Mathematical Analysis of the Navier Stokes Equations

Based on lectures by Edriss Titi at the University of Cambridge (2024)
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1. Integral equations

We begin by seeing how PDEs arise from physics, which will motivate the notion of a 'weak solution'. Suppose charge is distributed in an insulated ball $\Omega \subset \mathbb{R}^3$ according to the known function $f : \Omega \to \mathbb{R}$. An engineer would find the electric potential $\phi : \Omega \to \mathbb{R}$ by solving the Poisson equation

$$\begin{cases} \Delta\phi = -f & \text{on } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

To a physicist, the classical laws are summarised as 'the Lagrangian of the system must be minimised', i.e. ϕ minimises

$$J(\psi) = \int_\Omega \mathcal{L}(\psi, x)dx$$

over all twice-differentiable ψ such that $\psi = 0$ on $\partial\Omega$, where

$$\mathcal{L}(x) = \frac{1}{2}|\nabla\phi|^2 - f\phi$$

is the Lagrangian given by kinetic energy minus potential energy (and is not the subject of these notes). A necessary and sufficient condition for ϕ to be a minimiser is $J(\phi + \epsilon\gamma) \geq J(\phi)$ for all $\epsilon \neq 0$ and a fixed (but arbitrary) $\gamma \in C^\infty(\Omega)$ vanishing on the boundary. Thus,

$$\int_\Omega \frac{1}{2}|\nabla\phi|^2 + \frac{\epsilon}{2}|\nabla\gamma|^2 + \epsilon\nabla\phi\cdot\nabla\gamma - f\phi - \epsilon f\gamma \geq \int_\Omega \frac{1}{2}|\nabla\phi|^2 - f\phi$$

Rearranging,

$$\frac{\epsilon^2}{2}\int|\nabla\gamma|^2 + \epsilon\int\nabla\phi\cdot\nabla\gamma - \epsilon\int f\gamma \geq 0$$

Dividing by $|\epsilon| \neq 0$,

$$\text{sign}(\epsilon)\left(\frac{\epsilon}{2}\int|\nabla\gamma|^2 + \int\nabla\phi\cdot\nabla\gamma - f\gamma\right) \geq 0$$

Taking $\epsilon \to 0$, we deduce for all $\gamma \in C^\infty(\Omega)$ vanishing on the boundary,

$$\int_\Omega \nabla\phi\cdot\nabla\gamma = \int_\Omega f\gamma$$

Integrating by parts,

$$\int_\Omega -\Delta\phi\gamma = \int_\Omega f\gamma \tag{1.1}$$

It follows that, necessarily, $-\Delta\phi = f$ *almost everywhere*. Though we have not discussed sufficiency, it is not surprising that such a condition comes out of an integral equation. The moral of the story is that the fundamental equations in physics arise as integral equations, not PDEs. This suggests that thinking about PDE's in our current 'classical' sense might just be wrong.

2. Distributions

The physical problem in the previous section translates to condition involving integration against an arbitrary function γ in (1.1). The pointwise requirement, $-\Delta\phi = f$ a.e., arises as a result. The space of choices of γ is referred to as *test functions* for the problem. In this case we considered $\gamma \in C^\infty$, and often by density (e.g. in L^2), this is equivalent taking any other C^k space, $k \geq 0$, as our test functions. However, in the general setting to be developed, allowing γ to range over a larger class of functions is a stronger requirement. A typical choice of test functions is defined as follows:

Notation 2.1. Throughout, Ω is an open bounded subset in $\mathbb{R}^d$, unless stated otherwise.

Definition 2.2. The *space of test functions* $D(\Omega)$ is $C_c^\infty(\Omega)$ with the following topology: $\gamma_n \to 0$ in D iff

(i) $\exists K \subset \Omega$ compact with $\text{supp}(\gamma_n) \subset K$ for all n.

(ii) For any $\alpha \in \mathbb{N}^d$, $(\partial^\alpha \gamma_n) \to 0$ uniformly on K, where $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$.

Remark 2.3. When one speaks of a 'test function' without context, they generally refer to an element of $D(\Omega)$.

Definition 2.4. The *space of distributions* $D'(\Omega)$ is the dual of $D(\Omega)$. For $p \geq 1$, there is an embedding $L^p \hookrightarrow D'(\Omega)$ given by $f \to \phi_f$ where $\phi_f(\gamma) = \int f\gamma$, which makes sense by Holder's inequality. Elements of $D'(\Omega)$ are *distributions*.

Definition 2.5. If $\phi \in D'(\Omega)$ and $\phi = \phi_f$ in D' for some $f \in L^p$, $p \geq 1$, then we say ϕ is *represented by a function* and often do not distinguish between f and ϕ.

Thus if we write the equation

$$-\Delta\phi = f \text{ in } D' \tag{2.1}$$

for functions ϕ and f in some L^p space, we mean,

$$\int_\Omega -\Delta\phi\gamma = \int_\Omega f\gamma$$

for all $\gamma \in D(\Omega)$. However, $D'(\Omega)$ contains many distributions which are not given by integration against an L^p function.

Example 2.6. Consider $\delta \in D'(\Omega)$ given by $\delta(\gamma) = \gamma(0)$. Indeed, δ is continuous (if $\gamma_n \to 0$ in D, then $\gamma_n(0) \to 0$) and linear. However, one can easily show that there is no $p \geq 1$ and $f \in L^p$ such that $\delta(\gamma) = \int f\gamma$ for all $\gamma \in D(\Omega)$.

Notation 2.7. If X is a vector space with dual X', the action of X' on X is written

$$\langle f, x\rangle_{X',X} := f(x)$$

for $f \in X'$ and $x' \in X$. We often omit the subscript when the context is clear.

Remark 2.8. When X is a Hilbert space and $f \in X'$ is represented by $u_f \in X$ (Riesz Representation Theorem), we have $\langle f, x\rangle_{X',X} = \langle u_f, x\rangle_X$, whence this notation is derived.

As it stands, (2.1) doesn't make sense for every ϕ in D', as ϕ may not even be represented by a function. Our next task is to define the derivative of an element in D'. The way we do this follows a common blueprint in this field: for $f \in C^1(\Omega)$ and $\gamma \in D(\Omega)$, we have

$$\int_\Omega (\partial_1 f)\gamma dx = -\int_\Omega f(\partial_1\gamma)dx$$

by integration by parts. The LHS makes sense for $f \in C^2(\Omega)$, but the RHS makes sense in a much larger space e.g. $f \in L^p$, $p \geq 1$. We can go even further and write the previous display as

$$\langle T_{\partial_1 f}, \gamma\rangle = -\langle T_f, \partial_1\gamma\rangle$$

Thus we define,

Definition 2.9. If $T \in D'(\Omega)$ then the *distributional* or *weak derivative* $\partial_i T \in D'(\Omega)$ is defined by

$$\langle \partial_i T, \gamma \rangle := -\langle T, \partial_i \gamma \rangle$$

for $1 \leq i \leq d$.

Remark 2.10.

- Equation (2.1) now has a meaning for all ϕ and f in $D'(\Omega)$.
- The previous discussion shows distributions derivatives align with our usual notion of derivatives of functions i.e. $\partial_i T_f = T_{\partial_i f}$ when $f \in C^1(\Omega)$.

2.1. Sobolev spaces

Although the general setting of distributions comes up time and time again, a key tool in PDE theory is having the right spaces at hand. Sometimes, we do want to treat distributions which are represented by functions specially. The following spaces are of particular interest in that regard.

Definition 2.11. Let $p \geq 1$ and $m \in \mathbb{N}$. Suppose $f \in L^p(\Omega)$ and all $(\partial_\alpha f)_{|\alpha| \leq m}$ are distributions represented by L^p functions, where $|\alpha| := |\alpha_1| + \cdots + |\alpha_d| \leq m$. Then we say f belongs to the *Sobolev space* $W^{m,p}(\Omega)$, which is a Banach space equipped with the norm

$$\|f\|_{W^{m,p}} = \sum_{0 \leq |\alpha| \leq m} \|\partial_\alpha f\|_{L^p}$$

We will typically use the following special case of Sobolev spaces:

Definition 2.12. The *Sobolev Hilbert spaces* are

$$H^m(\Omega) := W^{m,2}(\Omega)$$

Example 2.13. The most common domain we deal with is $\Omega = (0,1)^d$. Solving over this domain is called a *periodic boundary condition* because solutions extend to periodic functions on $\mathbb{R}^d$.

For $\phi \in L^2(\Omega)$, we can Fourier expand:

$$\widehat{\phi}_k := \int_\Omega \phi(x) e^{-2\pi i k \cdot x} dx, \quad k \in \mathbb{Z}^d$$
$$\|\phi\|_{L^2}^2 = \sum_{k \in \mathbb{Z}^d} |\widehat{\phi}_k|^2$$

Moreover,

$$\begin{aligned}
\widehat{\partial_i \phi}_k &= \int_\Omega (\partial_i \phi) e^{-2\pi i k \cdot x} dx \\
&= \int_\Omega \phi (2\pi i k_i e^{-2\pi i k \cdot x}) dx \qquad \text{(definition of weak derivative)} \\
&= 2\pi i k_i \widehat{\phi}_k
\end{aligned}$$

Thus, if $\partial_i \phi \in L^2$,

$$||\partial_i \phi||_{L^2}^2 = 4\pi^2 \sum_{k \in \mathbb{Z}^d} k_i^2 |\widehat{\phi}_k|^2 \tag{2.2}$$

Moreover, (2.2) holds if $\partial_i \phi$ is not assumed to be in L^2, i.e. RHS finite implies LHS finite, and the justification of this is left to the reader. We also have that $\partial_i \phi \in L^2$ implies $\phi \in L^2$. Thus, generalising to higher orders, we can now state an equivalent characterisation of $H^m(\Omega)$.

Notation 2.14. Unless otherwise stated, $\Omega = (0, 1)^d$, i.e. we treat the case of periodic boundary conditions.

Definition 2.15. (Periodic characterisation of H^s) For $s \in \mathbb{R}_{\geq 0}$,

$$H^s(\Omega) := \left\{ \phi \in L^2 : \sum_{k \in \mathbb{Z}^d} |k|^{2s} |\widehat{\phi}_k|^2 < \infty \right\}$$

In particular, this definition agrees with the previous definition when s is an integer. The inner product is given by,

$$\langle \phi, \psi \rangle_{H^s} = \sum_{k \in \mathbb{Z}^d} |k|^{2s} \widehat{\phi}_k \widehat{\psi}_k$$

Theorem 2.16. Let $0 < s < t$. Then

(i) $H^t \subset H^s$

(ii) the inclusion map $i : H^t \to H^s$ is continuous

(iii) the inclusion map is compact

Proof. For (i) and (ii), note $||\cdot||_{H^s} \leq ||\cdot||_{H^t}$. For (iii), let (ϕ_n) be a sequence in H^t with $||\phi_n||_{H^t} \leq 1$. We are required to show (ϕ_n) has a H^s-convergent subsequence. By Banach-Alaoglu Theorem, there is a subsequence converging weakly to some $\phi \in H^t$. Without loss of generality, we may assume $\phi_n \to \phi$ weakly in H^t. This implies the convergence of the Fourier coefficients: for any k, $\widehat{\phi}_{n,k} \to \widehat{\phi}_k$ as $n \to \infty$. Then for $N \in \mathbb{N}$ fixed,

$$\begin{aligned}
||\phi_n - \phi||_{H^s}^2 &= \sum_{k\in\mathbb{Z}^d} |k|^{2s}|\widehat{\phi}_{n,k} - \widehat{\phi}_k|^2 \\
&= \sum_{|k|\leq N} |k|^{2s}|\widehat{\phi}_{n,k} - \widehat{\phi}_k|^2 + \sum_{|k|>N} |k|^{2s}|\widehat{\phi}_{n,k} - \widehat{\phi}_k|^2 \\
&\leq N^{2s} \sum_{|k|\leq N} |\widehat{\phi}_{n,k} - \widehat{\phi}_k|^2 + \sum_{|k|>N} \frac{1}{N^{2(t-s)}}|k|^{2s}|\widehat{\phi}_{n,k} - \widehat{\phi}_k|^2 \\
&\leq N^{2s} \sum_{|k|\leq N} |\widehat{\phi}_{n,k} - \widehat{\phi}_k|^2 + \frac{1}{N^{2(t-s)}}||\phi_n - \phi||_{H^t}^2
\end{aligned}$$

For N sufficiently large, the second term is arbitrarily small independently of n, since $||\phi_n||_{H^t} \leq 1$. For N fixed, the first term tends to zero as $n \to \infty$. The conclusion follows. □

Theorem 2.17. (Sobolev Interpolation) Let $a < b$ and $s = (1-\theta)a + \theta b$ for $\theta \in (0, 1)$. Then,

$$||\phi||_{H^s} \leq ||\phi||_{H^a}^{1-\theta}||\phi||_{H^b}^{\theta}$$

Proof. For $k \in \mathbb{Z}^d$,

$$|k|^{2s}|\widehat{\phi}_k|^2 = \left(|k|^{2a}|\widehat{\phi}_k|^2\right)^{1-\theta} \left(|k|^{2b}|\widehat{\phi}_k|^2\right)^{\theta}$$

Applying Holder's inequality with $p^{-1} = 1 - \theta$ and $q^{-1} = \theta$,

$$\begin{aligned}
||\phi||_{H^s}^2 &\leq \left(\sum_{k\in\mathbb{Z}^d} |k|^{2a}|\widehat{\phi}_k|^2\right)^{1-\theta} \left(\sum_{k\in\mathbb{Z}^d} |k|^{2b}|\widehat{\phi}_k|^2\right)^{\theta} \\
&= ||\phi||_{H^a}^{2(1-\theta)}||\phi||_{H^b}^{2\theta}
\end{aligned}$$

□

We state without proof the following results:

Theorem 2.18. (Sobolev L^p embedding) Let $\frac{1}{p} \geq \frac{1}{2} - \frac{s}{d}$ with $p < \infty$. Then there is a constant C depending only on Ω, s, p such that for all $\phi \in H^s$,

$$||\phi||_{L^p} \leq C||\phi||_{H^s}$$

In particular, $H^s \subset L^p$.

Theorem 2.19. (Sobolev C^k embedding) Let $k \geq 0$ and $s > k + \frac{d}{2}$ then there is a constant C depending only on Ω, s, k such that for all $\phi \in H^s$,

$$\|\phi\|_{C^k} \leq C\|\phi\|_{H^s}$$

where $\|\phi\|_{C^k} := \sum_{|\alpha|\leq k}\|\partial^\alpha \phi\|_\infty$. In particular, $H^s \subset C^k$.

Remark 2.20. This holds for more general domains, but in this case we haven't defined $H^s(\Omega)$ for s not an integer.

Example 2.21.

(i) Suppose $d = 2$. Then by interpolation,

$$||u||_{H^{1/2}} \leq ||u||_{L^2}^{1/2}||u||_{H^1}^{1/2}$$

Thus by embedding,

$$||u||_{L^4} \leq C||u||_{L^2}^{1/2}||u||_{H^1}^{1/2} \tag{2.3}$$

(ii) Suppose $d = 3$. Then by embedding,

$$||u||_{L^6} \leq C||u||_{H^1}$$

and

$$\begin{aligned} ||u||_{L^3} &\leq C'||u||_{H^{1/2}} \\ &\leq C'||u||_{L^2}^{1/2}||u||_{H^1}^{1/2} \qquad \text{(interpolation)} \end{aligned}$$

and

$$\begin{aligned} ||u||_{L^4} &\leq C''||u||_{H^{3/4}} \\ &\leq C''||u||_{L^2}^{1/4}||u||_{H^1}^{3/4} \end{aligned} \tag{2.4}$$

> Although these look like random inequalities now, we will use these estimates a lot when considering the Navier Stokes equations. The difficulty of the study of Navier Stokes in $d = 3$ compared to $d = 2$ can be summarised by the weight of the H^1 exponent in (2.4) being more than in (2.3). One can show that in $d = 3$, the only dimensionally consistent choice of exponents (with dimensionless constant) is that in (2.4), so there is no hope of improving it.

Our method for finding simple estimates in Example 2.21 was

(i) get $||\cdot||_{L^p} \leq C||\cdot||_{H^s}$

(ii) get $||\cdot||_{H^s} \leq ||\cdot||_{H^a}||\cdot||_{H^b}$

But (i) doesn't work for $p = \infty$. It turns out, if we 'skip the middleman', we can get (ii):

Theorem 2.22. (Agmon) Let $u \in L^2$ and $0 < a < d/2 < b$ such that $(1-\theta)a + \theta b = d/2$. Then,

$$||u||_{L^\infty} \leq C||u||_{H^a}^{1-\theta}||u||_{H^b}^{\theta}$$

Proof. Let $\hat{u}$ be the Fourier transform of u. By the Fourier inversion theorem,

$$\begin{aligned}
\|u\|_{L^\infty} &\leq C\int_{\mathbb{R}^d}|\hat{u}(k)|dk \\
&\leq C\int_{|k|\leq R}|\hat{u}(k)|dk + C\int_{|k|>R}|\hat{u}(k)|dk \\
&= C\int_{|k|\leq R}|\hat{u}(k)|\frac{(1+|k|^2)^{a/2}}{(1+|k|^2)^{a/2}} + C\int_{|k|\geq R}|\hat{u}(k)|\frac{(1+|k|^2)^{b/2}}{(1+|k|^2)^{b/2}} \\
&\leq C\|u\|_{H^a}\int_{|k|\leq R}(1+|k|^2)^{-a}dk + C\|u\|_{H^b}\int_{|k|\geq R}(1+|k|^2)^{-b}
\end{aligned}$$

where the last inequality comes from Cauchy-Schwarz. The remaining integrals are spherically symmetric, and one can easily show they are finite and that R can be chosen so that the two terms are equal. This yields the desired result. □

Definition 2.23. The r-dimensional vector field Sobolev space is defined by

$$(H^s)^r := \{f \in (L^2(\Omega))^r : ||f||_{H^s_r} < \infty\}$$

for $s \in \mathbb{R}_{\geq 0}$, where

$$||f||_{(H^s)^r} := \sum_{i=1}^{r} ||f_i||_{H^s}$$

Remark 2.24. This is the standard definition of a product of finitely many normed spaces. There are many ways to choose an equivalent norm to the above since all norms in $\mathbb{R}^r$ are equivalent. Usually we drop the r in notation since any result for H^s will obviously generalise to $(H^s)^r$.

