

CELL MODULES FOR THE TEMPERLEY-LIEB ALGEBRA IN MIXED CHARACTERISTIC

Charles Senécal
(joint w/ S. Martin and R. Spencer)

Representation Theory Down Under
June 6, 2025

III.
I. CELL MODULES FOR THE
TEMPERLEY-LIEB ALGEBRA
IN MIXED CHARACTERISTIC II.

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I. TEMPERLEY-LIEB ALGEBRA

Def Temperley-Lieb category $\mathcal{JL}^k(s)$ has:

- objects $\mathbb{Z}_{\geq 0}$
- $\text{Hom}_{\mathcal{JL}}(m, n)$ has k -basis all (m, n) -diagrams

eg $\begin{matrix} 6 \\ \uparrow \\ 10 \end{matrix}$

• composition: = δ

• monoidal structure: $m \otimes n = m + n$, $\boxed{d_1} \otimes \boxed{d_2} = \boxed{d_1} \boxed{d_2}$

The Temperley-Lieb algebra $TL_n^k(s)$ is $\text{End}_{\mathcal{JL}^k(s)}(n)$.

Representation theory of $TL_n^k(s)$ depends heavily on choice of

a commutative ring k

a distinguished element $s \in k$

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Multiple different cases:

- semisimple case, e.g. $k = \mathbb{C}[\delta], \mathbb{Q}[\delta], \mathbb{Z}[\delta]$ or $k = \mathbb{C}, \delta = 2$
- characteristic p case, e.g. $k = \mathbb{F}_p, \delta = 2$
- "root of unity" case, e.g. $k = \mathbb{Q}[\delta]/(m_\delta)$
- "mixed" case, e.g. $k = \mathbb{Z}[\delta]/(p, m_\delta)$

MIXED CHARACTERISTIC II.

Def (q -numbers)

For $n \in \mathbb{Z}$, $[n] \in \mathbb{Z}[\delta]$ def.

- by
- $[0] = 0$
 - $[1] = 1$
 - $\delta[n] = [n+1] + [n-1]$
 - $[-n] = -[n]$

Def Given some $\ell \in \mathbb{Z}_{>0}$, let $m_\delta \in \mathbb{Z}[\delta]$ be irreducible s.t. $m_\delta \mid [\ell]$ but $m_\delta \nmid [k] \ \forall 1 \leq k < \ell$.

$\Rightarrow \ell$ is minimal s.t. $\bar{\delta} \in k$ satisfies $[\ell] = 0$.

Fact If $m \mid n$, then $[m] \mid [n]$.

Def The pair (ℓ, p) is the mixed characteristic of (k, δ) .

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$TL_n^{\mathbb{K}}(S)$ is a cellular algebra. We are interested in two of its cellular bases:

III.

CELL MODULES

• Schur-Weyl duality: $TL_n^{\mathbb{K}}(S) \cong \text{End}_{U_q(SL_2)}(V^{\otimes n})$

cell modules are

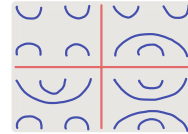
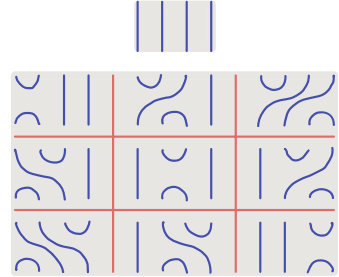
$$C(m) := \text{Hom}_{U_q(SL_2)}(\Delta(m), V^{\otimes n})$$

↑ has a cellular structure
[Andersen, Stroppel, Tubbenhauer]

Claim $S(m) \cong C(m)$

Thm (Spencer, 2023) If $(\mathbb{K}, S)$ has mixed characteristic (ℓ, p) with $\ell < \infty$, then the composition factors of $S(m)$ are $\{L(m(s)) \mid m(s) \leq n, S \text{ up-admissible}\}$, all with multiplicity one.

