curve B and a surjective morphism $\phi : X \to B$ such that the preimage of a generic point is a smooth elliptic curve. This map is called an *elliptic fibration*.

Remark 7.59. Generic point here means 'holds on a non-empty open subset'. A non-generic point is called a 'singular point'. There is a complete description of what the fibres at the singular points can be, see Section V.7 in [10].

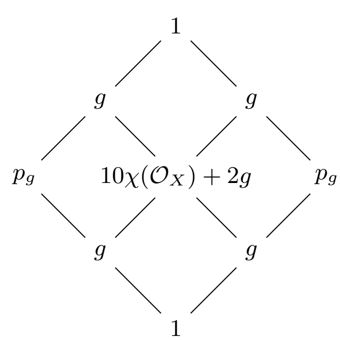

FIG 4. *Hodge diamond of an elliptic surface over a base curve B, where $g = g_B$ and $C = 10\chi(\mathcal{O}_X) + 2g$. For a proof, see Section 6.10 in [13].*

An elliptic surface can have Kodaira dimension $-\infty$, 0, or 1. This usually depends on the genus of the base curve B. For example, if $g_B \geq 2$ (i.e. a general type curve) then one can show $\kappa(X) = 1$.

Definition 7.60. A *properly elliptic surface* is an elliptic surface of dimension 1.

Theorem 7.61. Let X be a surface.

(i) If X is elliptic, then $c_1^2 = 0$ and $c_2 \geq 0$.

(ii) If X is minimal and $\kappa = 1$, then X is elliptic.

Proof. (i) is proved along the way to proving (ii). (ii) boils down to giving a canonical morphism into some projective space whose image is a curve, then showing that curve has genus 1. See Proposition IX.2 in Beauville [3]. The proof relies on the last third of II Algebraic Geometry more than the ideas we have developed. □

7.6. Surfaces with Kodaira dimension zero

Theorem 7.62. Let X be a minimal surface with $\kappa = 0$. Then X belongs to one of the following cases:

(i) $p_g = 0; q = 0$, then $2K \sim 0$ and X is a 'Enriques surface'.

(ii) $p_g = 0; q = 1$, then X is a 'hyperelliptic surface'.

(iii) $p_g = 1; q = 0$, then $K \sim 0$ and X is a '$K3$ surface'.

(iv) $p_g = 1; q = 2$, then X is an 'Abelian surface'. where $\sim$ denotes linear equivalence of divisors.

Proof. Theorem VIII.2 in [3]. □

A *hyperelliptic surface* is a product of two elliptic curves quotiented by a finite group of automorphisms. One can show that there are only seven possibilities for this group: C_2, C_2^2, C_3, C_3^2, C_4, $C_4 \times C_2$, or C_6. From the figure below, one can deduce the Chern numbers are $c_1 = c_2 = 0$.

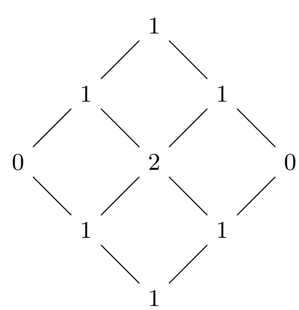

FIG 5. *Hodge diamond of a hyperelliptic surface*

A *K3 surface* is a surface with trivial canonical bundle and $q = 0$. They are named after the three K's of algebraic geometry: Kummer, Kähler, Kodaira. If X is K3, we have $c_1^2 = 0$ and $\chi(X) = 1 + p_g - q = 2$, thus by Noether's theorem we have $c_2 = 24$. By Fact 7.35 and Serre duality, we have the Betti numbers $b_1 = b_3 = 0$ and from the previous sentence we deduce $(b_0, b_1, b_2, b_3, b_4) = (1, 0, 22, 0, 1)$.

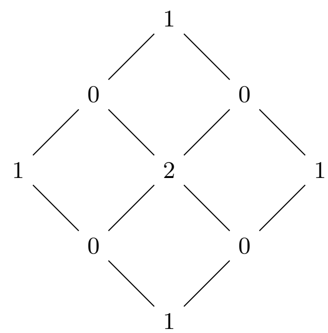

FIG 6. *Hodge diamond of a K3 surface*

Example 7.63. The following are (non-trivially) K3 surfaces,

- a smooth quartic in $\mathbb{P}^3$, e.g. $x^4 + y^4 + z^4 + w^4 = 0$,
- the intersection of a quadric and a cubic in $\mathbb{P}^4$,
- the intersection of three quadrics in $\mathbb{P}^5$.