Example 2.25. If $u \in H^s$ then $\nabla u := (\partial_1 u, \ldots, \partial_d u) \in (H^s)^d$.

Theorem 2.26. (Poincare) Let $u \in H^1(\Omega)$. Then,

$$||u - \overline{u}||_{L^2} \leq C||\nabla u||_{L^2}$$

where $\overline{u} = |\Omega|^{-1} \int_\Omega u(x)dx$ and $C = 4\pi^2$ is an absolute constant.

Proof. We may without loss of generality assume $\overline{u} = 0$ by subtracting a constant from u. In particular, $\widehat{u}_0 = 0$.

$$\begin{aligned} ||\partial_j u||_{L^2} &= \sum_{k \in \mathbb{Z}^d} |\widehat{\partial_j u}_k|^2 \\ &= \sum_{k \in \mathbb{Z}^d} |2\pi i k_j \widehat{u}_k|^2 \\ &= 4\pi^2 \sum_{k \in \mathbb{Z}^d} |k_j|^2 |\widehat{u}_k|^2 \\ \therefore ||\nabla u||_{L^2} &= 4\pi^2 \sum_{k \in \mathbb{Z}^d} |k|^2 |\widehat{u}_k|^2 \\ &\geq 4\pi^2 \sum_{k \in \mathbb{Z}^d} |\widehat{u}_k|^2 \qquad (\widehat{u}_0 = 0) \end{aligned}$$

□

Example 2.27. The Navier Stokes Equations in $\mathbb{R}^d$ are

$$\begin{cases} \partial_t u_k - \nu \Delta u + (u \cdot \nabla) u_k + \partial_k p = f_k, & k = 1, \ldots, d \\ \nabla \cdot u = 0 \end{cases}$$

where the velocity field u and scalar pressure p are unknown with periodic boundary conditions, f is the known forcing term, and ν is the known visocsity constant. Note $(u\cdot\nabla)u_k = \nabla\cdot(uu_k)$, so integrating the equation we have:

$$\int_\Omega \partial_t u_k - \nu\Delta u + \nabla\cdot(uu_k) + \partial_k p dx = \int_\Omega f_k dx$$
$$\therefore \frac{d}{dt}\overline{u_k} = \overline{f}_k$$
$$\therefore \frac{d}{dt}\overline{u} = \overline{f}$$

where the many terms vanish in the second equality by the periodic boundary condition (first applying the divergence theorem to the second and third terms). Assuming the forces have average zero, $\overline{u}$ is constant, and by choosing the write coordinates we may assume it is zero, thus $||u||_{L^2} \leq C||\nabla u||_{L^2}$.

We often want to use the esimate '$||u||_{L^2} \leq C||\nabla u||_{L^2}$', specifically without the $\overline{u}$ term appearing in Poincare's inequality. Thus we define spaces of average zero functions which will be key spaces in the future.

Remark 2.28. Defining the correct spaces in PDE theory is key. We started with spaces of continuous functions in earlier study which are tricky to work with, and in these notes we have discussed Sobolev spaces which are nicer to work with, and we will continue defining refinements of these spaces until we can solve the Navier Stokes equation over that space.

Definition 2.29. For a normed space $X \subset L^1$, define $\dot{X} = \{u \in X : \overline{u} = 0\}$ with the same norm as X.

Remark 2.30. Shortly, we will only consider the dot-spaces, since we always make this assumption in our treatment of the Navier Stokes equations.

Example 2.31. Poincare's inequality says the norms $||\cdot||_{H^1}$ and $||\nabla\cdot||_{L^2}$ are equivalent on $\dot{H}^1$. The latter is called the *Dirichlet norm.*

2.2. Fourier Series in $D'(\Omega)$

Definition 2.32. For $\mu \in D'(\Omega)$ and $k \in \mathbb{Z}^d$, the *Fourier coefficient* is

$$\widehat{\mu}_k := \langle \mu, e^{-2\pi i k\cdot x}\rangle_{D',D}$$

Remark 2.33. We don't say anything about convergence of $\sum_{k\in\mathbb{Z}^d}\widehat{\mu}_k e^{-2\pi ik\cdot x}$

Proposition 2.34. For $\mu \in D'$ and $\phi \in D$,

$$\langle \mu, \phi\rangle_{D',D} = \sum_{k\in\mathbb{Z}^d} \widehat{\mu}_k(\widehat{\phi}_k)^*$$

where $*$ denotes complex conjugation.

Proof. $\phi = \sum_{k\in\mathbb{Z}^d}\widehat{\phi}_k e^{-2\pi ik\cdot x}$, so.

$$\langle \mu, \phi\rangle_{D',D} = \sum_{k\in\mathbb{Z}^d}\langle \mu, \widehat{\phi}_k e^{-2\pi ik\cdot x}\rangle_{D',D} \qquad = \sum_{k\in\mathbb{Z}^d}\widehat{\mu}_k(\widehat{\phi}_k)^* \qquad (\mu \text{ continuous})$$

□

Definition 2.35. $H^{-1}(\Omega) := (\dot{H}^1)'$, the normed space dual.

Remark 2.36. In general, one can define the negative Sobolev spaces in this way, but we only need H^{-1}.

Proposition 2.37. For $\mu \in H^{-1}$, there exists a unique $\mu' \in \dot{D}'$ such that

$$\langle \mu, \phi\rangle_{H^{-1},\dot{H}^1} = \langle \mu', \phi\rangle_{D',D} \qquad \forall \phi \in \dot{D}$$

Proof. The statement is more complicated than the proof. Since $\dot{D} \subset \dot{H}^1$, we have $H^{-1} \subset \dot{D}'$, so $\mu \in \dot{D}'$ so under this inclusion $\mu' = \mu$ is a choice. It is unique because $\dot{D}$ is dense in $\dot{H}^1$. □

Proposition 2.38. Let $\mu \in H^{-1}$. Then

(i) $\forall \phi \in \dot{H}^1$,

$$\langle \mu, \phi\rangle_{H^{-1},\dot{H}^1} = \sum_{k\neq 0}\widehat{\mu}_k(\widehat{\phi}_k)^*$$

(ii) We have

$$||\mu||^2_{H^{-1}} = \sum_{k\neq 0}|k|^{-2}|\widehat{\mu}_k|^2$$

In particular, the RHS is finite. Conversely, every sequence $(b_k)_{k\neq 0} \in \mathbb{R}$ with $\sum_{k\neq 0}|k|^{-2}|b_k|^2 < \infty$ corresponds to a $\mu \in H^{-1}$ with $\widehat{\mu}_k = b_k$ for all $k \neq 0$.

Proof. (i) follows from Proposition 2.37, Proposition 2.34, and density of D in H^1. For (ii), we follow a similar idea to the proof that $(\ell^p)' = \ell^q$ for $p^{-1} + q^{-1} = 1$. First observe

$$\begin{aligned} ||\mu||_{H^{-1}} &= \sup_{||\phi||_{H^1}=1} \langle \mu, \phi \rangle_{H^{-1},H^1} \\ &= \sup_{||\phi||_{H^1}=1} \sum_{k\neq 0} \widehat{\mu}_k (\widehat{\phi}_k)^* && \text{(i)} \\ &\leq \sqrt{\left(\sum_{k\neq 0} |k|^{-2} |\widehat{\mu}_k|^2\right) \cdot 1} && \text{(Cauchy-Schwarz)} \end{aligned}$$

and conversely, define

$$\phi_N = \sum_{|k|\leq N} |k|^{-2} (\widehat{\mu}_k)^* e_k \in H^1$$

where $e_k(x) = e^{-2\pi i k\cdot x}$. Then

$$||\phi_N||_{H^1}^2 = \sum_{|k|\leq N} |k|^{-2} |\widehat{\mu}_k|^2$$

and

$$\langle \mu, \phi_N \rangle = \sum_{|k|\leq N} |k|^{-2} |\widehat{\mu}_k|^2 = ||\phi_N||_{H^1}^2$$

By definition,

$$\begin{aligned} &|\langle \mu, \phi_N \rangle| \leq ||\mu||_{H^{-1}} ||\phi_N||_{H^1} \\ \therefore \quad &||\phi_N||_{H^1} \leq ||\mu||_{H^{-1}} \end{aligned}$$

Taking $N \to \infty$, we deduce $\sum_{|k|\neq 0} |k|^{-2} |\widehat{\mu}_k|^2 < \infty$. Thus $\phi := \sum_{k\neq 0} |k|^{-2} (\widehat{\mu}_k)^*$ defines an element of $\dot{H}^1$. Applying μ, we obtain $||\mu||_{H^{-1}} \leq \sum_{k\neq 0} |k|^{-2} |\widehat{\mu}_k|^2$ as required. The last part of (ii) is straightforward. ∎

3. Time independent Stokes problem

We recall the Navier Stokes equations (NSE) in $\mathbb{R}^d$:

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases}$$

where u is the velocity field, ν is the viscosity, p is the scalar pressure, and f is the forcing term. The unknowns are u and p.

Suppose we are in a domain with length scale L, velocity scale U, pressure scale P, and time scale $T = L/U$. Then

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f$$

Looking at the scales of these terms:

$$\frac{U}{L/U} \qquad \nu \frac{U}{L^2} \qquad U\frac{U}{L} \qquad \frac{P}{L} \qquad F$$

Dividing by U^2/L,

$$1 \qquad \frac{\nu}{UL} \qquad 1 \qquad \frac{P}{U^2} \qquad \frac{FL}{U^2}$$

The *Reynolds number* $\mathrm{Re} = UL/\nu$ describes the behaviour of the flow. When $\mathrm{Re} >> 1$, the $-\nu\Delta u$ term can be neglected. We study the case $\mathrm{Re} << 1$ where the dimensionless terms can be neglected. In this case, we have the Stokes problem:

$$\begin{cases} -\nu \Delta u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases}$$

3.1. The Right Spaces

Define the function spaces on Ω,

$$\begin{aligned} T &:= \{\phi : \phi \text{ is a polynomial of trigonometric functions}\} \\ \mathcal{V} &:= \{\phi \in \dot{T}^d : \nabla \cdot \phi = 0\} \\ H &:= \{\phi \in (\dot{L}^2)^d : \nabla \cdot \phi = 0\} \\ V &:= \{\phi \in (\dot{H}^1)^d : \nabla \cdot \phi = 0\} \end{aligned}$$

where the product spaces are obtained via Definition 2.23, and define the norms

$$\begin{aligned} ||\phi||_H &= ||\phi||_{L^2} \\ ||\phi||_V &= ||\nabla \phi||_{L^2} \end{aligned}$$

Remark 3.1.

(i) $||\cdot||_V$ is an equivalent norm to $||\cdot||_{H^1}$ by Poincare's inequality.

(ii) $\mathcal{V}$ is dense in V and H.

(iii) We will often use the following pattern to prove results about H and V: Prove the result for $\mathcal{V}$, which is dense. Then extend the result to the whole space. Since $\mathcal{V}$ is a very nice spacec (e.g. C^∞ functions), this will be very handy.

(iv) We have $V \hookrightarrow H \hookrightarrow H' \hookrightarrow V'$, thus solving a PDE in V is a weaker requirement than solving it over H.

Definition 3.2.

- We say $u \in V'$ is a *weak solution* to the Stokes problem if
$$-\nu\Delta u + \nabla p = f \quad \text{ in } V'$$
for some pressure $p \in \dot{L}^2$ and $f \in V'$.
- We say $u \in H'$ is a *strong solution* to the Stokes problem if
$$-\nu\Delta u + \nabla p = f \quad \text{ in } H$$
for some pressure $p \in \dot{H}^1$ and $f \in H$.

H is a closed subspace of $(\dot{L}^2)^d$ and V is a closed subspace of $(\dot{H}^1)^d$, thus they are Hilbert spaces. In the case of V, the H^1 inner product is not inherited since we use a different (but equivalent norm). Thus,

Definition 3.3. The inner product on V is $((\phi,\psi)) := \int_\Omega \nabla\phi\cdot\nabla\psi$ for $\phi,\psi \in V$.

Notation 3.4. For $f,g \in L^2$, $(f,g) := \langle f,g\rangle_{L^2}$, i.e. we use round brackets to refer to L^2.

Our plan to solve the Stokes problem is as follows. We currently have unknowns u and p in the same equation. We will decouple the problem into two equations, one for u and one for p, then solve them separately. Our next theorem is what will allow us to do exactly that.

Theorem 3.5. (Helmholtz Decomposition for H^{-1}) Let $v \in (H^{-1})^d$. Then there exists a unique $w \in (H^{-1})^d$ with $\nabla\cdot w = 0$ and $p \in \dot{L}^2$ such that
$$v = w + \nabla p \tag{3.1}$$

Proof. We know elements of H^{-1} are uniquely determined by their Fourier coefficients (Proposition 2.34), so we'll determine what the coefficients of w and p must be, then show the series converge.

$$\widehat{v}_k = \widehat{w}_k + ik\widehat{p}_k, \qquad k \neq 0 \tag{3.2}$$

Taking the Euclidean dot product in $\mathbb{R}^d$,

$$\widehat{v}_k \cdot ik = \widehat{w}_k \cdot ik - |k|^2 \widehat{p}_k \tag{3.3}$$

Since $\nabla \cdot w = 0$, we have

$$ik \cdot \widehat{w}_k = 0, \qquad \forall k \neq 0 \tag{3.4}$$

Combining (3.3) and (3.4),

$$\begin{aligned} \widehat{v}_k \cdot ik &= -|k|^2 \widehat{p}_k \\ \therefore \quad \widehat{p}_k &= -\frac{\widehat{v}_k \cdot ik}{|k|^2} \end{aligned}$$

Putting this back into (3.2), we obtain

$$\widehat{w}_k = \widehat{v}_k - ik\frac{\widehat{v}_k \cdot ik}{|k|^2}$$

Thus p and w are uniquely determined, provided we show they exist in the correct spaces. Recall

$$\begin{aligned} v \in H^{-1} &\iff \sum_{k \neq 0} |k|^{-2} |\widehat{v}_k|^2 < \infty \\ p \in \dot{L}^2 &\iff \sum_{k \neq 0} |\widehat{p}_k|^2 < \infty, \ \widehat{p}_0 = 0 \end{aligned}$$

Indeed,

$$\begin{aligned} \sum_{k \neq 0} \left| \frac{\widehat{v}_k \cdot ik}{|k|^2} \right|^2 &\leq \sum_{k \neq 0} \frac{|\widehat{v}_k|^2 |ik|^2}{|k|^4} \\ &< \infty \end{aligned}$$

and $w \in H^{-1} \iff \sum_{k \neq 0} \frac{|\widehat{w}_k|^2}{|k|^2} < \infty, \ \widehat{w}_0 = 0$ is similarly shown. □

Remark 3.6.

(i) $w \perp \nabla p$ in H^1 so the decomposition is orthogonal.

(ii) $(H^{-1})^d = (H_\sigma^{-1})^d \oplus (H_g^{-1})^d$ where $(H_\sigma^{-1})^d = \{v : \nabla \cdot v = 0\}$ and $(H_g^{-1})^d = \{\nabla p : p \in \dot{L}^2\}$.

(iii) The projection $P_\sigma : (H^{-1})^d \to (H_\sigma^{-1})^d$ is called the *Helmholtz projection.*

We now state the corresponding Helmholtz decomposition theorems for other spaces. The proofs are very similar to the H^{-1} case because in each case, Fourier coefficients determine the distribution. One then does a similar computation to show the distributions obtained lie in the correct spaces, which we omit.

Theorem 3.7. (Helmholtz for L^2) Let $v \in (L^2)^d$. Then there exists a unique $w \in H$ and $p \in H^1$ such that

$$v = w + \nabla p$$

with $(w, \nabla p)_{H^1} = 0$.

Notation 3.8. We now omit the dot on in notation, so all function spaces are average zero unless otherwise stated, e.g. $D = \dot{D}$ and $L^2 = \dot{L}^2$. Only average zero functions will be of use to us, but in most cases the results we give have non-average zero analogues (e.g. subtract the average).

Theorem 3.9. (Helmholtz for D') Let $\mu \in (D')^d$. Then there exists a unique $w \in (D')^d$, $p \in D'$ such that

$$\mu = w + \nabla p$$

with $\nabla \cdot w = 0$.

