

Centralizers of products of $\mathcal{L}U_q(\mathfrak{sl}_2)$ -modules at roots of unity

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(Quantum) Schur-Weyl duality

- * $U_q(\mathfrak{sl}_2)$: Lusztig's divided powers quantum group.
- * Set $V = \Delta(1)$ the fundamental rep. of $U_q(\mathfrak{sl}_2)$.

Quantum Schur-Weyl duality for $U_q(\mathfrak{sl}_2)$: for $\beta = q + q^{-1}$,

$$TL_n(\beta) \overset{\text{uses}}{\curvearrowright} V^{\otimes n} \overset{\text{via}}{\curvearrowright} U_q(\mathfrak{sl}_2)$$

R-matrices *coproduct*

we have $TL_n(\beta) \cong \text{End}_{U_q(\mathfrak{sl}_2)}(V^{\otimes n})$ & $U_q(\mathfrak{sl}_2) \cong \text{End}_{TL_n(\beta)}(V^{\otimes n})$.

Our question: what about $\text{End}_{U_q(\mathfrak{sl}_2)}(M \otimes V^{\otimes n})$ for
 $M = T(i), L(i), \Delta(i), \dots$?

(ℓ, p) -Jones-Wenzl idempotents $q^{2\ell} = 1$

We have $V^{\otimes i} \cong T(i) \oplus U$.

$\leadsto$ There should be an idempotent of $TL_i(\beta)$ which projects on the direct summand $T(i)$.

- Burruall, Libedinsky & Sentinelli
- Martin & Spencer
- Sutton, Tubbenhauer, Wedrich & Zhu

If $n < \ell$ or $n \equiv -1 \pmod{\ell}$, set ${}_{\ell}^{\infty} JW_n = JW_n$. Else,

$${}_{\ell}^{\infty} JW_n := \text{[red box with } n \text{]} = \text{[grey box with } n \text{]} + \lambda_n \text{[diagram with two boxes } m_i \text{ and a crossing]}$$

Thm: ${}_{\ell}^{\infty} JW_n$ is well-defined & $\text{Im}({}_{\ell}^{\infty} JW_i) \cong T(i)$.

Endomorphisms of $T(i) \otimes V^{\otimes n}$

$$q^{2l} = 1$$

* $T(i)$ is a direct summand of $V^{\otimes i}$, so

start with the module $V^{\otimes i+n} = V^{\otimes i} \otimes V^{\otimes n}$.

* Then $\sum_{i \in \mathbb{Z}} \mathbb{N}_i \in \text{TL}_{i+n}(\beta) \cong \text{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V^{\otimes i+n})$ is the projection on the direct summand $T(i) \otimes V^{\otimes n}$.

$$\Rightarrow \text{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(T(i) \otimes V^{\otimes n}) \cong \sum_{i \in \mathbb{Z}} \mathbb{N}_i \cdot \text{TL}_{i+n}(\beta) \cdot \sum_{i \in \mathbb{Z}} \mathbb{N}_i$$

(Flores & Peltola) Valenced Temperley-Lieb algebra

or boundary seam algebra

(Morin-Duchesne,
Rasmussen & Ridout, Spencer)

The blob algebra (or one boundary Temperley-Lieb algebra)

Let's first introduce the blob category (following Tsuchiya-Lehrer-Zhang, who called it the Temperley-Lieb category of type B):

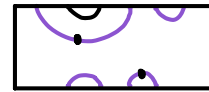
Def: $\mathcal{TLB}(\beta, \beta_1, \beta_2)$ is the $\mathbb{C}$ -linear category with

- objects: $\mathbb{N}_{\geq 0}$

- morphisms:

$$\text{Hom}_{\mathcal{TLB}(\beta, \beta_1, \beta_2)}(n, m) = \{ \text{marked TL diagrams} \} / \text{relations}$$

Eg $n=4, m=6$.



relations: $\bigcirc = \beta$, $\bigcirc \bullet = \beta_1$, $\downarrow = \beta_2 \downarrow$

Def: The blob algebra $\text{TLb}_n(\beta, \beta_1, \beta_2)$ is $\text{End}_{\mathcal{TLB}(\beta, \beta_1, \beta_2)}(n)$.

The blob algebra (or one boundary Temperley-Lieb algebra)

Def: $\text{TLbn}(\beta, \beta_1, \beta_2)$ is the unital, associative algebra generated by $\{1, e_0, e_1, \dots, e_{n-1}\}$ with the relations

$$e_i^2 = \beta e_i, \quad e_i e_{i \pm 1} e_i = e_i \quad \text{for } i, i \pm 1 \neq 0$$

$$e_i e_j = e_j e_i \quad \text{if } |i - j| > 1$$

$$e_1 e_0 e_1 = \beta_1 e_1$$

$$\bigcirc = \beta_1$$

$$e_0^2 = \beta_2 e_0$$

$$\downarrow = \beta_2 \downarrow$$

In the diagrammatic presentation,

$$e_0 = \downarrow \mid \dots \mid \quad \& \quad e_i = \mid \dots \mid \bigcup_{i \atop i+1}^i \mid \dots \mid \quad \text{for } 1 \leq i \leq n-1$$

The blob algebra action

For representations M, N of $\mathcal{LU}_q(\mathfrak{sl}_2)$, define

$$\check{R}_{M,N} := \text{flip}_{M,N} \circ R_{M,N} : M \otimes N \rightarrow N \otimes M$$

and define $\check{R}_0^2, \check{R}_i \in \text{End}_{\mathcal{LU}_q(\mathfrak{sl}_2)}(L(i) \otimes V^{\otimes n})$ by

(double braiding) $\check{R}_0^2 := \check{R}_{V, L(i)} \circ \check{R}_{L(i), V} \otimes 1_V^{\otimes n-1}$

(braiding) $\check{R}_i := 1_{L(i)} \otimes 1_V^{\otimes i-1} \otimes \check{R}_{V,V} \otimes 1_V^{\otimes n-i-1}$

Prop: We have a morphism of algebras

$$\text{TLb}_n(q+q^{-1}, 1, \beta_2, i) \longrightarrow \text{End}_{\mathcal{LU}_q(\mathfrak{sl}_2)}(L(i) \otimes V^{\otimes n})$$

$$e_0 \mapsto \alpha_i (q^i - \check{R}_0^2) \quad \alpha_i \in \mathbb{C}$$

$$e_i \mapsto q - q^{\frac{1}{2}} \check{R}_i$$

Endomorphisms of $L(i) \otimes V^{\otimes n}$

Conclusion:

- * When the map $TLb_n(q+q^{-1}, 1, \beta_{2,i}) \rightarrow \text{End}(L(i) \otimes V^{\otimes n})$ is surjective, this endomorphism algebra is isomorphic to a quotient of the blob algebra.
- * This leads to 2 immediate questions:
 - For what values of i and n is it surjective?
 - Can we describe the kernel?

Thank you!

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- * L. Sutton, D. Tubbenhauer, P. Wedrich, J. Zhu. SL_2 tilting modules in the mixed case. arXiv:2105.07724.