An *Enriques surface* is a surface with $q = 0$ and K non-trivial but $2K \sim 0$. This can be shown to imply $p_g = 0$, and given that we deduce the Chern numbers $c_1^2 = 0$ and $c_2 = 12$. One can show all Enriques surfaces are elliptic and arise as quotient of a K3 surface by a group of order 2 acting without fixed points. They have $p_g = q = 0$, thus serve as a class of counterexamples to the obvious strengthening of Castelnuovo's criterion. A calculation similar to that done with K3 surfaces gives $c_1 = 0$ and $c_2 = 12$.

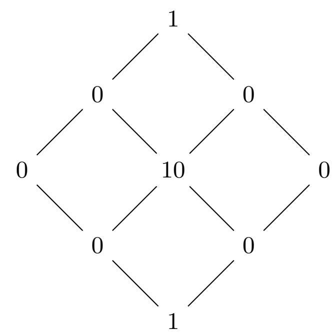

Fig 7. *Hodge diamond of an Enriques surface*

Example 7.64. Consider the K3 surface X given by $x^4 + y^4 + z^4 + w^4 = 0$ in $\mathbb{P}^3$ and let T be the order 4 automorphism $(x, y, z, w) \to (x, iy, -z, -iw)$. Then T^2 has eight fixed points. Blowing up these eight points and taking the quotient by T^2 gives a K3 surface with a fixed-point free involution T, and the quotient of this by T is an Enriques surface.

A *complex torus* is any complex surface homeomorphic to a power of S^1. An *Abelian variety* is any variety which is a complex torus. They are a generalisation of elliptic curves and are studied in algebraic number theory. Together with the K3 surfaces, abelian surfaces are the only 'Calabi-Yau' manifolds of dimension two (manifolds with $K \sim 0$). In particular, $c_1 = 0$. One can show $p_g = 1$ and $q = 2$, thus $\chi(\mathcal{O}_X) = 0$ so by Noether's theorem $c_2 = 0$.

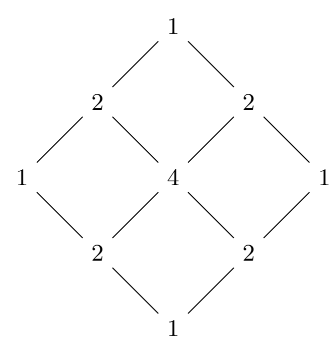

Fig 8. *Hodge diamond of an Abelian surface*

7.7. Compact complex surfaces

In this section we address the the following question: what about surfaces which are not varieties?

Definition 7.65.

- A *complex manifold* is a manifold whose charts in $\mathbb{C}^n$ and the transition maps are holomorphic.
- A complex manifold is *compact* if the image of the atlas is a compact topological space.
- A *complex surface* is a complex manifold of dimension two.
- A complex manifold is *algebraic* if it is a variety.

Example 7.66.

(i) $\mathbb{CP}^2$ is a compact complex manifold.

(ii) A *Hopf surface* is a compact complex surface defined as follows: quotient $\mathbb{C}^2\backslash\{0\}$ by group action $z \to \frac{1}{2}z$. This gives a non-algebraic surface.

Remark 7.67. One can more generally define canonical line bundles, plurigenera, Hodge numbers, and all the other invariants discussed for compact complex manifolds. All of these definitions agree with the corresponding definitions for varieties.

The most surprising fact in this article is that the classification of compact complex surfaces is essentially the same as the classification of algebraic surfaces we have given here:

Class	κ	Algebraic?	c_1^2	c_2	Smallest n s.t. $K^n \sim 0$
rational	$-\infty$	Always	8, 9	4, 3	
class VII		Never	≤ 0	≥ 0	
ruled of genus $g \geq 1$		Always	8(1-g)	4(1-g)	
Enriques	0	Always	0	12	2
hyperelliptic		Always	0	0	2
Primary Kodaira		Never	0	0	1
Secondary Kodaira		Never	0	0	2, 3, 4, 6
K3		Sometimes	0	24	1
Abelian		Always	0	0	1
Elliptic	1	Always	0	≥ 0	
General type	2	Sometimes	> 0	> 0	

FIG 9. *Enriques-Kodaira classification of minimal compact complex manifolds*

Remark 7.68. Mumford [18] showed the analogous classification of surfaces in positive characteristic is similar to Figure 7.7, except in characteristic 2 and 3 there a couple more cases to consider.