Theorem 3.10. (Helmholtz for D) Let $\phi \in (D)^d$. Then there exists a unique $w \in (D)^d$, $p \in D$ such that

$$\phi = w + \nabla p$$

Proof. We deduce this from L^2 Helmholtz. Since $\phi \in L^2$, we have $\phi = w + \nabla p$ with $w \in V$ and $p \in H^1$. It suffices to show $w, p \in D$. Since $w \in H^1$ and $\phi \in H^1$, we have $\nabla p \in H^1$. Taking a derivative,

$$\partial_j \phi = \partial_j w + \partial_j \nabla p$$

Let $\partial_j \phi = w' + \nabla p'$ be the L^2-Helmholtz decomposition of $\partial_j \phi$. Then

$$(\partial_j w - w') + \nabla(\partial_j p - p') = 0$$

Therefore, by uniqueness of L^2 Helmholtz, for some $q \in H^1$,

$$\begin{aligned} \partial_j w - w' &= \nabla q \\ \therefore \quad 0 &= \Delta q \\ \therefore \quad 0 &= |k|^2 \widehat{q}_k, \qquad k \in \mathbb{Z}^d \end{aligned}$$

Thus $q = 0$, so $w' = \partial_j w$ and $p' = \partial_j p$. In particular, $w \in H^2$. We now repeat the argument to get $\nabla p \in H^2$, and inductively this continues. □

Theorem 3.11. (de Rham) For $l \in (D')^d$ and $\phi \in (D)^d$, let l_i and ϕ_i denote the projections onto the ith component in the image. Define $\langle l, \phi \rangle_d := \sum_{i=1}^d \langle l_i, \phi_i \rangle_{D',D}$. Then,

$$\langle l, \phi \rangle_d = 0 \quad \forall \phi \in \mathcal{V} \iff l = \nabla p \text{ for some } p \in \dot{D}'$$

Proof.
" $\Leftarrow$" if $l = \nabla p$ then

$$\begin{aligned} \langle \nabla p, \phi \rangle_d &= -\langle p, \nabla \cdot \phi \rangle_{D',D} && \text{(integration by parts)} \\ &= 0 && (\phi \in \mathcal{V}) \end{aligned}$$

" $\Rightarrow$" Each $l_i \in D$ has a Fourier expansion, thus we obtain a Fourier-like expansion of l,

$$l = \sum_{k \neq 0} \widehat{l}_k e^{-2\pi i k \cdot x}$$

where each $\widehat{l}_k \in \mathbb{C}^d$. Let $\phi \in \mathcal{V}$, then $\nabla \cdot \phi = 0$ so taking Fourier coefficients,

$$\begin{aligned} ik \cdot \widehat{\phi}_k &= 0 && (\forall k \neq 0) \\ \therefore \quad k \cdot \widehat{\phi}_k &= 0 \end{aligned}$$

We also have

$$
\begin{aligned}
0 &= \sum_{j=1}^{d} \langle l_j, \phi_j \rangle_{D',D} \\
&= \sum_{j=1}^{d} \sum_{k \neq 0} \widehat{l}_{j_k} (\widehat{\phi}_{j_k})^* \\
&= \sum_{k \neq 0} \widehat{l}_k \cdot \widehat{\phi}_k \qquad (3.5)
\end{aligned}
$$

Now we choose ϕ in a clever way. Let $c \in \mathbb{C}^d$ and consider $\phi = ce^{-2\pi i K \cdot x}$ for a fixed $K \neq 0$. Then

$$
\begin{aligned}
\phi \in \mathcal{V} &\iff \nabla \cdot ce^{-2\pi i K \cdot x} = 0 \\
&\iff K \cdot c = 0 \qquad \text{(Fourier coefficients)}
\end{aligned}
$$

and for such ϕ, by (3.5),

$$0 = \widehat{l}_K \cdot c$$

Thus $\widehat{l}_K \perp K^\perp$ (in $\mathbb{R}^d$) so $\widehat{l}_K$ is parallel to K, i.e. $\widehat{l}_K = \mu_K K$ for all $K \neq 0$. But for any $p \in D'$,

$$\widehat{\nabla p}_K = iK\widehat{p}_K, \qquad K \neq 0$$

so $l = \nabla p$ where $\widehat{p}_k = \mu_k$ for all k. Lastly one checks there indeed exists $p \in D'$ with $\widehat{p}_k = \mu_k$. □

Our next result is an application of this theorem. We show that every $\mu \in V'$ extends to an element of D'. Note that $V \not\subset D$ and $D \not\subset V$, so it is more accurate to say $\mu|_{D \cap V}$ extends to an element on D'. We note also that $D \cap V = \{\phi \in D : \nabla \cdot \phi = 0\} =: D_\sigma$.

Corollary 3.12. Let $\mu \in V'$. Then there exists $\mu' \in (D')^d$ such that

$$\langle \mu', \phi \rangle_{(D')^d,(D)^d} = \langle \mu, \phi \rangle_{V',V} \qquad (3.6)$$

for all $\phi \in D_\sigma$. Moreover μ' is unique up to a gradient ∇p, for some $p \in D'$.

Proof. By Helmholtz for D, we have the orthogonal decomposition $(D)^d = D_\sigma \oplus D_g$, where $D_g := \{\nabla p : p \in D\}$. We can define μ_0 on D_σ by (3.6) and $\mu'(d_\sigma + d_g) = \mu_0(d_\sigma)$. Since $\mu_0 \in (D_\sigma)'$, we have $\mu' \in (D')^d$.

If $\langle \mu', \phi \rangle = 0$ for all $\phi \in D_\sigma \supset \mathcal{V}$, then by de Rham, $\mu' = \nabla p$. □

3.2. Existence and Uniqueness

Theorem 3.13. The following problem has a unique solution (u, p):

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } H^{-1} \\ f \in H^{-1}, \nu > 0 & \text{fixed} \\ u \in V, p \in L^2 & \text{unknown} \end{cases}$$

Moreover, $||u||_V = \frac{1}{\nu}||P_\sigma f||_{H^{-1}} \leq \frac{1}{\nu}||f||_{H^{-1}}$

Proof. Let $f \in H^{-1}$, then $f = f_\sigma + \nabla q$ with $f_\sigma \in H^{-1}$ and $q \in L^2$ by Helmholtz. So,

$$-\nu\Delta u + \nabla p = f_\sigma + \nabla q$$

Note that $\nabla \cdot \Delta u = \Delta(\nabla \cdot u) = 0$, thus by uniqueness of Helmholtz decomposition,

$$\begin{cases} -\nu\Delta u = f_\sigma \\ p = q \end{cases}$$

Taking Fourier coefficients,

$$-\nu|k|^2\widehat{u}_k = \widehat{f}_{\sigma k}$$
$$\therefore \quad \widehat{u}_k = \frac{-1}{\nu|k|^2}\widehat{f}_{\sigma k}$$

which uniquely determines u and

$$\begin{aligned} ||u||_V^2 &= \sum |k|^2|\widehat{u}_k|^2 \\ &= \frac{1}{\nu^2}\sum_{k\neq 0}\frac{|\widehat{f}_{\sigma k}|^2}{|k|^2} \\ &= \frac{1}{\nu^2}||f_\sigma||_{H^{-1}}^2 \\ \therefore \quad ||u||_V &\leq \frac{1}{\nu}||f||_{H^{-1}} \end{aligned}$$

the last inequality coming from the fact an orthogonal projection can only decrease norm. □

Remark 3.14. When $f \in H^{-1}$ we showed the Stokes equation holds in H^{-1}. If we assumed $f \in L^2$, the equality of Fourier series again implies the equation holds in L^2. Moreover, repeating the method of estimating $||u||_V$, we obtain $||u||_{H^2} = \frac{1}{\nu}||f_\sigma||_{L^2}$. More generally, $||u||_{H^s} = \frac{1}{\nu}||f_\sigma||_{H^{s-2}}$. We see that the solution is more regular than the input data and this phenomenon is known as *elliptic regularity.*

Example 3.15. We won't always be able to find the Fourier coefficients of the solution to a PDE (hence deduce uniqueness). Here is another way to deduce uniqueness. Suppose u_i, p_i, $i = 1, 2$ are solutions. Then $u = u_1 - u_2 \in V$ and $p = p_1 - p_2 \in L^2$ solve

$$-\nu\Delta u + \nabla p = 0$$

in H^{-1}. By Helmholtz decomposition, $\nabla p = 0$ and $\Delta u = 0$. We need $u = 0$. Let $\phi \in \mathcal{V}$, then

$$\begin{aligned} 0 &= \langle \Delta u, \phi \rangle_{H^{-1},H^1} \\ &= \langle \nabla u, \nabla \phi \rangle_{H^{-1},H^1} \end{aligned}$$

by definition of the derivative. But $\nabla u \in L^2$, so

$$0 = \int_\Omega \nabla u \cdot \nabla \phi$$

Since $\mathcal{V}$ is dense in V, we can take a sequence $\phi_n \to \phi$ in V, thus $\nabla \phi_n \to \nabla \phi$ in L^2. Therefore

$$0 = \int_\Omega |\nabla w|^2$$

and the conclusion follows.

3.3. Spectral Theory for the Solution Operator

Suppose we have the Stokes problem

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } H \\ f \in H & \text{fixed} \\ u \in V, p \in L^2 & \text{unknown} \end{cases}$$

which has the stronger assumption that $f \in H$ and $\nu = 1$. Then by Helmholtz decomposition, $p = 0$ (as $f = f_\sigma$). Thus the problem is equivalent to

$$\begin{cases} -\Delta u = f & \text{in } H \\ f \in H & \text{fixed} \\ u \in V & \text{unknown} \end{cases}$$

Theorem 3.13 shows the solution map

$$\begin{aligned} S : H &\to H^2_\sigma \\ f &\to u \end{aligned}$$

is well defined and in light of Remark 3.14, an isometry ($||u||_{H^2} = ||f||_{L^2}$). In this section, we prove several properties of the operator S.

Proposition 3.16. $S : H \to H^2_\sigma$ is onto.

Proof. Let $v \in H^2_\sigma$, then seek $h \in H$ such that

$$-\Delta v = h$$

But $\Delta v \in H$ so there is nothing to do. □

We can also view S as a map $S : H \to H$.

Proposition 3.17. $S : H \to H$ is self adjoint.

Proof. Let $f, g \in L^2$.

$$\begin{aligned} (S(f), g) &= (u, g) && (S(f) = u) \\ &= (u, -\nabla v) && (v = S(g)) \\ &= (\nabla u, \nabla v) && \text{(IBP)} \\ &= (-\Delta u, v) && \text{(IBP)} \\ &= (f, S(g)) \end{aligned}$$

□

Proposition 3.18. $S : H \to H$ is positive definite

Proof.

$$\begin{aligned}(S(f), f) &= (u, -\Delta u) \\ &= (\nabla u, \nabla u) \\ &\geq 0\end{aligned}$$

with equality iff $\nabla u = 0$ i.e. $u = 0$. □

Corollary 3.19. There is an orthonormal basis $(\phi_k) \subset H$ of eigenfunctions of $S : H \to H$,

$$S(\phi_k) = \mu_k \phi_k$$

with $\mu_1 \geq \mu_2 \geq \cdots > 0$ and $\mu_k \to 0$ as $k \to \infty$.

Proof. Spectral Theorem for Compact Self-Adjoint operators on a Hilbert space. □

Remark 3.20.

- Although $S : H \to H^2_\sigma$ is an isometry, $S : H \to H$ is not, so indeed the eigenvalues are not all norm 1. Going forwards, we consider $S : H \to H$ unless stated otherwise.
- We have $S\phi_k = \mu_k \phi_k$ and we know S maps into H^2_σ, thus $\phi_k \in H^2_\sigma$. In other words, the eigenfunctions are more regular than general elements of H.

Lemma 3.21. Let $u \in H^2$, $v \in V$. Then

$$(-\Delta u, v) = (\nabla u, \nabla v)$$

Proof. This is integration by parts, which we have used already, but we give a justification here. Suppose $v \in \mathcal{V}$, then

$$\begin{aligned}(-\Delta u, \phi) &= \langle -\Delta u, \phi \rangle_{D',D} \\ &= \langle \nabla u, \nabla \phi \rangle_{D',D} && \text{(definition)} \\ &= (\nabla u, \nabla \phi)\end{aligned}$$

The last equality is because ∇u is represented by a function. For any $v \in V$, we can take $v_n \in \mathcal{V}$ with $v_n \to v$ in V, then taking limits we are done.

□

Definition 3.22. Define the Stokes operator $A := S^{-1} : H_\sigma \to H$ and its domain $D(A) = H^2_\sigma$.

Remark 3.23. This looks complicated, but recall $A = -\Delta$.

Proposition 3.24. The Stokes operator A can be extended to a map on all of V, $\overline{A} : V \to V'$ and

$$||\overline{A}u||_{V'} = ||u||_V$$

Proof. By Lemma 3.21, for all $u \in D(A)$, $v \in V$,

$$(Au, v) = ((u, v))$$

Au is an element of V' and represented by an L^2 function (because $u \in H^2_\sigma$), so the L^2-action is the V'-action, i.e. $(Au, v) = \langle Au, v\rangle_{V',V}$. Talking about L^2-action of Au only makes sense for $u \in H^2_\sigma = D(A)$, but the RHS makes sense for all $u \in V$. Thus we define

$$\overline{A} : V \to V'$$
$$(\overline{A}u)(v) = ((u, v))$$

Then $\overline{A}|_{D(A)} = A$ and by Cauchy-Schwarz on V, $||Au||_{V'} \le ||u||_V$. But $||(Au)(u)||_V = ||u||^2$, so in fact $||Au||_{V'} = ||u||_V$.

□

Notation 3.25. We drop the overline in $\overline{A}$ in notation and consider A itself as a map from V to V'.

The eigenfunctions of A are (ϕ_k), the eigenfunctions of S. The corresponding eigenvalues are $\lambda_k := \mu_k^{-1}$, satisfying $\lambda_1 \le \lambda_2 \le \cdots \to \infty$. A well known result that we won't use or prove quantifies this rate:

Theorem 3.26. (Weyl asymptotics) $\lambda_k \sim k^{2/d}$

Definition 3.27. For $\alpha > 0$, A^α is defined on the eigenfunction basis (ϕ_k), by

$$A^\alpha \phi_k := \lambda_k^\alpha \phi_k$$

We define the domains or *spectral Sobolev spaces*,

$$D(A^\alpha) := \{u \in D' : \nabla \cdot u = 0, \quad ||u||_{A^\alpha} < \infty\}$$

where

$$||u||_{A^\alpha}^2 := \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |\langle u, \phi_k \rangle_{D',D}|^2$$

Remark 3.28. Let $\alpha \geq 0$. Then $D(A^\alpha) \subset H$ so for $u \in D(A^\alpha)$, $||u||_{A^\alpha} = \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |(u, \phi_k)|^2 = ||A^\alpha u||_{L^2}$, as usual making the canonical identification between a distribution in H and the L^2 function representing it. Moreover $D(A^\alpha)$ is a Hilbert space with inner product $\langle f, g \rangle_{A^\alpha} = \sum_{k=1}^{\infty} \lambda_k^{2\alpha} (f, \phi_k)(g, \phi_k)^*$

Example 3.29. We show $D(A^1) = H_\sigma^2$, thus consistent with our previous definition, i.e.

$$u \in H_\sigma^2 \iff u \in H, \quad \sum_1^{\infty} \lambda_k^2 |(u, \phi_k)|^2 < \infty$$

"$\Rightarrow$" If $u \in H_\sigma^2$, then $u \in H$ and

$$||Au||_H^2 = ||\sum_1^{\infty} (u, \phi_k) \lambda_k \phi_k||_{L^2}^2 = \sum_1^{\infty} \lambda_k^2 |(u, \phi_k)|^2 < \infty$$

"$\Leftarrow$" This is a delicate situation. $|| \cdot ||_{H^2}$ is defined in terms of the Fourier coefficients, but we have an eigenfunction expansion, and going between the two could be messy. It's better to use what we know: $u \in H$. Let $u_k := (u, \phi_k)$, then

$$\begin{aligned} \sum_k \lambda_k^2 |u_k|^2 &= ||\sum_k \lambda_k u_k \phi_k||_{L^2} \\ &= ||\sum_k A(u_k \phi_k)||_{L^2} \qquad (3.7) \\ &< \infty \end{aligned}$$

We cannot simply take the A out because we don't know $\sum_k u_k \phi_k \in H_\sigma^2$ yet. The idea is to show $-\Delta u = \sum_k A(u_k \phi_k)$ directly. We know the RHS is in L^2 by the above display, so we're done (after, $||\Delta u||_{L^2} = ||u||_{H^2}$ by writing both down in terms of Fourier coefficients and the LHS is finite). Consider the action on a test function $\gamma \in D$.

$$\begin{aligned}\langle \sum_k A(u_k\phi_k), \gamma\rangle_{D',D} &= \sum_k \langle A(u_k\phi_k), \gamma\rangle_{D',D} \\ &= \sum_k (Au_k\phi_k, \gamma) \qquad (Au_k\phi_k \text{ represented by } L^2) \\ &= \sum_k (u_k\phi_k, A\gamma) \qquad (A \text{ self adjoint}) \\ &= \langle \sum_k u_k\phi_k, -\Delta\gamma\rangle_{D',D} \qquad (\text{reversing steps})\end{aligned}$$

Thus $\sum_k A(u_k\phi_k) = -\Delta u$ in D', therefore $-\Delta u$ is represented by an L^2-function, i.e. $-\Delta u \in L^2$.

The reader is encouraged to think about why the above method doesn't generalise easily to non-integer powers of A. However, the generalisation is true.

Definition 3.30. On a space X, the (possibly infinite) norms $||\cdot||, ||\cdot||' : X \to [0,\infty]$ are said to be *equivalent* if there exist constants c, c' such that for all $u \in X$,

$$c||u|| \leq ||u||' \leq c'||u||$$

Remark 3.31. This implies any convergent sequence in $||\cdot||$ is convergent in $||\cdot||'$ and vice versa, thus the topologies are the same.

Proposition 3.32. Let $\alpha > 0$, then the norms $||\cdot||_{A^\alpha}$ and $||\cdot||_{H^{2\alpha}}$ are equivalent on H. In particular,

$$D(A^\alpha) = H^{2\alpha}_\sigma$$

Proof. Statements of this kind are difficult to prove on general domains. On the torus the eigenfunctions of A are simply the exponentials, $e_k = e^{-2\pi i k\cdot x}$ for $k \in \mathbb{Z}^d$ all along. The eigenvalues are $\lambda_k = |k|^2$, thus this result is trivial by putting these into the definitions of the norms (Do it!). $\square$

4. Steady State NSE

The steady state NSE is given by the equation in H^{-1}:

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \tag{4.1}$$

for $f \in H^{-1}$, $\nu > 0$ given, $u \in V$, $p \in L^2$ to be determined, and $d = 3$. We don't know yet whether this equation makes sense, since $(u \cdot \nabla)u$ may not be in H^{-1}. We now justify this in $\mathbb{R}^3$.

Notation 4.1. For a distribution u, write $|u| := \|u\|_{L^2}$ and $\|u\| := \|u\|_V := \|\nabla u\|_{L^2}$ whenever it makes sense.

Proposition 4.2. Let $v, u \in (H^1)^3$. Then

(i) $(u \cdot \nabla)v \in L^{6/5}$

(ii) $\|(u \cdot \nabla)v\|_{L^{6/5}} \le C|u|^{1/2}\|u\|^{1/2}\|v\|$

(iii) $\|(u \cdot \nabla)v\|_{H^{-1}} \le C'|u|^{1/2}\|u\|^{1/2}\|v\|$

for some absolute constants C, C'.

Proof. The appearance of 6/5 seems mysterious so we first see where that comes in. The juice of this proposition lies in (iii), since it tells us $(u \cdot \nabla)v \in H^{-1}$ and also a bound on the norm. Let $w \in H^1$, then

$$\begin{aligned} |\langle (u \cdot \nabla)v, w \rangle_{H^{-1},H}| &= \left| \int (u \cdot \nabla)v \cdot w \right| \\ &\le \int |(u \cdot \nabla)v||w| \end{aligned}$$

By Sobolev embedding theorem, we know $H^1 \hookrightarrow L^6$ in $d = 3$ (we will use this fact again and again), thus $\|w\|_{L^6} \le C\|w\|_{H^1}$ and so we apply Holder with $p = 1/6$ to get

$$\begin{aligned} |\langle (u \cdot \nabla)v, w \rangle_{H^{-1},H}| &\le C\|(u \cdot \nabla)v\|_{L^{5/6}}\|w\|_{H^1} \\ \therefore \quad \|(u \cdot \nabla)v\|_{H^{-1}} &\le C\|(u \cdot \nabla)v\|_{L^{5/6}} \end{aligned}$$

We now understand why bounding this quantity is desirable. It remains to show (ii).