• diagram basis

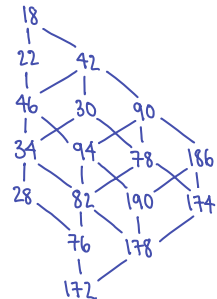


cell modules $S(m)$,
 $0 \leq m \leq n$, $m \equiv n \pmod{2}$

e.g. $(\ell, p) = (3, 2)$

In TL_{200} ,

$S(18) =$



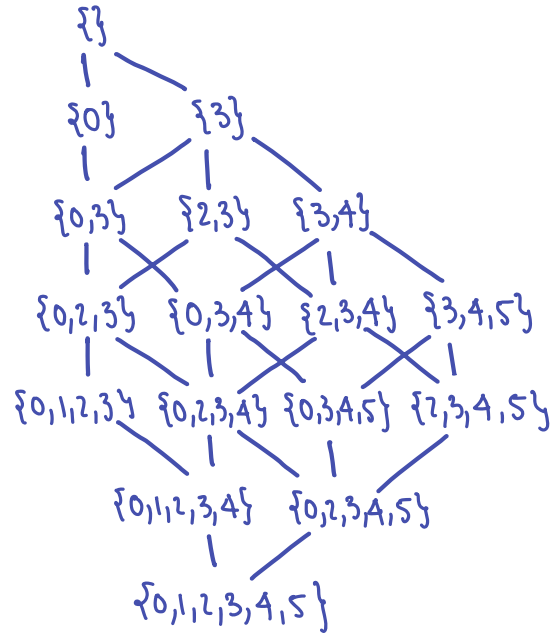
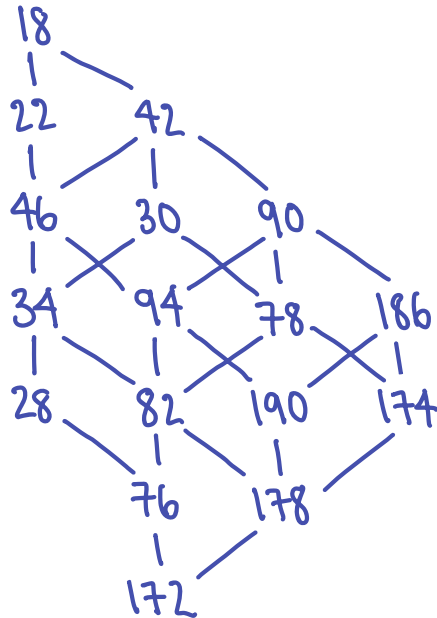
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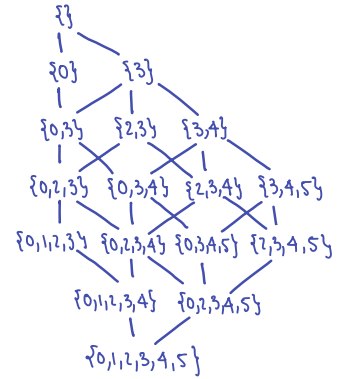
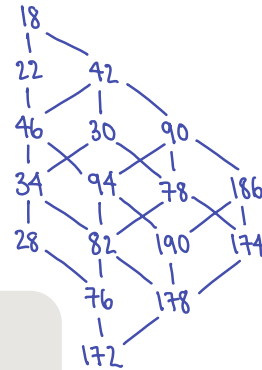


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Thm (Martin, Spencer, S.)

The lattice of submodules of $S(m)$ is given by the lattice of the poset $(\{s \mid s \text{ is up-admissible, } m(s) \leq n\}, \supseteq)$.



Proof Use Schur-Weyl duality and produce a correspondence

$$\left\{ \begin{array}{l} \text{composition factor} \\ L(m(s)) \text{ in } C(m) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{choice of morphism} \\ f: \Delta(m) \rightarrow T(m(s)) \hookrightarrow V^{\otimes n} \end{array} \right\}$$

Translates the submodule structure problem into the language of morphisms between tiltings $\rightsquigarrow$ up/down morphisms between (ℓ, p) -JW idempotents (Sutton-Tubbenhauer-Wedrich-Zhu, 2023).

JANTZEN FILTRATIONS

Characteristic p case Look at $S(m)$ for $TL_n^{\mathbb{Z}}(2)$.

Def For $k \geq 1$, $S(m)_k := \{x \in S(m) \mid \langle x, S(m) \rangle \subseteq p^k \mathbb{Z}\}$
 $\overline{S(m)}_k := S(m)_k \otimes_{\mathbb{Z}} \mathbb{F}_p$

Def The Jantzen filtration is

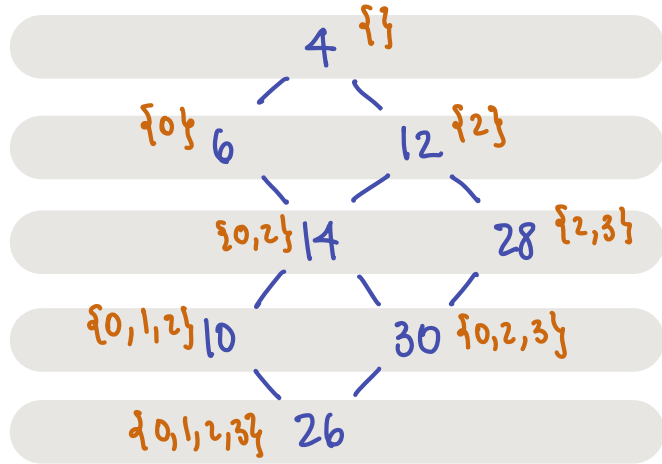
$$\overline{S(m)} = \overline{S(m)}_0 \supseteq \overline{S(m)}_1 \supseteq \dots \supseteq \overline{S(m)}_k \supseteq \overline{S(m)}_{k+1} = 0.$$

Thm (Jantzen) G the Gram matrix of $\langle \cdot, \cdot \rangle$. Then

$$v_p(\det G) = \sum_{k \geq 1} \dim_{\mathbb{F}_p} \overline{S(m)}_k$$

JANTZEN FILTRATIONS

eg in $TL_{30}^{\mathbb{Z}}(2)$, $l=p=2$, consider $S(4)$:



Prop (S.) A composition factor $L(m(s))$ of $S(n,m)$ falls in the i^{th} layer iff $|s|=i$.

→ What about the mixed case?

Problems $\mathbb{Z}[S]$ is not a PID. What are the analogues of $p\mathbb{Z}$ and v_p ?

JANTZEN FILTRATIONS

$$\overline{S(m)} = \overline{S(m)}_0 \supseteq \overline{S(m)}_1 \supseteq \overline{S(m)}_2 \supseteq \overline{S(m)}_3 \supseteq \dots$$

$$\mathbb{Z} \supseteq (p) \supseteq (p^2) \supseteq (p^3) \supseteq \dots$$

}

$\mathbb{Z}[S]_{\downarrow}$

$$(m_{\delta, p}) \supset (m_{\delta, p^2}) \supset (m_{\delta, p^3}) \supset (m_{\delta, p^4}) \supset (m_{\delta, p^5}) \supset \dots$$

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$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

JANTZEN FILTRATIONS

eg $(\ell, p) = (3, 2)$
 $L(178)$ in $S(30)$:



All composition factors of
 $S(30)$ in TL_{50000} :