Example 7.69. The Hopf surface is class VII.

As well as the definitions of the invariants for varieties agreeing with those for complex compact surfaces, the theorems we have seen for algebraic surfaces also hold more generally for complex compact surfaces. In particular, the Noether and BMY inequalities, which restrict the Chern numbers of surfaces. A good question is whether these are the best inequalities possible. The following result gives an answer: half of the time, almost always.

Theorem 7.70. (Persson) Let $n, m \in \mathbb{N}$ such that

(i) $n + m \equiv 0 \mod 12$

(ii) $n \leq 2m$ (note: not $3m$)

(iii) $5n - m + 36 \geq 0$

then there exists a minimal compact complex surface of general type with $c_1^2 = n$ and $c_2 = m$, except possibly if $n - 2m + 3k = 0$ where $k = 2$, or $k = 19$, or k is odd and less than 16.

Proof. Theorem 9.1 in [10].

□

Remark 7.71.

- (i) is a requirement of Noether's theorem. (iii) is Noethers inequality. (ii) is half of the BMY inequality.
- It is currently unknown whether there exist positive integers n, m satisfying (i), (iii), and $n \leq 3m$ such that no compact complex surface has Chern numbers $c_1^2 = n$ and $c_2 = m$.

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Index

Abelian variety, 63
affine variety, 7
algebraic, 64
arbitrary, 15
arithmetic genus, 55

Betti numbers, 55
birational map, 47
birational morphism, 47
birationally equivalent, 47
blow-ups, 47

canonical bundle ω_X, 45
canonical divisor, 44
canonical divisor K , 45
category, 3
Chern numbers, 56
coherent, 17
cohomology, 32
compact, 64
complex manifold, 64
complex surface, 64
complex torus, 63
contravariant functor, 4
Cousin problem, 30

de-Rham cohomology groups, 55
decent, 10
determinant sheaf, 16
differential map, 19
direct image functor , 34
direct limit, 34
direct sum of $\mathcal{R}$ by I, 15
divisors, 22
dominates, 48
dual sheaf, 20

elliptic, 60
elliptic fibration, 60
Enriques surface, 62
Euler characteristic, 37, 50
exact sequence of abelian groups, 5
exact sequence of sheaves, 13
exact sequence of sheaves of rings, 13
exterior product, 6

factorial, 22
fake projective plane, 56
field of rational functions $k(X)$, 23
finitely generated, 16
finitely presented, 16
flabby, 31
functor, 3

general type, 46
genus, 37
geometric genus, 55
germ equivalent, 9
global sections functor, 17
group of invertible fractional ideal sheaves, 23
group of principal parts, 29

higher direct image, 35
Hodge numbers, 57
Hopf surface, 64
hyperelliptic surface, 61

ideal sheaf, 18
image sheaf, 14
inclusion map, 12
injective, 12, 32
intersection multiplicity, 49
intersection number, 50
intersection sheaf, 50

inverse image, 35
invertible, 15
irreducible divisor, 22
irregularity, 55
isomorphism, 12

K3 surface, 62
kernel, 13
Kodaira dimension, 45

line bundle, 15
local parameter, 7
locally free, 15

minimal, 48
morphism, 7, 11

ordinary, 30

patching condition, 10
Picard group, 20
plurigenus, 45
presentation, 16
presheaf, 11
presheaf of abelian groups, 8
principal sections, 30
projective variety, 7
properly elliptic surface, 60
pull-back sheaf, 35
push-forward sheaf, 34

quasi-affine variety, 7
quasi-coherent, 17
quasi-projective variety, 7
quotient sheaf, 14

rank n, 15
rational, 58
rational function, 7
rational map, 47
rationally equivalent, 26
regular function, 7
restriction map, 8
ruled, 58

section, 9
sheaf, 11
sheaf exterior product, 15
sheaf of $\mathcal{R}$-modules, 14
sheaf of abelian groups, 10
sheaf of differential forms, 45
sheaf of differential i-forms, 57
sheaf of differentials, 19
sheaf of fractional ideals, 23
sheaf of principal parts, 30
sheaf tensor product, 15
sheafification, 14
short exact sequence of abelian groups, 5
stalk, 9
subsheaf, 12
support, 27
surface, 46
surjective, 12

tensor product of R-modules, 6
tensor product sheaf, 14
the sheaf of regular functions, 11
topological Euler characteristic, 55
torsion, 26
torsion-free, 27

vanishing set , 7
variety, 7