For $i = 1, 2, 3$, $[(u \cdot \nabla)v]_i = u \cdot \nabla v_i$ so pointwise $|(u \cdot \nabla v)| \le |u||\nabla v|$ by Cauchy Schwarz in $\mathbb{R}^3$. Thus,

$$\begin{aligned} \int |(u \cdot \nabla)v|^{6/5} dx &\le \int |u|^{6/5} |\nabla v|^{6/5} dx \\ &\le \|u\|_{L^3}^{3(2/5)} \|\nabla v\|_{L^2}^{2(3/5)} && \text{(Holder, } p = 5/2) \\ \therefore \quad \|(u \cdot \nabla)v\|_{L^{6/5}} &\le \|u\|_{L^3}\|\nabla v\|_{L^2} \end{aligned}$$

By Sobolev embedding theorem, $H^{1/2} \hookrightarrow L^3$ in $d = 3$ so $\|u\|_{L^3} \leq C\|u\|_{H^{1/2}}$. Moreover by the Sobolev interpolation inequality $\|u\|_{H^{1/2}} \leq |u|^{1/2}||u||^{1/2}$. Altogether,

$$\|(u \cdot \nabla)v\|_{L^{6/5}} \leq C|u|^{1/2}\|u\|^{1/2}\|v\|$$

and we're done. □

Remark 4.3. Although we have spelled out the details in the proof of the last proposition, usage of Sobolev embedding and interpolation will be bread and butter going forwards.

To solve (4.1) we employ the same decoupling strategy. Since each term is in H^{-1}, we can apply the Helmholtz projection to get

$$\nu Au + P_\sigma(u \cdot \nabla)u = f_\sigma$$
$$\nabla p = f_g$$

and so p is easily found and the challenge lies in finding u. Thus we seek to solve the equation

$$\nu Au + B(u, u) = f \quad \text{ in } V' \tag{4.2}$$

where $A = -\Delta$ is the Stokes operator and $B : V \times V \to V'$ by $B(u, v) := P_\sigma[(u \cdot \nabla)v]$, $f \in V'$ is arbitrary, and $u \in V'$ to be determined. The B term is the non-linearity in the equation.

Proposition 4.4. For $u, v, w \in V$,

(i)
$$||B(u, v)||_{V'} \leq C|u|^{1/2}||u||^{1/2}||v||$$

(ii)
$$\langle B(u, v), w\rangle_{V',V} = -\langle B(u, w), v\rangle_{V',V}$$

In particular, $\langle B(u, v), v\rangle_{V',V} = 0$

(iii) $\langle B(u, v), w\rangle_{V',V} = \int(u \cdot \nabla)v \cdot w$

(iv) $\|B(u, v)\|_{L^2} \leq \tilde{C}\|u\|\|v\|_{H^{3/2}}$

Proof. (i) is a restatement of Proposition 4.2 (iii) after applying a projection. For (ii), first consider $u, v, w \in \mathcal{V}$. We first show that $B(u, v)$ is given by what we expect (i.e. the P_σ doesn't cause issues). Indeed,

$$\begin{aligned}\int (u\cdot\nabla)v\cdot w &= ((u\cdot\nabla)v, w) \\ &= ((u\cdot\nabla)v, P_\sigma w) && (\nabla\cdot w = 0) \\ &= (P_\sigma(u\cdot\nabla)v, w) \\ &= (B(u,v), w)\end{aligned}$$

Since $B(u,v)$ is represented by a L^2 function ($u, v \in \mathcal{V}$), we conclude

$$\langle B(u,v), w\rangle_{V',V} = \int (u\cdot\nabla)v\cdot w$$

Now,

$$\begin{aligned}\int (u\cdot\nabla)v\cdot w &= \int \sum_{1\le i,j\le d} u_j(\partial_j v_i)w_i \\ &= \int \sum_{1\le i,j\le d} (\partial_j u_j v_i)w_i && (\nabla\cdot u = 0) \\ &= -\int \sum_{1\le i,j\le d} u_j v_i \partial w_i && \text{(IBP)} \\ &= -\int (u\cdot\nabla)w\cdot v\end{aligned}$$

and so we have the result for $u, v, w \in \mathcal{V}$.

Extending this to $u, v, w \in V$ requires some standard labour. We note that if $u_n, v_n, w_n \in \mathcal{V}$ converge to u, v, w in V respectively, then by Proposition 4.2 (iii),

$$\lim_n \int (u_n\cdot\nabla)v_n\cdot w_n = \int (u\cdot\nabla)v\cdot w$$

and $\langle B(u_n,v_n), w_n\rangle_{V',V} \to \langle B(u,v), w\rangle_{V',V}$ by (i). Thus we have (iii),

$$\langle B(u,v), w\rangle_{V',V} = \int (u\cdot\nabla)v\cdot w$$

Moreover for the same reason,

$$\lim_n \int (u_n\cdot\nabla)w_n\cdot v_n = \int (u\cdot\nabla)w\cdot v$$

Thus

$$\langle B(u,v),w\rangle_{V',V} = -\langle B(u,w),v\rangle_{V',V}$$

Lastly for (iv),

$$|\int (u\cdot\nabla)v\cdot w| \le \|u\|_{L^6}\|\nabla v\|_{L^3}|w|$$

so done after applying Sobolev L^3 embedding. □

Remark 4.5. (ii) is useful because it allows us to bound $\langle B(u,v),w\rangle$ without controlling the derivatives of v. (iv) tells us when $B(u,v)$ is in L^2, and the condition $v\in H^{3/2}$ is slightly stronger than what you might guess (e.g. $v\in V$).

4.1. Existence of weak solutions

To solve (4.2) we do 'formal manipulations' (e.g. swapping derivatives and limits) to arrive at a solution, then afterwards justify what we found actually solves the original equation. Recall (ϕ_k) is an eigenbasis of A and let $u=\sum_{k=1}^{\infty} a_k\phi_k$, $f=\sum_{k=1}^{\infty} f_k\phi_k$, then (4.2) is

$$vA\left(\sum_1^\infty a_k\phi_k\right) + B\left(\sum_1^\infty a_k\phi_k, \sum_1^\infty a_k\phi_k\right) = \sum_1^\infty f_k\phi_k$$

Swapping sums,

$$\sum_1^\infty \nu\lambda_k a_k\phi_k + \sum_{k,m}^\infty a_k a_m B(\phi_k,\phi_m) = \sum_1^\infty f_k\phi_k$$

By linear independence of the (ϕ_k),

$$\nu\lambda_n a_n + \sum_{l,m=1}^\infty a_l a_m\langle B(\phi_l,\phi_m),\phi_n\rangle_{V',V} = f_n, \quad \text{for all } n \tag{4.3}$$

This is an infinite system of quadratic equations for the (a_n). Solving this head-on would be difficult so we use a technique called *Galerkin approximation*. We solve the finite system

$$\nu\lambda_n a_n + \sum_{l,m=1}^N a_l a_m\langle B(\phi_l,\phi_m)\phi_n\rangle_{V',V} = f_n, \quad \text{for } n\le N \tag{4.4}$$

or, at least show solutions exist, and further show that they converge to the solutions of (4.3) as $N \to \infty$. The following lemma will tell us solutions to the finite system exist, and our next task will be to set the scene so this lemma can be applied.

Lemma 4.6. Let D_R be a closed ball in $\mathbb{R}^N$ and $g : D_R \to \mathbb{R}^N$ continuous and for all $v \in \partial D_R$,

$$\langle g(v), v\rangle_{\mathbb{R}^N} < 0$$

Then there exists a $v^* \in D_R$ such that $g(v^*) = 0$.

Proof. By scaling, we can assume $R = 1$. Suppose $g \neq 0$ and consider $F(v) = \frac{g(v)}{\|g(v)\|_{\mathbb{R}^N}}$. By Brouwer fixed point theorem, F has a fixed point, i.e. $F(w) = w$ for some w. Thus $\|w\|_{\mathbb{R}^N} = 1$ but

$$0 < \langle F(w), w\rangle_{\mathbb{R}^N} = \frac{1}{\|g(w)\|_{\mathbb{R}^N}}\langle g(w), w\rangle_{\mathbb{R}^N} < 0$$

a contadiction. ∎

We use a technique called 'a-priori estimates', we assume a solution of (4.4) exists and show it must have certain properties. In particular, bound the norm. Let $u_N = \sum_1^N a_n\phi_n$ be a solution of (4.4), where $a_n = a_n(N)$. Then u_N satisfies

$$vAu_N + P_N B(u_N, u_N) = P_N f \tag{4.5}$$

where P_N denotes the L^2-projection onto $(\phi_k)_1^N$, noting that A commutes with P_N. Considering the action on u_N,

$$\begin{aligned} v((u_N, u_N)) + \langle B(u_N, u_N), P_N u_N\rangle_{V',V} &= \langle f, P_N u_N\rangle_{V',V} \\ \therefore \quad \nu((u_N, u_N)) + \langle B(u_N, u_N), u_N\rangle &= \langle f, u_N\rangle \end{aligned}$$

By Proposition 4.4, the second term vanishes. Thus,

$$\|u_N\| \leq \frac{1}{\nu}\|f\|_{V'} =: R_1$$

Thus the solutions to of all the Galerkin systems (i.e. for any N) are contained in a ball in V. This already tells us u_N has a weakly convergent subsequence provided solutions exist, and this will be important later. We now show solutions indeed exist. Applying $(\nu A)^{-1}$ to (4.5),

$$u_N + (\nu A)^{-1} P_N B(u_N, u_N) - (\nu A)^{-1} P_N f = 0$$

and define

$$
\begin{aligned}
&H_n := \operatorname{span}(\phi_1, \ldots, \phi_n) \\
&g : H_N \to H_N \\
&g(v) = (\nu A)^{-1} P_N f - (\nu A)^{-1} P_N B(v, v) - v
\end{aligned}
$$

We seek v^* with $g(v^*) = 0$.

Claim $((g(w), w)) < 0$ when $\|w\| = 2R_1$.

Proof.

$$
\begin{aligned}
\langle Ag(w), w\rangle_{V',V} &= -\|w\|^2 + \underbrace{\frac{1}{\nu}\langle B(w,w), w\rangle}_{=0} + \frac{1}{\nu}\langle P_N f, w\rangle \\
&= -\|w\|^2 + \frac{1}{\nu}\langle P_N f, w\rangle \\
&\leq -4R_1^2 + \frac{1}{\nu}\|f\|_{V'}(2R_1) \\
&= -2R_1^2 < 0
\end{aligned}
$$

We recall $\langle Av, v\rangle_{V',V} = ((g(w), w))$ and the claim is proven. □

Thus by Lemma 4.6 (noting all norms in finite dimensions are equivalent), there exists v^* with $g(v^*) = 0$ and moreover the solution u_N satisfies $\|u_N\| \leq 2R_1$. We summarise the results of this section so far with the following theorem:

Theorem 4.7. The Galerkin approximation system (4.5) has a solution u_N for every N with $\|u_N\| \leq 2R_1$ and there is a subsequence u_{N_k} such that

$$
\begin{aligned}
u_{N_k} &\to u \quad \text{in } V \text{ weakly} \\
u_{N_k} &\to u \quad \text{in } H \text{ strongly}
\end{aligned}
$$

for some $u \in V$.

Proof. The first assertion is what we already showed. The subsequence is obtained by Banach-Alaoglu and the compact embedding $V \hookrightarrow H$. □

Theorem 4.8. The 3D steady state NSE (4.2) has a solution in V' given by $u \in V$, the limit in Theorem 4.7.

Proof. Without loss of generality, relabel the sequence u_N so that $u_N \to u$ weakly in V and strongly in H. It suffices to show for every k,

$$\nu\langle Au, \phi_k\rangle_{V',V} + \langle B(u,u), \phi_k\rangle_{V',V} = \langle f, \phi_k\rangle_{V',V} \tag{4.6}$$

We have

$$\begin{aligned} \nu\langle Au_N, \phi_k\rangle_{V',V} + \langle P_N B(u_N,u_N), \phi_k\rangle_{V',V} &= \langle P_N f, \phi_k\rangle_{V',V} \\ \therefore \quad \nu((u_N,\phi_k)) + \langle B(u_N,u_N), P_N\phi_k\rangle &= \langle f, P_N\phi_k\rangle \end{aligned}$$

so taking N large enough ($N > k$),

$$\nu((u_N,\phi_k)) + \langle B(u_N,u_N), \phi_k\rangle = \langle f, \phi_k\rangle$$

Since $u_N \to u$ weakly in V, we have $((u_N,\phi_k)) \to ((u,\phi_k))$. To establish (4.6), it remains to show $\langle B(u_N,u_N),\phi_k\rangle \to \langle B(u,u),\phi_k\rangle$. Dealing with this non-linearity is our main obstruction.

Claim $\int (u_N\cdot\nabla)u_N\phi_k \to \int (u\cdot\nabla)u\phi_k$ as $N\to\infty$

$$\begin{aligned} \left|\int (u_N\cdot\nabla u_N - u\cdot\nabla u)\phi_k\right| &\le \left|\int (u_N-u)\nabla u_N\phi_k\right| + \left|\int (u\cdot\nabla)(u_N-u)\phi_k\right| \\ &\le \|u_N-u\|_{L^3}|\nabla u_N|\|\phi_k\|_{L^6} + \|u\|_{L^3}|\nabla(u_N-u)|\|\phi_k\|_{L^6} \\ &\le \left(C\|u_N-u\|_{H^{1/2}}\|u_N\| + C\|u\|_{H^{1/2}}\|u_N-u\|\right)\|\phi_k\|_{L^6} \\ &\le \left(\|u_N-u\|^{1/2}|u_N-u|^{1/2}\|u_N\| + \|u\|^{1/2}|u_N-u|^{1/2}\|u\|\right)C\|\phi_k\|_{L^6} \\ &\le \underbrace{|u_N-u|^{1/2}}_{\to 0}\underbrace{\left(\|u_N-u\|^{1/2}\|u_N\| + \|u\|^{3/2}\right)}_{\text{bounded by Theorem 4.7}}C\|\phi_k\|_{L^6} \end{aligned}$$

□

Remark 4.9. This section reflects a common proof pattern. Come up with approximate solutions which one would expect converge to the real solution, then show they indeed do. The difficult in the latter lies in dealing with the non-linear terms.

4.2. Existence of strong solutions

The steady state NSE in L^2 is given by

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in } L^2$$

for $f \in L^2$ known, $u \in D(A) = H^2_\sigma$ to be determined, and $d = 3$. Using Helmholtz in L^2 we can carry out the usual decoupling and obtain,

$$\nu Au + B(u,u) = f \quad \text{in } H \tag{4.7}$$

for $f \in H$.

Remark 4.10. For (4.7) to make sense, one expects each term to be in H, and indeed this is true as follows. Since $u \in D(A)$, we have $Au \in H$. The non-linear term $B(u,u)$ is more subtle. We recall that $H^2 \subset C$ by the Sobolev embedding theorem with $d = 3$, so in particular $u \in L^\infty$. Moreover,

$$\int |(u \cdot \nabla)u|^2 \leq \int |u|^2 |\nabla u|^2 \leq \|u\|_\infty^2 \|\nabla u\|_{L^2}^2 < \infty$$

and so $B(u,u) \in H$.

Equation (4.7) is equivalent to

$$\nu(Au, \phi_k) + (B(u,u), \phi_k) = (f, \phi_k)$$

for all k. From Theorem 4.8, we have a $u \in V$ such that

$$\nu\langle Au, \phi_k\rangle_{V',V} + \langle B(u,u), \phi_k\rangle_{V',V} = \langle f, \phi_k\rangle_{V',V}$$

and in light of Remark 4.10, if $u \in D(A)$, then $Au, B(u,u) \in H$ so the V' action is, by definition, given by the L^2 inner product and we're done. Thus there is only one thing to do:

Proposition 4.11. The u_N are uniformly bounded in $D(A)$. Consequently, $u \in D(A)$, where u is the limit from Theorem 4.8.

Proof. We carry out the Galerkin method again, using the $u_N \in H_N := \text{span}(\phi_k)_1^N$ obtained in Theorem 4.7 satisfying

$$\nu Au_N + P_N B(u_N, u_N) = P_N f \tag{4.8}$$

We recall Theorem 4.8 which states $\|u_N\| \leq 2R_1$ where the constant R_1 depends only on f and ν. Since $u_N \in H_N \subset D(A)$, we can take the inner product of (4.8) with Au_N,

$$\begin{aligned}
&|\nu Au_N|^2 + (P_N B(u_N,u_N), Au_N) = (P_N f, Au_N)\\
\therefore\quad &|\nu Au_N|^2 + (B(u_N,u_N), P_N Au_N) = (f, P_N Au_N)\\
\therefore\quad &|\nu Au_N|^2 + (B(u_N,u_N), Au_N) = (f, Au_N)\\
\therefore\quad &\nu^2|Au_N|^2 \le |f||Au_N| + \left|\int (u_N\cdot\nabla)u_N\cdot Au_N\right|\\
\therefore\quad &\qquad\qquad \le |f||Au_N| + \|u_N\|_{L^6}\|\nabla u_N\|_{L^3}|Au_N|\\
\therefore\quad &\qquad\qquad \le |f||Au_N| + C\|u_N\|\|\nabla u_N\|^{1/2}|\nabla u_N|^{1/2}|Au_N|
\end{aligned}$$

Dividing by $|Au_N|$,

$$\nu^2|Au_N| \le |f| + C\|u_N\|^{3/2}\|\nabla u_N\|^{1/2}$$

Note that $\|\nabla v\|^2 = \sum_{k\neq 0}|k|^2|\widehat{\nabla v}_k|^2 = \sum_{k\neq 0}|k|^4|\widehat{v}_k|^2 = \|v\|_{H^2}$ and recall that the H^2-norm is equivalent to the A-norm (Proposition 3.32), thus

$$\begin{aligned}
\nu^2|Au_N| &\le |f| + C'\|u_N\|^{3/2}|Au_N|^{1/2}\\
&\le |f| + C'(2R_1)^{3/2}|Au_N|^{1/2}
\end{aligned}$$

We see this inequality can only hold true if Au_N is bounded independently of N, since the quadratic $\nu^2x^2 - C'(2R_1)^{3/2}x - |f|$ is eventually positive. In particular,

$$|Au_N|^{1/2} \le \frac{1}{2\nu^2}\left(C'(2R_1)^{3/2} + \sqrt{C^2(2R_1)^3 + 4\nu^2|f|^2}\right)$$

The last assertion in the statement follows from Banach-Alaoglu and comparing Fourier coefficients of the limits. □

Remark 4.12. This 'quadratic trick' is good to notice, but it wont suffice later and we'll upgrade it to a much more flexible technique called 'Young's inequality'.

4.3. Uniqueness of weak solutions

Theorem 4.13. The solution to the steady state NSE (4.2) is unique for $f \in V'$ fixed provided $\nu\lambda_1^{1/2} > 2CR_1$, where C is the constant from Proposition 4.2 (ii).

Proof. Suppose u and v solve (4.2) and $w = u - v$. Then

$$\nu Aw + B(u,w) + B(w,u) - B(w,w) = 0 \quad \text{in } V'$$

Acting on w,

$$\begin{aligned} \nu\langle vAw, w\rangle_{V',V} + 0 + \langle B(w,u), w\rangle_{V',V} - 0 = 0 \\ \therefore \quad \nu\|w\|^2 = -\langle B(w,u), w\rangle \\ \therefore \quad \nu\|w\|^2 \le C|w|^{1/2}\|w\|^{1/2}\|u\|\|w\| \end{aligned}$$

where the last inequality comes from Proposition 4.4. By Proposition 3.32, for all $\psi \in V$,

$$\begin{aligned} \|\psi\|^2 &= \sum_1^\infty |(\psi, \phi_k)|^2 \lambda_k \\ &\ge \lambda_1 |\psi|^2 \end{aligned}$$

Thus,

$$\begin{aligned} \nu\|w\|^2 &\le \frac{C}{\lambda_1^{1/2}}\|u\|\|w\|^2 \\ &\le \frac{C(2R_1)}{\lambda_1^{1/2}}\|w\|^2 \end{aligned}$$

Hence $w = 0$ or $\nu\lambda_1^{1/2} \le C(2R_1)$. □

4.4. Weak solutions are strong solutions

Let $u \in V$ be a solution to (4.2), i.e.

$$\nu Au + B(u,u) = f \qquad \text{in } V'$$

Remark 4.14. The assumption $u \in V$ could be seen as modest since we exhibited a solution which is in V, and in the previous section gave a condition for this solution to be unique.

In this section we will show, under the assumption $f \in H$, we have

$$\nu Au + B(u,u) = f \qquad \text{in } H$$

i.e. weak solutions are strong solutions. We recall from Section 4.2 that this follows if $u \in D(A)$. Thus, our approach is to show that any weak solution lies in $D(A)$. We do this in two steps, first showing $u \in D(A^{3/4})$, then using that to show $u \in D(A)$.

Remark 4.15. The assumption $f \in H$ is clearly necessary.

Let u be an arbitrary weak solution and define $u_N = \sum_{k=1}^{N}(u, \phi_k)\phi_k$ (not the Galerkin approxmation!).

Lemma 4.16. The norms $\|u_N\|_{D(A^{3/4})}$ are uniformly bounded. In particular, $u \in D(A^{3/4})$.

Proof. Note that given the first assertion, the second follows easily by applying Banach-Alaoglu and comparing Fourier coefficients of the limit and u. For the first assertion, observe u_N satisfies

$$\nu A u_N + P_N B(u,u) = P_N f$$

Since the u_N are smooth, we can take the action with $A^{1/2}$,

$$(\nu A u_N, A^{1/2}u_N) + (B(u,u), A^{1/2}u_N)^{1} = (f, A^{1/2}u_N)$$

noting the projections are self adjoint, commute with powers of A, and $P_N u_N = u_N$. Since $A^{1/4}$ (or any power of A) is self adjoint, we have

$$\nu(A^{3/4}u_N, A^{3/4}u_N) \leq -(B(u,u), A^{1/2}u_N) + |f||A^{1/2}u_N| \tag{4.9}$$

Observe

$$\begin{aligned} |(B(u,u), A^{1/2}u_N)| &= \left| \int (u \cdot \nabla)u \cdot A^{1/2}u_N \right| \\ &\leq \|u\|_{L^6}|\nabla u|\|A^{1/2}u_N\|_{L^3} && \text{(Holder)} \\ &\leq C\|u\|^2\|A^{1/2}u_N\|_{L^3} \end{aligned}$$

By Sobolev embedding $\|\cdot\|_{L^3} \leq C'\|\cdot\|_{H^{1/2}}$, and by Proposition 3.32, the $A^{1/4}$ norm and $\|\cdot\|_{H^{1/2}}$ norm are equal. Thus $\|\cdot\|_{L^3} \leq C'\|\cdot\|_{A^{1/4}}$

$$\begin{aligned} \therefore \quad |(B(u,u), A^{1/2}u_N)| &\leq C''\|u\|^2|A^{3/4}u_N| \\ &= C''\|u\|\|u_N\|_{D(A^{3/4})} \end{aligned}$$

[1] Technically, $B(u,u)$ is not necessarily in L^2 (yet!), so having it in an L^2 inner product doesn't make perfect sense. However, we're effectively working in $\mathbb{R}^N$ here, so there are no issues.

Into (4.9),

$$\nu|A^{3/4}u_N|^2 \leq C''\|u\|\|u_N\|_{D(A^{3/4})} + |f||A^{1/2}u_N|$$

Recall $|A^{1/2}u_N| = \|u_N\| \leq 2R_1$, thus

$$\nu\|u_N\|^2_{D(A^{3/4})} \leq C''\|u\|\|u_N\|_{D(A^{3/4})} + |f|(2R_1)$$

We now have a uniform estimate on $\|u_N\|_{D(A^{3/4})}$ by the quadratic trick; the polynomial $\nu x^2 - C''\|u\|x - |f|(2R_1)$ is positive for x sufficiently large, thus $\|u_N\|_{D(A^{3/4})} \leq K$ for some constant $K = K(C'', |f|, R_1, \nu)$. □

Proposition 4.17. The norms $\|u_N\|_{D(A)}$ are uniformly bounded. In particular, $u \in D(A)$ and u solves (4.7) in H.

Proof. We again note that the second claim follows from the first using a standard application of Banach-Alaoglu and the discussion at the beginning of the section. We have

$$\begin{aligned} &\langle \nu Au, Au_N\rangle_{V',V} + \langle B(u,u), Au_N\rangle_{V',V} = \langle f, Au_N\rangle_{V',V} \\ \therefore\quad &\nu|Au_N|^2 + \langle B(u,u), Au_N\rangle_{V',V} = (f, Au_N) \end{aligned}$$

$$\begin{aligned} \therefore\quad \nu|Au_N|^2 &\leq -\langle B(u,u), Au_N\rangle + |f||Au_N| \\ &\leq \|u\|_{L^6}\|\nabla u\|_{L^3}|Au_N| + |f||Au_N| \\ &\leq C\|u\|\|\nabla u\|_{H^{1/2}}|Au_N| + |f||Au_N| \\ &\leq C(2R_1)\|u\|_{H^{3/2}}|Au_N| + |f||Au_N| \\ &\leq C(2R_1)K|Au_N| + |f||Au_N| \\ \therefore\quad \nu|Au_N| &\leq 2CKR_1 + |f| \end{aligned}$$

Thus $\|u_N\|_{D(A)}$ is uniformly bounded. □

5. Time dependent Stokes problem

We consider (as always, in $d = 3$),

$$\begin{cases} \frac{du}{dt} - \nu\Delta u + \nabla p = f \\ \nabla \cdot u = 0 \\ u(x,0) = u_0(x) \in H \\ f \in H \quad \text{known, independent of } t \end{cases} \tag{5.1}$$

We think of $u(t)$ being a spatial distribution (or function of x) for every t, i.e. $u(t) \in D'(\Omega)$ acts on spatial test functions $\phi(x) \in D(\Omega)$ for every t. Because of this, we write $\frac{du}{dt}$ or $\dot{u}$ to denote differentiation with respect to time, rather than a partial derivative.

Remark 5.1.

- We will define exactly what we mean by 'differentiation with respect to time' shortly. We will not use a limit definition as this allows for monstrosities such as the Cantor function. What we would like is that the Fundamental Theorem of Calculus holds, but we are yet to see how to integrate u (which takes values in a Banach space).
- Setting the scene in this way makes every value of t important, we haven't loosened anything to 'almost everywhere'. Specifically, the initial condition $u(x,0) = u_0(x)$ is enforced despite being on a measure zero set.

5.1. Bochner integration

Notation 5.2. In this section $(E, \mathcal{E}, \mu)$ denotes a measure space.

Definition 5.3. Let X be a Banach space with its Borel σ-algebra and $f : E \to X$ measurable.

- If f is a simple function, i.e. $f(x) = \sum_1^n a_j 1_{E_j}(x)$, where $(a_j) \in X$ and $E_j \in \mathcal{E}$, then define its *Bochner integral*

$$\int_E f(x) d\mu(x) := \sum_1^n a_j \mu(E_j)$$

- We say f is *(Bochner) integrable* if there exist simple functions (f_k) such that

$$\int_E \|f(x) - f_k(x)\|_X d\mu(x) \to 0 \quad \text{as } k \to \infty$$

where this integral is taken in the classical sense, and in this case define its Bochner integral

$$\int_E f(x) d\mu(x) = \lim_{k\to\infty} \int f_k(x) d\mu(x)$$

Proposition 5.4. Let X be a separable Banach space and $f : E \to X$ measurable. Then,

$$f \text{ integrable } \iff \int_E \|f(x)\|_X d\mu < \infty$$

Motivated by this result we define,

Definition 5.5. Let $1 \leq p < \infty$ then

$$L^p(E, X) := \left\{ f : E \to X : \int_E \|f(x)\|_X^p d\mu(x) < \infty \right\}$$

.

$$L^\infty(E, X) := \{ f : E \to X : \text{ess sup}_{x \in E} f(x) < \infty \}$$

Proposition 5.6. Let $f : E \to X$ measurable then for all $T \in X'$, $\langle T, f(x) \rangle_{X',X}$ is measurable and if f is integrable then

$$\langle T, \int_E f(x) d\mu \rangle_{X',X} = \int_E \langle T, f(x) \rangle_{X',X} d\mu$$

Remark 5.7. This result tells us Bochner integration is similar to $\mathbb{R}^d$ differentiation in the following sense. In $\mathbb{R}^d$, elements of the dual $(\mathbb{R}^d)'$ can be viewed as coordinate projections in some coordinate system (c.f. dual basis), so this result phrased in $\mathbb{R}^d$ would look like: for $f : E \to \mathbb{R}^d$,

$$\left(\int_E f(x) dx \right)_i = \int_E f(x)_i \, dx$$

i.e. taking the component of an integral is like integrating that component.

We now specialise to the case when $E = (0, T) \subset \mathbb{R}$ is a time interval with the usual Lesbegue sets and measure, and so our integrals are over time. This will be the only case relevant to us.

Theorem 5.8. If X is reflexive, then the dual of $L^p((0, T), X)$ is $L^q((0, T), X^*)$ where $1/p + 1/q = 1$ and $p \in [1, \infty)$.

Remark 5.9. Typically, we take $X = H$, V, or V', which are all Hilbert spaces hence reflexive. An important consequence of this theorem is that the $L^p((0, T), X)$ spaces are dual spaces, so Banach-Alaoglu can be applied. Moreover, if $p \neq 1, \infty$ then $L^p((0, T), X)$ is reflexive so the weak and weak* topologies are the same, and hence we use these terms interchangeably.

Proposition 5.10. For $p < q$, $L^q((0,T),X) \subset L^p((0,T),X)$.

Definition 5.11. Let X be a Banach space and $u \in L^1((0,1),X)$. We say u is *differentiable (in the absolutely continuous sense)* if there exists $g \in L^1((0,1),X)$ such that

$$u(t) = u_0 + \int_0^t g(s)ds \tag{5.2}$$

for some $u_0 \in X$ and almost all $t \in (0,1)$ and we write $\dot{u} := \frac{du}{dt} := g$ for the *derivative of* u.

Remark 5.12.

- The definition generalises over to any open interval (a,b) in the obvious way. It also generalises to closed intervals without change since (5.2) need only holds almost everywhere.
- Recall from real analysis that a function $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous if and only if the Fundamental Theorem of Calculus holds almost everywhere, whence this definition is inspired. It is natural for us to demand this property since our results so far have always relied from extracting information from integrals (e.g. distributions are typically integration against a function).
- The condition is enforced for almost all t, not all t, because $L^1((0,1),X)$ are 'equal' if they are equal almost everywhere.

Lemma 5.13. Let X Banach and $u, g \in L^1((0,1),X)$. The following are equivalent:

(i) u is differentiable with $\dot{u} = g$.

(ii) $\forall \phi \in D(0,1)$ (test functions on $(0,1)$),

$$\int_0^1 u(t)\phi(t)dt = -\int_0^1 g(t)\phi'(t)dt$$

(iii) For all $T \in X'$, the weak derivative of the scalar function $t \to \langle T, u(t)\rangle_{X',X} \in D'(0,1)$ (distributions on $(0,1)$) is

$$\frac{d}{dt}\langle T, u(t)\rangle_{X',X} = \langle T, g(t)\rangle_{X',X}$$

i.e. for all $\phi \in D(0,1)$

$$\int_0^1 \langle T, g(t)\rangle \phi(t)dt = -\int_0^1 \langle T, u(t)\rangle \phi'(t)dt$$

Remark 5.14.

- (ii) is like saying g is the weak derivative of u with respect to the t variable, except this doesn't make sense since distributions are $\mathbb{R}^d$-valued, whereas $t \to u(t)$ is X-valued.
- (iii) is further similarity to $\mathbb{R}^d$ in the sense of Remark 5.7, saying that the derivative of a component is the component of the total derivative.

5.2. Existence of weak solutions

After applying the usual Helmholtz decoupling to (5.1), we are required to solve the equation

$$\begin{cases} \dot{u} + \nu A u = f & \text{in } V' \\ u(0) = u_0 \end{cases} \tag{5.3}$$

with $f \in V'$ independent of t and $u_0 \in H$ known, $u(t) \in V$ for all $t > 0$. Our first idea is to guess a solution using formal (but sensible) manipulations. Write $u(t) = \sum_{k=1}^{\infty} a_k(t)\phi_k$ where (ϕ_k) are the usual eigenfunctions of A and the a_k are real valued functions to be determined. Then, formally commuting derivatives with sums, (5.3) becomes

$$\sum_1^\infty \dot{a_k}(t)\phi_k + \nu \sum_1^\infty a_k \lambda_k \phi_k = \sum_1^\infty f_k \phi_k$$

Thus,

$$\dot{a_k}(t) + \nu \lambda_k a_k = f_k \qquad \forall k \tag{5.4}$$

$$\therefore a_k(t) = e^{-\nu\lambda_k t} a_k(0) + \frac{f_k}{\nu \lambda_k}(1 - e^{-\nu\lambda_k t}) \qquad \forall k$$

where $a_k(0)$ is known via u_0. We now ask whether the partial sums $u_N = \sum_{k=1}^{N} a_k(t)\phi_k$ converge. Although we have the a_k explicitly, we showcase a new technique. Multiplying (5.4) by $a_k(t)$ we have

$$\frac{1}{2}\frac{d}{dt}a_k^2 + \frac{\nu\lambda_k}{2}a_k^2 \leq \frac{|f_k|^2}{2\nu\lambda_k}$$
$$\therefore \quad \frac{d}{dt}a_k^2 + \nu\lambda_k a_k^2 \leq \frac{|f_k|^2}{\nu\lambda_k} \tag{5.5}$$

We now apply an inequality/idea called *Gronwall's inequality*. We know that classical equations of the form

$$y : \mathbb{R} \to \mathbb{R}$$
$$y' + cy = d \tag{5.6}$$

with $c, d \in \mathbb{R}$ have general solution $y = Ae^{Bt} + C$ for some constants A, B, C. The hope is that if we instead have inequality

$$y + cy \leq d$$

then y must satisfy $y \leq Ae^{Bt} + C$ for some constants A, B, C. In our context, this gives an exponential bound on $|a_k|$ and d is indeed a constant, but in the version we prove d need not be constant in time.

Lemma 5.15. (Gronwall) Suppose y is absolutely continuous and solves the classical equation

$$\begin{cases} y'(t) + cy(t) \leq d(t) \\ y(0) = y_0 \end{cases}$$

Then

$$y \leq e^{-ct}y_0 + \int_0^t e^{-c(t-s)}d(s)ds$$

In particular if d is constant,

$$y \leq e^{-ct}y_0 + \frac{d}{c}\left(1 - e^{-ct}\right)$$

Proof. This is easy, we do it the same way we would solve (5.6).

$$e^{ct}y' + e^{ct}y \leq e^{ct}d$$
$$\therefore \quad \frac{d}{dt}\left(e^{ct}y\right) \leq e^{ct}d(t)$$
$$\therefore \quad y \leq e^{-ct}y_0 + \int_0^t e^{-c(t-s)}d(s)ds$$

where for the last inequality we applied F.T.C, using absolute continuity of y. □

Applying Gronwall to (5.5),

$$a_k(t)^2 \leq e^{-\nu\lambda_k t} a_k(0)^2 + \frac{|f_k|^2}{(\nu\lambda_k)^2}(1 - e^{-\nu\lambda_k t}) \tag{5.7}$$

$$\leq e^{-\nu\lambda_1 t} a_k(0)^2 + \frac{1}{\nu^2\lambda_1}\frac{|f_k|^2}{\lambda_k} \cdot 1 \tag{5.8}$$

Summing over k,

$$\sum_k^N a_k(t)^2 \leq e^{-\nu\lambda_1 t}\sum_k^N a_k(0)^2 + \frac{1}{\nu^2\lambda_1}\sum_k^N \frac{|f_k|^2}{\lambda_k}$$

Recall $V' = D(A^{-1/2})$ and the $A^{-1/2}$ norm is the V' norm, so we have

$$\sum_k^N a_k(t)^2 \leq e^{-\nu\lambda_1 t}|u_0|^2 + \frac{1}{\nu^2\lambda_1}\|f\|_{V'}^2$$

Thus the series $v := \sum a_k(t)\phi_k$ converges in H for every t and moreover its H-norm is uniformly bounded in t by a constant. We can be more specific than this.

Definition 5.16. Let $T > 0$ and X a Banach space. The space of continuous functions from $[0, T]$ to X, denoted $C([0, T], X)$ is a Banach space with norm

$$\|f\| = \sup_{t\in[0,T]} \|f(t)\|_X$$

Proposition 5.17. The sequence $u_N := \sum_1^N a_k(t)\phi_k$ converges in $C([0, T], H)$.

Proof. Let $0 \leq t < T$ arbitrary. Summing (5.8),

$$\sum_{k=n}^m |a_k(t)|^2 \leq e^{-\nu\lambda_1 t}\sum_{k=n}^m |a_k(0)|^2 + \frac{1}{\nu^2\lambda_1}\sum_{k=m}^n \frac{|f_k|^2}{\lambda_k}$$

$$\leq \sum_{k=n}^m |a_k(0)|^2 + \frac{1}{\nu^2\lambda_1}\sum_{k=n}^m \frac{|f_k|^2}{\lambda_k}$$

The RHS is independent of t and converges to zero as $n \wedge m \to \infty$ since both series converge. □

We are throwing a lot of information away going from (5.7) to (5.8) by replacing λ_k with λ_1. Indeed, we can do better. Multiplying (5.7) by λ_k and bounding the exponentials by 1,

$$\lambda_k|a_k|^2 \leq \lambda_k|a_k(0)|^2 + \frac{|f_k|^2}{\nu^2\lambda_k}$$

Thus,

$$\|v(t)\|^2 \leq \|u_0\|^2 + \frac{1}{\nu^2}\|f\|_{V'}^2$$

so $v(t) \in V$ provided $u_0 \in V$, which we did not assume. If we didn't assume $u_0 \in V$, it wouldn't really make sense to try show $v(t) \in V$ for all t since $v(0) = u_0$. Surprisingly, we can show the next best thing:

Proposition 5.18. The constructed weak solution $v = \sum_k a_k(t)\phi_k$ is in V for all $t > 0$, assuming only $u_0 \in H$.

Proof. We repeat our previous steps but don't discard the exponential. From (5.7) we have

$$\lambda_k|a_k(t)|^2 \leq \lambda_k e^{-\nu\lambda_k t}|a_k(0)|^2 + \frac{|f_k|^2}{\nu^2\lambda_k} \tag{5.9}$$

$$\therefore \quad \|v(t)\|^2 \leq \sum_k (\lambda_k e^{-\nu\lambda_k t})|a_k(0)|^2 + \frac{\|f\|_{V'}^2}{\nu^2}$$

The real function $s \to se^{-\nu ts}$ is bounded for all $t > 0$, say by the constant M_t. Therefore

$$\|v(t)\| \leq M_t|u_0| + \frac{\|f\|_{V'}^2}{\nu^2}$$

i.e. $v(t) \in V$ for all $t > 0$. □

Repeating the approach in Proposition 5.17, one shows

Proposition 5.19. The sequence $u_N := \sum_1^N a_k(t)\phi_k$ converges to v in $C([t_0, T], V)$ for any $0 < t_0 < T$. Moreover, $\|v(t)\| \leq K(|u_0|, t_0, \|f\|_{V'})$ for all $t > t_0$.

Notation 5.20. $C_{\text{loc}}((0,T), V)$ is the space of locally bounded continuous functions on $(0,T)$,

$$C_{\text{loc}}((0,T), V) := \{f : \text{for all } K \subset (0,T) \text{ compact, } f|_K \text{ is continuous}\}$$

and the topology is defined by: $\phi_n \to 0$ in $C_{\text{loc}}((0,T),V)$ if for all compact K, $\phi_n \to 0$ uniformly on K.

Remark 5.21. We wont use this space much, but Proposition 5.19 can be summarised as $u_N \to v$ in $C_{\text{loc}}((0,T),V)$ (as well as the bound on $\|v\|$).

We understand now that the behaviour can be bad at zero. What about integration? From (5.9),

$$\begin{aligned}\lambda_k \int_0^T a_k(t)^2 dt &\leq \int_0^T \lambda_k e^{-\nu\lambda_k t} a_k(0)^2 + \frac{|f_k|^2}{\nu^2\lambda_k}T \\ &\leq \frac{1}{\nu}(1-e^{-\nu\lambda_k T})a_k(0)^2 + \frac{|f_k|^2}{\nu^2\lambda_k}T\end{aligned}$$

When we integrate, the problem disappears. We can now sum over k to get,

$$\int_0^T \|u_N\|^2 dt \leq \frac{1}{\nu}|u_0|^2 + \frac{T}{\nu^2}\|f\|_{V'}$$

Thus u_N is bounded in $L^2((0,T),V)$. Again, by being more careful, we show:

Proposition 5.22. The sequence $u_N := \sum_1^N a_k(t)\phi_k$ converges to v in $L^2((0,T),V)$ for any $T>0$.

Proof.

$$\begin{aligned}\|u_n - u_m\|^2_{L^2((0,T),V)} &= \int_0^T \sum_{k=n}^m \lambda_k a_k(t)^2 dt \\ &\leq \frac{1}{\nu}\sum_{k=n}^m a_k(0)^2 + \frac{T}{\nu^2}\sum_{k=n}^m \frac{|f_k|^2}{\lambda_k}\end{aligned}$$

The RHS converges to zero as $n \wedge m \to \infty$ since both series converge. □

Corollary 5.23. Au_N converges to Av in $L^2((0,T),V')$.

Proof.

$$\begin{aligned}\|Au_N - Av\|^2_{L^2((0,T),V')} &= \int_0^T \|Au_N - Av\|^2_{V'} ds \\ &\leq \|A\|^2_{V'} \int_0^T \|u_N - v\|^2_V ds \\ &\to 0\end{aligned}$$

□

We are yet to show v solves the Stokes problem in V'. From (5.4), we know the u_N solve the finite dimensional problem:

$$\dot{u_N} + \nu A u_N = P_N f$$

Integrating,

$$u_N(t) - u_N(0) + \nu \int_0^t A u_N(s) ds = P_N f t \tag{5.10}$$

We consider each term separately as $N \to \infty$. Since $u_N \to v$ in $C([0,T], H)$, we have $u_N(t) \to v(t)$ and $u_N(t) \to u_0$ in H, and therefore in V'. Also $P_N f \to f$ in V'. The next lemma deals with the remaining term:

Lemma 5.24. $\int_0^t A u_N(s) \to \int_0^t A v(s) ds$ in V' as $N \to \infty$ for all $t \in [0,T]$.

Proof.

$$\begin{aligned}
\| \int_0^t A u_N(s) - A v(s) ds \|_{V'} &\leq \int_0^t \| A u_N(s) - A v(s) \|_{V'} ds \\
&\leq T^{1/2} \left(\int_0^t \| A u_N - A v \|_{V'}^2 ds \right)^{1/2} \\
&\to 0 \qquad\qquad \text{(Corollary 5.23)}
\end{aligned}$$

□

Thus the limit of (5.10) in V' as $N \to \infty$ is

$$\begin{aligned}
& v(t) - u_0 + \nu \int_0^t A v(s) ds = f t \quad \text{ in } V' \\
\therefore \quad & v(t) = u_0 + \int_0^t -\nu A v(s) + f ds \\
\therefore \quad & \dot{v} = -\nu A v(s) + f \quad \text{ in } V'
\end{aligned}$$

by definition of $\dot{v}$ and $v(0) = u_0$ as desired. Note that we used the fact $-\nu A v + f \in L^1((0,T), V')$, which is true since $Av \in L^2((0,T), V)$ and $f \in V'$ is independent of time.

We summarise this section:

Theorem 5.25. (Stokes weak existence) The time dependent Stokes problem (5.3) has a weak solution $v \in L^2((0,T),V) \cap C([0,T],H) \cap C_{\text{loc}}((0,T),V)$ for all $T > 0$.

5.3. Uniqueness of weak solutions

Lemma 5.26. (Lions-Magenes) If $u \in L^2((0,T),V)$ and $\dot{u} \in L^2((0,T),V')$ then

$$\langle \dot{u}, u \rangle_{V',V} = \frac{1}{2}\frac{d}{dt}|u(t)|^2$$

Moreover, $|u(t)|^2$ is absolutely continuous on $(0,T)$ and,

$$|u(t)|^2 - |u(s)|^2 = 2\int_s^t \langle \dot{u}(r), u(r) \rangle_{V',V} dr$$

for all $t, s \in (0,T)$.

Suppose $u_1, u_2 \in L^2((0,T),V)$ solve (5.3) with different initial conditions and $w := u_1 - u_2$. Then

$$\dot{w} + \nu A w = 0 \quad \text{in } V'$$

with $w(0) = u_1(0) - u_2(0)$ and $w \in L^2((0,T),V)$.

$$\langle \dot{w}, w \rangle_{V',V} + \nu \|w\|^2 = 0$$

$$\therefore \quad \frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w\|^2 = 0 \qquad \text{(Lions-Magenes)}$$

By Poincare, $\|w(t)\| \geq \lambda_1 |w(t)|$, so

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\lambda_1|w(t)|^2 \leq 0$$

$$\therefore \quad |w(t)|^2 \leq e^{-2\nu\lambda_1 t}|w(0)|^2 \qquad \text{(Gronwall)}$$

We summarise with the following theorem

Theorem 5.27. (Stokes weak well-posedness) If $u_1, u_2 \in L^2((0,T),V)$ are weak solutions to the time dependent Stokes equation (5.3), then

$$|u_1(t) - u_2(t)| \leq e^{-\nu\lambda_1 t/2}|u_1(0) - u_2(0)|$$

It follows that u_1 and u_2 converge to the same equilibrium as $t \to \infty$. Moreover, well-posedness holds in the $C([0,T],H)$ norm. In particular, the solutions are uniquely determined by their initial conditions.

We can also establish well-posedness for the $L^2([0,T],V)$ norm,

Proposition 5.28. Let $u_1, u_2 \in L^2((0,T),V)$ weak solutions to (5.3). Then,

$$\|u_1 - u_2\|_{L^2((0,T),V)} \le \frac{1}{\sqrt{2\nu}} |u_1(0) - u_2(0)|$$

Proof. We have

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w\|^2 = 0$$

Integrating,

$$\frac{1}{2}(|w(T)|^2 - |w(0)|^2) + \nu \int_0^T \|w\|^2 ds = 0$$

$$\therefore \quad \nu \int_0^T \|w\|^2 ds \le \frac{1}{2}|w(0)|^2$$

□

6. The Navier Stokes equation

6.1. Existence of weak solutions

The 3D NSE is given by

$$\begin{cases} \dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \\ \nabla \cdot u = 0 \\ u(0) = u_0 \end{cases}$$

for some known $u_0 \in H$ and $f \in V'$ independent of time. Applying the usual Helmholtz decoupling, we seek to solve,

$$\begin{cases} \dot{u} + \nu A u + B(u,u) = f \quad \text{ in } V' \\ u(0) = u_0 \in H \end{cases} \tag{6.1}$$

where we have asserted the equation should hold in V' as this section is devoted to weak solutions. We combine all of the methods used to solve the previous equations. Write $u = \sum_1^\infty a_k(t)\phi_k$, $u_N = \sum_1^N a_k(t)\phi_k \in H_N$ as usual. Following the formal manipulations at the beginning of Section 5.2, we have

$$\dot{a_k} + \nu\lambda_k a_k + \sum_{l,m\ge 1} a_l a_m (B(\phi_l, \phi_m), \phi_k) = \langle f, \phi_k \rangle_{V',V}$$

Unlike the previous case, we cannot solve this explicitly so we use Galerkin. The Galerkin system is given by:

$$\begin{cases} \dot{a_k} + \nu\lambda_k a_k + \sum_{l,m\le N} a_l a_m (B(\phi_l,\phi_m),\phi_k) = \langle f,\phi_k\rangle \\ a_k(0) = (u_0,\phi_k) \end{cases} \tag{6.2}$$

for $k \le N$.

Lemma 6.1. (Picard-Lindelof) Let $f(t,s) : [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ be continuous in t and locally Lipschitz in s, and $t_0 \in (0,1)$. Then the classical ODE

$$\begin{cases} \dot{z} = f(t,z) \\ z(t_0) = z_0 \end{cases}$$

has a unique C^1 solution in the interval $[t_0-\epsilon, t_0+\epsilon]$ for some $\epsilon > 0$.

Example 6.2. Not necessarily global, e.g. $\dot{z} = z^2$ blows up in finite time.

Rewriting the Galerkin system, we have

$$\begin{aligned} \dot{u_N} &= -\nu A u_N - P_N B(u_N,u_N) + P_N f \\ &= F(u_N) \end{aligned} \tag{6.3}$$

This equation is in the finite dimensional space H_N, so Picard-Lindelof may be used. F is trivially continuous in time as it is constant. Moreover $A + P_N f$ is Lipschitz in space because it is affine. It remains to show $P_N B(\cdot,\cdot)$ is too.

$$\begin{aligned} \|P_N(B(x,x) - B(y,y))\| &\le \|B(u,u-v)\| + \|B(u-v,v)\| \\ &\le C(\|u\|^{1/2}|u|^{1/2} + \|v\|^{1/2}|v|^{1/2})\|u-v\| \end{aligned}$$

by Proposition 4.4, noting all norms are equivalent in finite dimensions. It follows that $P_N B(\cdot,\cdot)$ is locally Lipschitz. Applying Picard-Lindelof to $F : [-1,T]\times\mathbb{R}^3 \to \mathbb{R}^3$, where T is arbitrarily large, we see (6.3) has unique short time solutions with initial condition specified at any $t_0 \in [0,\infty)$.

Proposition 6.3. There is a unique global solution to the Galerkin system (6.3) for each N.

Proof. Fix N and let y be the unique solution of (6.3) around $t = 0$ and suppose the largest interval of existence of u is $[0,\tau)$. Note the interval must be half-open otherwise it is closed and we can extend it by Picard-Lindelof. Then $\lim_{t\to\tau}\|y(t)\|_\infty = \infty$ otherwise we can extend it to the limit. Since norms in N-dimensions are equivalent, we have $\lim_{t\to\tau}|y(t)| = \infty$. We show this is impossible.

Let (6.3) act on y, then

$$\begin{aligned} &(\dot{y}, y) + \nu(Ay, y) + (P_N B(y,y), y) = (P_N f, y) \\ \therefore \quad &\frac{1}{2}\frac{d}{dt}|y|^2 + \nu\|y\|^2 + 0 \le \|f\|_{V'}\|y\| \end{aligned} \tag{6.4}$$

We wish to use Gronwall, but we have both $|y|$ and $\|y\|$ in the mix and the $\|y\|$ terms have different powers. To proceed, we do a magic trick that we will use time and time again. Recall the trivial inequality $a^2+b^2 \ge 2ab$. We want both $\|y\|^2$ terms to have related coefficients[2], thus we apply this inequality to $a = \sqrt{\nu}\|y\|$ and $b = \frac{1}{\sqrt{\nu}}\|f\|_{V'}$ to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|y|^2 + \nu\|y\|^2 \le \frac{1}{2\nu}\|f\|_{V'}^2 + \frac{1}{2}\nu\|y\|^2 \\ \therefore \quad &\frac{1}{2}\frac{d}{dt}|y|^2 + \frac{1}{2}\nu\lambda_1|y|^2 \le \frac{1}{2\nu}\|f\|_{V'}^2 \end{aligned}$$

Thus by Gronwall,

$$|y(t)|^2 \le e^{-\lambda_1\nu t}|u_0|^2 + \frac{\|f\|_{V'}^2}{2\nu^2\lambda_1} \tag{6.5}$$

In particular, $|y|$ is bounded, a contradiction. □

We conclude from uniqueness of Lindelof-Picard, there is a unique solution to (6.3) for $t \in [0,\infty]$ which we call u_N that lives in H_N, thus determining coefficients $(a_{k,N})_{k=1}^N$ such that $u_N = \sum_1^N a_{k,N}(t)\phi_k$. However, $\sum_1^N a_{k,N+1}(t)\phi_k$ also solves (6.3) so we deduce there is a single choice of coefficients $(a_k)_1^\infty$ such that $u_N = \sum_1^N a_k(t)\phi_k$.

From (6.5) we also learn, for any $t > 0$ arbitrary,

$$\begin{aligned} &|u_N(t)|^2 \le |u_0|^2 + \frac{\|f\|_{V'}^2}{2\nu^2\lambda_1} \\ \therefore \quad &\|u_N\|_{C([0,T],H)} \le K_0 \end{aligned} \tag{6.6}$$

[2] Or for a physicist, the same dimensions

for some constant $K_0 = K_0(u_0, f, \nu, \lambda)$. By Banach-Alaoglu, there is a subsequence u_{N_j} converging weakly to some $v \in L^\infty([0,T], H)$[3]. The NSE has a non-linear term unlike the Stokes equation, and weak convergence does not play nice with non-linearities.

Example 6.4. $y_n = \sin nx \to 0$ weakly in $C([0,1])$ but $y_n^2 = \frac{1}{2} - \frac{1}{2}\cos 2nx \to \frac{1}{2}$ weakly, not 0.

We aim to get strong convergence (perhaps along a further subequence) in some norm. To do this, we will apply a compactness theorem in which we require bounds on more norms. We just estimated the $|u_N(t)|$ norms, and next we try $\|u_N(t)\|$. Recall that in the Proposition 6.3 we quickly discarded estimates involving $\|y\|$ for $|y|$. This is our first lead. Again, letting (6.4) act on u_N,

$$
\begin{aligned}
(\dot{u_N}, u_N) + \nu(Au_N, u_N) + (B(u_N, u_N), u_N) &= \langle f, u_N\rangle_{V',V} \\
\therefore \quad \frac{1}{2}\frac{d}{dt}|u_N|^2 + \nu\|u_N\|^2 + 0 &\leq \|f\|_{V'}|u_N| \\
&\leq \|f\|_{V'}\frac{\|u\|}{\lambda_1^{1/2}} \\
&\leq \frac{\|f\|_{V'}^2}{2\lambda_1\nu} + \frac{1}{2}\nu\|u\|^2 \\
\frac{d}{dt}|u_N|^2 + \nu\|u_N\|^2 &\leq \frac{\|f\|_{V'}^2}{\lambda_1\nu}
\end{aligned}
$$

Integrating from 0 to T,

$$
\begin{aligned}
|u_N(T)|^2 - |u_N(0)|^2 + \nu\int_0^T \|u_N(s)\|^2 ds &\leq \frac{\|f\|_{V'}^2}{\nu\lambda_1}T \\
\therefore \quad |u_N(T)|^2 + \nu\int_0^T \|u_N(s)\|^2 ds &\leq \underbrace{|u_0|^2 + \frac{\|f\|_{V'}^2 T}{\nu\lambda_1}}_{=:\overline{K_1}}
\end{aligned}
$$

Thus $\|u_N\|_{L^2((0,T),V)} \leq \overline{K_1}$ for all N, noting $\overline{K_1}$ is independent of N. By Banach-Alaoglu, we have further convergent subsequence converging v in $L^2((0,T),V)$ weak$*$. We summarise our two results so far, and will see a useful theorem applies.

[3] Banach Alaoglu can only be applied to a dual space, so we cannot apply it to $C([0,T],H)$.

Lemma 6.5. Let u_N be the unique global solutions to (6.3) described in Proposition 6.3. Then there is a subsequence u_{N_j} and $v \in L^\infty((0,T),H) \cap L^2((0,T),V)$ such that

$$\begin{cases} u_{N_j} \to v & L^\infty((0,T),H) \text{ weak}* \\ u_{N_j} \to v & L^2((0,T),V) \text{ weak}* \end{cases} \tag{6.7}$$

Our candidate solution to (6.1) is v, but one will have a hard time showing it indeed solves the equation using only the weak* convergence we have established. We will now try to prove strong convergence in a suitable space.

Theorem 6.6. (Aubin-Lions Compactness) Suppose we have a sequence of Banach space embeddings:

$$X_1 \xhookrightarrow{\text{compact}} X_0 \xhookrightarrow{\text{cont.}} X_{-1}$$

For $1 \le p, q \le \infty$, let

$$W = \{u \in L^p((0,T),X_1) : \dot{u} \in L^q((0,T),X_{-1})\}$$

a subspace of $L^p((0,T),X_1)$. Then,

(i) if $p < \infty$ then W is compactly embedded in $L^p((0,T),X_0)$.

(ii) If $p = \infty$ and $q > 1$ then W is compactly embedded in $C([0,T],X_0)$.

Corollary 6.7. We have $u_N \to v$ in $L^2((0,T),H)$ strongly along a subsequence

Proof. We apply Aubin-Lions. Recall $V \xhookrightarrow{\text{cpct.}} H \xhookrightarrow{\text{cont.}} V'$. We know that u_N is bounded in $L^2((0,T),V)$. The second condition, $\dot{u_N} \in L^q((0,T),V')$, is trivial for all q, because the Picard-Lindelof solutions u_N are in $C^1([0,T],V')$ for any $T > 0$ (Proposition 6.3)[4] and $C([0,T],V') \subset L^q((0,T),V')$ for any q. The conclusion follows.

□

To take limits in (6.3), we need to know $\dot{u_N}$ converges to $\dot{v}$ and in what sense.

[4]In fact they are in $C^1([0,T],C^\infty)$ since all norms in finite dimensions are equivalent.

Proposition 6.8. The norms $\|\dot{u_N}\|_{L^{4/3}((0,T),V')}$ are uniformly bounded in N and in particular we have $\dot{u_N} \to \dot{v}$ weakly in $L^{4/3}((0,T),V')$.

Proof. From (6.3), we have

$$\|\dot{u_N}\|_{V'} \le \nu\|Au_N\|_{V'} + \|P_N B(u_N,u_N)\|_{V'} + \|f\|_{V'}$$

We show each term is bounded. Observe (or recall from Proposition 3.24), $\|Au_N\|_{V'} = \sup_{w\in V,\|w\|=1}((u,w)) = \|u\|$, so

$$\int_0^T \|Au_N\|_{V'}^2 = \int_0^T \|u_N\|^2 \le \overline{K_1}^2$$

where we recall $\overline{K_1}^2$ is the constant bounding $\|u_N\|_{L^2((0,T),V)}$. Lastly, for the non-linearity we use our usual estimate

$$\begin{aligned}\|P_N B(u_N,u_N)\|_{V'} &\le C\|u_N\|^{1/2}|u_N|^{1/2}\|u_N\| \\ &\le CK_0^{1/2}\|u_N\|^{3/2}\end{aligned}$$

Thus,

$$\int_0^T \|P_N B(u_N,u_N)\|_{V'}^{4/3} \le C^{4/3}K_0^{2/3}\int_0^T \|u_N\|^2$$

is uniformly bounded in N. The last statement is Banach Alaoglu. □

Remark 6.9. The only obstruction to getting $\dot{u_N}$ bounded in $L^2((0,T),V')$ is having $P_N B(u_N,u_N)$ bounded in $L^2((0,T),V')$. To obtain this, we would need a uniform bound on

$$C^2K_0\int_0^T \|u_N\|^3$$

i.e. u_N bounded in L^3, which we do not have. In 2D, one follows our methods and gets u_N bounded in L^4, thus in L^3.

We summarise what we have:

Lemma 6.10. Let u_N be the unique global solutions to (6.3) described in Proposition 6.3. Then there is a subsequence u_{N_j} and $v \in L^\infty((0,T),H) \cap$

$L^2((0,T),V)$ with $\dot{v} \in L^{4/3}((0,T),V')$ such that

$$\begin{cases} u_{N_j} \to v & L^\infty((0,T),H) \text{ weak}* \\ u_{N_j} \to v & L^2((0,T),V) \text{ weak}* \\ \dot{u_{N_j}} \to \dot{v} & L^{4/3}((0,T),V') \text{ weak}* \\ u_{N_j} \to v & L^2((0,T),H) \text{ strongly} \end{cases} \tag{6.8}$$

Theorem 6.11. The limit v from Lemma 6.10 solves the 3D NSE (6.1) weakly (i.e. in V').

Proof. Without loss of generality, assume the subsequence in Lemma 6.10 is the whole sequence. Equation (6.3) states

$$\dot{u_N} + \nu A u_N + P_N B(u_N, u_N) = P_N f$$

We analyse each term, applying Lemma 6.10. We know $\dot{u_N} \to \dot{v}$ in $L^{4/3}((0,T),V')$ weakly. Also $Au_N \to Av$ in $L^2((0,T),V')$ weakly since $u_N \to v$ in $L^2((0,T),V)$ weakly. Further, $P_N f \to f$ in $L^2((0,T),V')$. It remains to deal with the non-linearity. Assuming this for now, we have

$$\dot{v} + \nu A v + B(v,v) = f \qquad \text{in } L^{4/3}((0,T),V')$$

and so by standard measure theory, we have, for almost all $t \in (0,T)$,

$$\dot{v}(t) + \nu A v(t) + B(v,v)(t) = f \qquad \text{in } V'$$

as desired. We now turn to the non-linearity. It is straightforward to check $B(u_N,u_N)$ is bounded in $L^{4/3}((0,T),V')$.

<u>Claim</u> $P_N B(u_N,u_N) \to B(v,v)$ in $L^2((0,T),V')$ weakly.

We need, for all $w \in V$,

$$\lim_{N\to\infty} \left| \int_0^T \langle P_N B(u_N,u_N) - B(v,v), w\rangle_{V',V} ds \right| = 0$$

Suffices to show for $w = \phi_k$.

$$\Big|\int_0^T \langle P_N B(u_N, u_N) - B(v,v), \phi_k\rangle_{V',V} ds\Big|$$

$$\leq \int_0^T |\langle B(u_N - v, u_N), \phi_k\rangle| + |\langle B(v, u_N - v), \phi_k\rangle| \qquad (N > k)$$

$$\leq C\int_0^T |u_N - v|^{1/2}\|u_N - v\|^{1/2}\|u_N\|\|\phi_k\| + \|v\|\|u_N - v\|^{1/2}|u_N - v|^{1/2}\|\phi_k\|$$

$$\leq 2C\|\phi_k\| \int_0^T |u_N - v|^{1/2}(\|u_N\| + \|v\|)^{3/2}$$

$$\leq 2C\|\phi_k\| \left(\int_0^T |u_N - u|^2\right)^{1/4} \left(\int_0^T (\|u_N\| + \|v\|)^2\right)^{3/4} \qquad \text{(Holder)}$$

$$\leq 2C\|\phi_k\|\|u_N - v\|^{1/2}_{L^2((0,T),H)} \left(\int_0^T (\|u_N\| + \|v\|)^2\right)^{3/4}$$

where to get the second inequality we switched $u_N - v$ with ϕ_k. It remains to show the above display tends to zero. Indeed, $\|u_N - v\|_{L^2((0,T),H)} \to 0$ by Lemma 6.10 and the other term is bounded since $\|u_N\|_{L^2((0,T),V)}$ is.

□

Remark 6.12.

- We showed that (6.1) holds for almost all t. In our approach, this is the best we can do because we defined v to be a limit of L^p functions, which are only defined almost everywhere. In the next section, we show v has a continuous representative, so we can talk about properties of v holding everywhere. However, we do not show that $\dot{v}$ has a continuous representative, so we cannot say the (6.1) holds for all t.
- Weak solutions are not unique in general.

6.2. Regularity of weak solutions

Notation 6.13. In this section, v is any weak solution of (6.1), not necessarily the solution constructed in Theorem 6.11.

Proposition 6.14. Suppose $v \in L^2((0,T),V)$. Then there is a representative of v in $C([0,T],V')$.

Proof.

$$
\begin{aligned}
\|v(t)-v(s)\|_{V'} &= \|\nu \int_s^t Av(u)du + \int_s^t B(v,v)du + (t-s)f\|_{V'} \\
&\leq \nu \int_s^t \|v(u)\|du + \int_s^t \|B(v,v)\|_{V'}du + |t-s|\|f\|_{V'} \\
&\leq \nu\sqrt{|t-s|}\sqrt{\int_s^t \|v(u)\|^2 du} + |s-t|^{1/4}\left(\int_s^t \|B(v,v)\|_{V'}^{4/3}\right)^{3/4} + \int |t-s|\|f\|_{V'} \\
&\to 0 \quad \text{as } s \to t.
\end{aligned}
$$

□

Remark 6.15. From the above proof we see that since we're on a bounded interval $(0,T)$, the last inequality implies Holder continuity with exponent $\frac{1}{4}$. From now on, we use v to refer to the continuous representative, not the L^p-equivalence class. We skipped showing that $B(v,v) \in L^{4/3}((0,T),V')$, but this is easy to check.

We now show an even stronger form of continuity.

Notation 6.16. Define H_{weak} to be H equipped with its weak topology. Thus v is H_{weak}-continuous if $t \to t_0$ then $(v(t),h) \to (v(t_0),h)$ for all $h \in H$.

Proposition 6.17. Suppose $v \in L^2((0,T),V) \cap L^\infty((0,T),H)$. Then $v \in C([0,T],H_{\text{weak}})$.

Proof.

Claim $\exists K > 0$ such that $|v(t)| \leq K$ for all $t \in [0,T]$.

By assumption we have a constant K and $E \subset [0,T]$ with $v(t) < K$ and $|E| = 0$ for all $t \notin E$. Suppose $t_0 \in E$ and take $t_n \in E^c$ such that $t_n \to t_0$. By the previous proposition, $v(t_n) \to v(t_0)$ in V'. But also, $|v(t_n)| \leq K$ implies $v(t_n)$ converges weakly in H (possibly along a subsequence, which we relabel). Thus $v(t_0) \in H$ and by the standard weak convergence result, $|v(t_0)| \leq \liminf |v(t_n)| \leq K$ as desired.

For all $w \in V$,

$$(v(t),w) = \langle v(t),w\rangle_{V',V} \to \langle v(t_0),w\rangle_{V',V} = (v(t_0),w).$$

as $t \to t_0$, since $v \in C([0,T],V')$. It remains to extend this to all $w \in H$. Indeed, let $w_n \in V$ such that $w_n \to w$ in H. Then,

$$\begin{aligned} |(v(t)-v(t_0),w)| &\le |(v(t)-v(t_0),w-w_n)| + |(v(t)-v(t_0),w_n)| \\ &\le 2K|w-w_n| + |(v(t)-v(t_0),w_n)| \end{aligned}$$

$$\begin{aligned} \therefore \qquad \limsup_{t\to t_0} |(v(t)-v(t_0),w)| &\le 2K|w-w_n| + 0 \\ &\to 0 \end{aligned}$$

as $n \to \infty$. □

Remark 6.18. This proposition didn't directly use the fact v was a solution of the 3D NSE, only that $v \in C([0,T],V')$. The weak solution we found in the previous subsection satisfies the hypotheses of this proposition.

6.3. The Energy Inequality

Notation 6.19. In this section, v is the weak solution of (6.1) obtained in Theorem 6.11.

We have

$$\dot{v} + \nu A v + B(v,v) = f \qquad \text{in } L^{4/3}((0,T),V')$$

Then, acting on v,

$$\int_0^T \langle \dot{v}, v\rangle_{V',V} + \nu\|v\|^2 + 0dt = \int_0^T \langle f, v\rangle_{V',V} dt$$

$$\therefore \qquad \underbrace{\frac{1}{2}|v(t)|^2 + \nu\int_0^T \|v(t)\|^2 dt}_{\text{final energy}} = \underbrace{\int_0^T \langle f, v\rangle dt + \frac{1}{2}|u_0|^2}_{\text{input energy}}$$

This is a nice result for the physicists. Unfortunately, it is wrong. We are not allowed to act on v since we only know $v \in L^2((0,T),V)$, where as $L^{4/3}((0,T),V') = (L^4((0,T),V))^*$. In 2D, our previous methods would give $v \in L^4((0,T),V)$, so this would work. Instead, we have:

Definition 6.20. A solution v to (6.1) is a *Leray-Hopf solution* if $v \in L^2([0,T],V) \cap L^\infty([0,T],H)$

$$\frac{1}{2}|v(t)|^2 + \nu \int_{t_0}^{t} \|v(s)\|^2 ds \leq \int_{t_0}^{t} \langle f, v \rangle ds + \frac{1}{2}|v(t_0)|^2, \quad \forall t > t_0$$

for $t_0 = 0$ and almost all $t_0 \in (0, T)$. In words, energy can only be dissipated over time.

Theorem 6.21. (Energy Inequality) The Galerkin solution v is Leray-Hopf.

Proof. The argument given above works for the Galerkin system u_N solving (6.3) since the u_N are even smooth with smooth derivatives, so certainly in L^4. Thus,

$$\frac{1}{2}|u_N(t)|^2 + \nu \int_{t_0}^{t} \|u_N(s)\|^2 ds = \int_{t_0}^{t} \langle f, u_N \rangle ds + \frac{1}{2}|u_N(t_0)|^2$$

Assume u_N converges to v in $L^2((0,T), H)$ (else take a subsequence). Then there is a set $E \subset [0, T]$ of full measure such that $u_N(t) \to v(t)$ in H for all $t \in E$. Moreover, $0 \in E$ since $u_N(t) \to u_0 = v(0)$. Therefore,

(i) $|u_N(t)| \to |v(t)|$ for all $t \in E$

(ii) $\int_{t_0}^{t} \langle f, u_N \rangle dt \to \int_{t_0}^{t} \langle f, v \rangle dt$

(iii) $u_N \to v$ in $L^2((0,T), V)$ weakly, so

$$\begin{aligned} \liminf_N \int_{t_0}^{t} \|u_N(s)\|^2 ds &\geq \int_{t_0}^{t} \liminf_N \|u_N(s)\|^2 ds && \text{(Fatou)} \\ &\geq \int_{t_0}^{t} \|v(s)\|^2 ds && \text{(Weak convergence)} \end{aligned}$$

Hence for $t, t_0 \in E$,

$$\frac{1}{2}|v(t)|^2 + \nu \int_{t_0}^{t} \|v(s)\|^2 ds \leq \int_{t_0}^{t} \langle f, v \rangle ds + \frac{1}{2}|v(t_0)|^2$$

Lastly, we upgrade to '$\forall t$'. Suppose $t \notin E$ and take $t_n \in E$ with $t_n \to t$. We know v is H_{weak}-continuous, so $v(t_n) \to v(t)$ weakly in H thus $|v(t)|^2 \leq \liminf_n |v(t_n)|^2$. The integrals are continuous in t with t_0 fixed, thus

$$\frac{1}{2}|v(t)|^2 + \nu \int_{t_0}^{t} \|v(s)\|^2 ds \leq \int_{t_0}^{t} \langle f, v \rangle ds + \frac{1}{2}|v(t_0)|^2$$

□

Remark 6.22.

- The inequality says final energy is less than input energy. Thus there is some unknown dissipation in the system.
- We used the fact our solutions came from Galerkin. It is unknown whether there is a weak solution u to (6.1) with $u \in L^\infty((0,T),H) \cap L^2((0,T),V)$ with $\dot{u} \in L^{4/3}((0,T),V')$ (i.e. the regularity of the Galerkin solution shown in Lemma 6.10) which is not a Leray-Hopf solution. Without regularity assumptions, one can show there are solutions with the inequality the other way around, i.e. energy is *increasing* over time.

6.4. Existence of strong solutions

We seek to solve

$$\begin{cases} \dot{u} + \nu A u + B(u,u) = f & \text{in } H \\ u(0) = u_0 \end{cases} \tag{6.9}$$

with $f \in H$ and $u_0 \in V$ known. From the sections on weak solutions, we know the Galerkin system

$$\begin{cases} \dot{u_N} + \nu A u_N + P_N B(u_N,u_N) = P_N f \\ u_N(0) = P_N u_0 \end{cases} \tag{6.10}$$

has a unique solution for each N. We recall from (6.6) that we have a uniform bound $\sup_N \sup_{0\le t\le T} |u_N(t)| \le K_0$. The hope is that now, with the assumption $f, u_0 \in H$, we can find a uniform bound on $\|u_N(t)\|$. Unfortunately, this is too much to ask[5], and instead what we can obtain is a uniform bound on a short time interval:

Proposition 6.23. (Young's inequality) If $a,b \ge 0$ and $1/p + 1/q = 1$ then $ab \le \frac{a^p}{p} + \frac{b^q}{q}$

Lemma 6.24. The norms $\|u_N\|_{C((0,T^*),V)}$ are uniformly bounded on a short time interval i.e. there is a $T^* \in (0,T]$ and constant K' such that

$$\sup_N \sup_{0\le t\le T^*} \|u_N\| \le K'$$

[5] If you manage to do it, email me before you email anyone else.

Proof. Previously we wanted to bound $|u_N|$ so we let the equation (6.10) act on u_N. To bound the V-norm, we act on Au_N.

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \nu|Au_N|^2 + (B(u_N,u_N),Au_N) = (f,Au_N) \tag{6.11}$$

and we recall

$$\begin{aligned}|(B(u_N,u_N),Au_N)| &\le \|u_N\|\|\nabla u_N\|_{L^3}|Au_N| \\ &\le \|u_N\|C\|u_N\|^{1/2}|Au_N|^{1/2}|Au_N| \\ &\le C\|u_N\|^{3/2}|Au_N|^{3/2} \\ &\le \frac{C^4}{4\nu^{3/2}}\|u_N\|^6 + \frac{3\nu}{4}|Au_N|^2 \qquad \text{(Young } p=4)\end{aligned}$$

where the fudging constants have been inserted so that the units match. Or alternatively, so that we have a 'νAu_N' term. We also apply Young's inequality in a clever way to the other term,

$$|(f,Au_N)| \le |f||Au_N| \le \frac{2|f|^2}{\nu} + \frac{\nu}{8}|Au_N|^2$$

which prevents the $|Au_N|^2$ terms cancelling in a moment. Thus (6.11) becomes

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \nu|Au_N|^2 \le \frac{2|f|^2}{\nu} + \frac{\nu}{8}|Au_N|^2 + \frac{1}{4\nu^{3/2}}\|u_N\|^6 + \frac{3\nu}{4}|Au_N|^2$$

$$\therefore \quad \frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \frac{\nu}{8}|Au_N|^2 \le \frac{2|f|^2}{\nu} + \frac{1}{4\nu^{3/2}}\|u_N\|^6 \tag{6.12}$$

$$\therefore \quad \frac{d}{dt}\|u_N\|^2 \le C_1(\|u_N\|^2 + C_2)^3$$

for some constants $C_1, C_2 > 0$ depending on $|f|$ and ν but not N. Everything in the proof so far has been similar to what we have done before, and now we do something tricky. Let $z_N = \|u_N\|^3 + C_2$. Then,

$$\dot{z_N} \le C_1 z_N^3$$

It remains to show z_N is bounded independently of N on some short time interval. Since z_N is continuous, we can find $T_N > 0$ such that

$$z_N(t) \le 2z_N(0), \qquad \forall t \in [0,T_N]$$

We define T_N to be the supremum time with this property. If T_N is finite, we must have equality by continuity. Moreover,

$$\begin{aligned}
\dot{z_N}(t) &\leq 8C_1|z_N(0)|^3, \qquad t \in [0, T_N] \\
&\leq 8C_1(\|u_0\|^2 + C_2)^3 \\
\therefore \qquad z_N(t) &\leq 8C_1(\|u_0\|^2 + C_2)^3 t + z_N(0) \\
\therefore \qquad 2z_N(0) &\leq 8C_1(\|u_0\|^2 + C_2)^3 T_N + z_N(0) \\
\therefore \qquad T_N &\geq \frac{z_N(0)}{8C_1(\|u_0\|^2 + C_2)^3}
\end{aligned}$$

Thus, regardless of whether it is finite or not, we have $T_N \geq T^*$ where

$$T^* := \frac{C_2}{8C_1(\|u_0\|^2 + C_2)^3}$$

Hence, on $[0, T^*]$, we have $\|u_N(t)\| \leq 2\|u_0\|^2 + C_2 =: K'$.

□

Lemma 6.25. The norms $\|u_N\|_{L^2((0,T^*),D(A))}$ are uniformly bounded. Consequently, $Au_N \to Av$ in $L^2((0, T^*), D(A))$ weakly along a subsequence.

Proof. Some information was discarded during the proof of the previous lemma. Returning to (6.12) and integrating,

$$\begin{aligned}
(\|u_N(T^*)\|^2 - \|u_0\|^2) + \frac{\nu}{4}\int_0^{T^*} |Au_N|^2 ds &\leq cT^* \\
\therefore \qquad \nu \int_0^{T^*} |Au_N|^2 &\leq \|u_0\|^2 + cT^*
\end{aligned}$$

for some constant c depending on on K', $|f|$, and ν. We know that $Au_N \to Av$ in $L^2((0, T^*), V')$ weakly along a subsequence from Theorem 6.11, hence by Banach-Alaoglu $Au_N \to Av$ in $L^2((0, T^*), D(A))$ along a further subsequence. □

Corollary 6.26. We have $u_N \to v$ strongly in $L^2((0, T^*), V)$ along a subsequence, where v is the same limit from Lemma 6.10. Moreover, $u_N \to v$ in $L^2((0, T^*), D(A))$ weakly

Proof. We note that everything from the section on weak solutions still holds, in particular Lemma 6.10. We have $D(A) \overset{\text{cpct.}}{\hookrightarrow} V \overset{\text{cont.}}{\hookrightarrow} H$. The sequence u_N is bounded in $L^2((0,T^*), D(A))$ by the previous lemma and $\dot{u_N} \in L^q((0,T^*), H)$ for any q for the same reason as in Corollary 6.7. Thus, by Aubin-Lions, u_N has a $L^2((0,T^*), V)$ convergent subsequence and the conclusion follows. The 'Moreover' is Banach Alaoglu on the previous lemma. □

Solving the NSE (6.9) in H is to say that $\dot{v}(t) + \nu Av(t) + B(v,v)(t) = f$ in H for almost all $t \in [0,T^*]$. This is equivalent to

$$\dot{v} + \nu Av + B(v,v) = f \qquad \text{in } L^2((0,T^*), H)$$

since equal in L^p is equivalent to being equal almost everywhere. The previous Corollary 6.26 gives v and Av in $L^2((0,T^*), H)$, but we do not yet know about $\dot{v}$ and $B(v,v)$. We recall in the case of weak solutions, we only had $B(v,v)$ in $L^{4/3}$, which is weaker than in L^2. In this case we will obtain what we want, and a little more.

Lemma 6.27. We have $P_N B(u_N,u_N)$ uniformly bounded in $L^4((0,T^*), H)$ and $B(v,v) \in L^4((0,T^*), H)$. Moreover, $P_N B(u_N,u_N) \to B(v,v)$ in $L^4((0,T^*), H)$ weakly along a subsequence.

Proof. Since we are in finite dimensions, we know $P_N B(u_N,u_N) \in H$.

$$\begin{aligned} |P_N B(u_N,u_N)| &\le \sup_{\substack{w\in H \\ |w|=1}} (B(u_N,u_N), w) \\ &= \sup_{|w|=1} \int_\Omega (u_N \cdot \nabla) u_N \cdot w \end{aligned}$$

Note that unlike in the weak case, we put the L^2 norm on w and not the L^6 norm.

$$\begin{aligned} &\le \|u_N\|_{L^6} \|\nabla u_N\|_{L^3} \\ &\le C\|u_N\| \|\nabla u_N\|_{H^{3/2}} \\ &\le C\|u_N\| \|u_N\|^{1/2} |u_N|^{1/2} \\ &\le C(K')^{3/2} |Au_N|^{1/2} \end{aligned}$$

where the last inequality is Lemma 6.24. Thus $P_N B(u_N,u_N)$ is uniformly bounded in $L^4((0,T^*), H)$. Recall from Theorem 6.11 that $P_N B(u_N,u_N) \to$

$B(v,v)$ in $L^2((0,T^*),V')$ along a subsequence. Hence, after applying a standard Banach-Alaoglu argument we have $P_N B(u_N,u_N) \to B(v,v)$ in $L^4((0,T^*),H)$ weakly along a subsequence. $\square$

Corollary 6.28. We have $\dot{u_N}$ uniformly bounded in $L^2((0,T^*),H)$. Consequently, $\dot{u_N} \to \dot{v}$ in $L^2((0,T^*),H)$ weakly along a subsequence.

Proof. We know

$$|\dot{u_N}| \leq \nu|Au_N| + |P_N B(u_N,u_N)| + |f|$$

Each term on the RHS is bounded in $L^2((0,T^*),H)$, hence so is $\dot{u_N}$. The usual Banach-Alaoglu argument gives $\dot{u_N} \to \dot{v}$ in $L^2((0,T^*),H)$ along a subsequence, noting $\dot{u_N} \to \dot{v}$ in $L^{4/3}((0,T^*),V')$ along a subsequence from Theorem 6.11. $\square$

Corollary 6.29. The limit of the Galerkin system v satisfies (6.9) for $t \in [0,T^*]$, where $T^* > 0$ is a constant depending only on $\|u_0\|$, $|f|^2$, and ν. Moreover, $v \in L^2((0,T^*),D(A)) \cap C((0,T^*),V)$ and $\dot{v} \in L^2((0,T^*),H)$.

Proof. Let $w \in L^2((0,T^*),H)$ arbitrary act on the Galerkin equation (6.10). The previous lemmas give weak convergence of each term as $N \to \infty$, and we see (6.9) holds. $\square$

6.5. Regularity of strong solutions

Let v be any solution to (6.9) in $L^2((0,T),D(A))$, i.e.

$$\dot{v} + \nu Av + B(v,v) = f \qquad \text{in } L^2((0,T),D(A))$$

Acting on $v \in L^2((0,T),D(A))$,

$$\int_0^T (\dot{v},v) + \nu(Av,v) + (B(v,v),v)dt = \int_0^T (f,v)dt$$

$$\therefore \qquad \int_0^T \frac{1}{2}\frac{d}{dt}|v|^2 + \nu\|v\|^2 + 0dt = \int_0^T (f,v)dt$$

$$\therefore \qquad \frac{1}{2}v(T)^2 + \nu\int_0^T \|v(t)\|^2 dt + \int_0^T (f,v)dt + |u_0|^2$$

gives the *Energy equality*. Note this is the same proof given at the beginning of Section 6.3 (which is now correct).

We now prove another regularity result.

Proposition 6.30. The map $t \to \|v(t)\|$ is continuous and moreover $v \in C([0,T], V_{\text{weak}})$.

Proof. Letting (6.9) act on $Av \in L^2((0,T^*),H)$,

$$(\dot{v}, Av) + \nu|Av|^2 + (B(v,v),Av) = (f,Av)$$

$$\therefore \quad \frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu|Av|^2 + (B(v,v),Av) = (f,Av) \qquad \text{(Lions-Magenes)}$$

Integrating for s,t arbitrary,

$$\|v(s)\|^2 - \|v(t)\|^2 = -2\nu\int_t^s |Av|^2 + 2\int_t^s (B(v,v),Av) + \int_t^s (f,Av)$$

and so the first claim is proven provided $(B(v,v),Av) \in L^1$ (as the others are).

$$\begin{aligned}|(B(v,v),Av)| &\le \|v\|\|v\|^{1/2}|Av|^{1/2}|Av| \\ &= \|v\|^{3/2}|Av|^{3/2} \\ &\le (K')^{3/2}|Av^{3/2}|\end{aligned}$$

Hence $(B(v,v),Av) \in L^{4/3}(0,T) \subset L^1(0,T)$.

For the second claim, the proof is identical to Proposition 6.17 with the Sobolev spaces shifted up by one.

□

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6.6. Uniqueness of strong solutions

Let u,v be two solutions to (6.9) on the interval $(0,T)$ satisfying the regularity conclusions in Corollary 6.29 and $w = u - v$. Then,

$$\dot{w} + \nu Aw + B(u,w) + B(w,u) - B(w,w) = 0 \qquad \text{in } L^2((0,T),H)$$

Acting on $w \in L^2((0,T),H)$,

$$
\begin{aligned}
& (\dot{w}, w) + \nu(Aw, w) + (B(w,u), w) = 0 \\
\therefore \quad & \frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + (B(w,u),w) = 0 \\
\therefore \quad & \frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 \leq |(B(w,u),w)| \\
& \qquad \leq |w|^{1/2}\|w\|^{1/2}\|u\|\|w\| \\
& \qquad \leq |w|^{1/2}\|w\|^{3/2}\|u\|
\end{aligned}
$$

We are faced with the same situation that we have seen many times before, where we wish to remove $\|w\|$ from the picture so that Gronwall can be applied. To do this, we apply Young's inequality on the RHS with a fudging constant $\nu^{3/4}$ so that the coefficients of $\|w\|^2$ on each side are comparable (or so that 'dimensions match'):

$$\nu^{3/4}\|w\|^{3/2}(\nu^{-3/4}|w|^{1/2}\|u\|) \leq \frac{3}{4}\nu\|w\|^2 + \frac{1}{4}(\nu^{-3}|w|^2\|u\|^4)$$

and so we have

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|w|^2 \leq \frac{1}{4\nu^3}|w|^2\|u\|^4 \\
\therefore \quad & |w(t)|^2 \leq e^{C\int_0^t \|u\|^4 d\tau}|w(0)|^2 \qquad \text{(Gronwall)}
\end{aligned}
$$

for some constant C depending on ν. We conclude that if $w(0) = 0$ (i.e. $u(0) = v(0)$) then $w = 0$ and even if we don't assume this we have the stability estimate:

Theorem 6.31. (Strong NSE well posedness) Let u, v be two solutions to (6.9) on $(0,T)$ except possibly with different initial conditions. Then for all $t \in (0,T)$,

$$|v(t) - u(t)| \leq c|v(0) - u(0)|$$

for some constant c depending on ν and T (or alternatively, a function growing like $e^{C\int_0^t \|u\|^4 d\tau}$). In particular, solutions are uniquely determined by their initial condition.

Remark 6.32. In 2D, one shows weak solutions are unique and strong solutions exist globally. For 3D, this yields the Millenium Prize.

6.7. Weak solutions are strong solutions

In the previous section, we showed that assuming $f \in H$, the weak solution found in Section 6.1 is actually a strong solution. In this section, we show any weak solution with the same level of regularity is also a strong solution.

Theorem 6.33. Let $u \in C([0,T], H_{\text{weak}}) \cap L^2((0,T),V)$ and $\dot{u} \in L^{4/3}((0,T),V')$ with $u(0) = u_0 \in V$ be a solution to (6.1) satisfying the energy inequality for all t:

$$\frac{1}{2}|u(t)|^2 + \nu \int_0^t \|u(s)\|^2 ds \leq \int_0^t (f, u(s))ds + \frac{1}{2}|u_0|^2$$

Suppose that a (unique) solution v to (6.9) exists on $[0,T]$ with $v(0) = u_0$. Then $u(t) = v(t)$ for all $t \in [0,T]$.

Proof. Let $w(t) = u(t) - v(t) \in V'$ then

$$\dot{w} + \nu A w + B(v,w) + B(w,v) - B(w,w) = 0$$

holds in $L^{4/3}((0,T),V')$ – the weakest space each term belongs to. Following the idea of the proof of Theorem 6.31, we would want to act on w now. Unfortunately, we can't as we don't know $w \in L^4((0,T),V) = (L^{4/3}((0,T),V'))^*$. We use 'Serrin's trick'[6]

$$\begin{aligned} &\dot{u} + \nu A u + B(u,u) = f \qquad \text{in } L^{4/3}((0,T),V') \\ \therefore \quad &\langle \dot{u}, v\rangle_{V',V} + \nu((u,v)) + \langle B(u,u), v\rangle_{V',V} = (f,v) \end{aligned} \tag{6.13}$$

since $v \in L^\infty((0,T),V) \subset L^4((0,T),V)$ and this holds in $L^1(0,T)$ since each term is integrable. Also,

$$\dot{v} + \nu A v + B(v,v) = f \qquad \text{in } L^2((0,T),H)$$

but $u \in L^2((0,T),V) \subset L^2((0,T),H)$ so we can take the action:

$$\langle \dot{v}, u\rangle + \nu((v,u)) + \langle B(v,v), u\rangle = (f,u) \tag{6.14}$$

Summing (6.13) and (6.14),

$$\langle \dot{u}, v\rangle + \langle \dot{v}, u\rangle + 2\nu((v,u)) + \langle B(u,u), v\rangle + \langle B(v,v), u\rangle = (f, u+v)$$

[6] More accurately due to a graduate student of Serrin

in $L^{4/3}((0,T),V')$. Note $\frac{d}{dt}(u,v) = \langle \dot{u}, v\rangle + \langle \dot{v}, u\rangle$ by Lions-Magenes[7] and

$$\begin{aligned}\langle B(u,u),v\rangle + \langle B(v,v),u\rangle &= \langle B(u,u),v\rangle - \langle B(v,u),v \\ &= \langle B(w,u),v\rangle \\ &= \langle B(w,w),v\rangle\end{aligned}$$

Hence

$$\frac{d}{dt}(u,v) + 2\nu((v,u)) - \langle B(w,v),w\rangle = (f,u+v) \tag{6.15}$$

Now,

$$\begin{aligned}|w|^2 &= |u|^2 + |v|^2 - 2(u,v) \\ &\leq 2\int_0^t (f,u+v)ds - 2\nu\int_0^t \|u(s)\|^2 + \|v(s)\|^2 ds - 2(u,v) + 2|u_0|^2\end{aligned}$$

by the energy inequality on u and v. Substituting (6.15),

$$\begin{aligned}&= 4\nu\int((v,u)) - 2\int\langle B(w,v),w\rangle - 2\nu\int\|u(s)\|^2 + \|v(s)\|^2 \\ &= 2\int\langle B(w,v),w\rangle - 2\nu\int\|w\|^2\end{aligned}$$

Then,

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 - \langle B(w,v),w\rangle \leq 0$$

which is what we would have if we let the original equation (6.33) act on w! Now we estimate as usual,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 &\leq \|w\|\|v\|\|w\|^{1/2}|w|^{1/2} \\ &\leq \frac{3\nu\|w\|^2}{4} + \frac{\|v\|^4|w|^2}{4} \\ \therefore \quad \frac{d}{dt}|w|^2 + \frac{1}{2}\nu\|w\|^2 &\leq \frac{1}{2}\|v\|^4|w|^2 \\ \therefore \quad \frac{d}{dt}|w|^2 &\leq \frac{1}{2}\|v\|^4|w|^2\end{aligned}$$

and we conclude by Gronwall that $|w| \equiv 0$. □

[7] This requires an extension of Lions-Magenes to L^p,L^q with $1/p + 1/q = 1$ as well as the polarisation identity