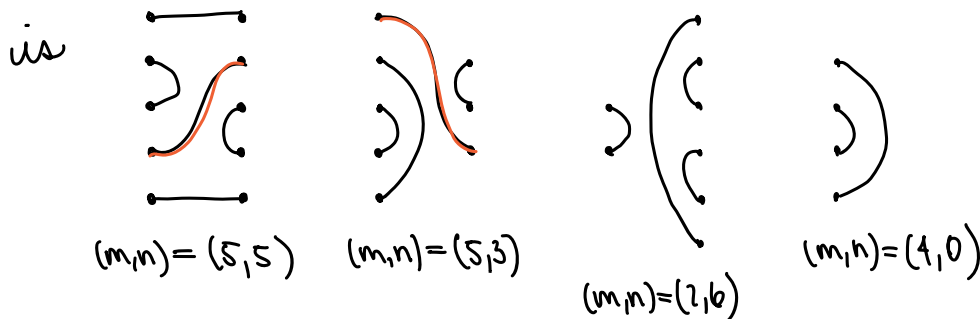


Cell Modules for the Temperley-Lieb Algebra in Mixed Characteristic

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§1 Temperley-Lieb Algebras

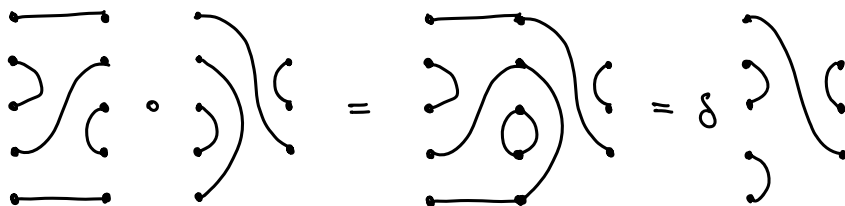
Def Let $m, n \in \mathbb{Z}_{\geq 0}$ of the same parity. An (m, n) -diagram



— are propagating lines

Def The Temperley-Lieb category $\mathcal{H}^k(\delta)$ is the monoidal category where

- objects are $\mathbb{Z}_{\geq 0}$
- $\text{Hom}_{\mathcal{H}^k(\delta)}(m, n)$ has basis all (m, n) -diags.
- composition:



- $m \otimes n = m + n$

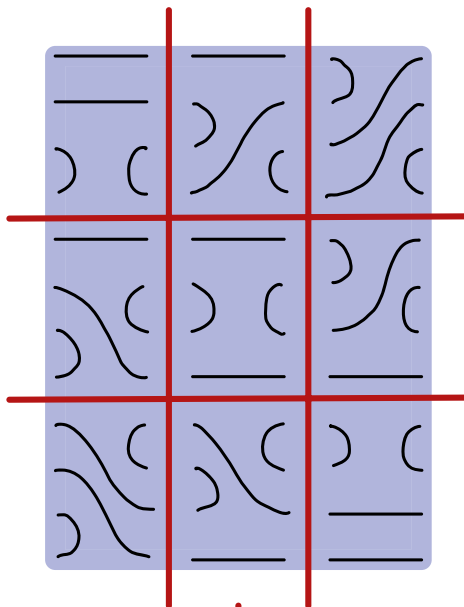
The Temperley-Lieb algebra $\text{TL}_n^k(\delta)$ is $\text{End}_{\mathcal{H}(\delta)}(n)$.

$TL_n^k(\delta)$ is a **cellular algebra** :

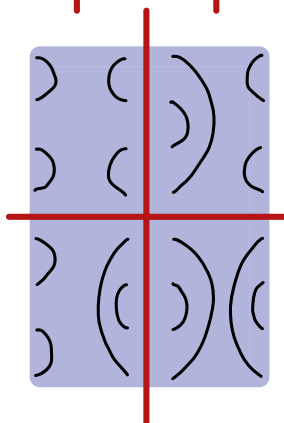
eg Take $n=4$. A basis of TL_4 :



4 prop. lines



2 prop. lines



0 prop. lines

Def Fix $n \in \mathbb{Z}_{\geq 0}$. The (left) cell modules for $TL_n(\delta)$, denoted

$S(n, m)$ for $\Lambda = \{0 \leq m \leq n \mid m \equiv n \pmod{2}\}$ has

- basis all epic (n, m) -diagrams

eg $S(4, 2) \quad \{ \equiv, \overline{\equiv}, \equiv \} \quad \text{not } \rangle \langle$

- multiplication is usual concatenation +

$$\rangle \langle \circ \overline{\equiv} = \rangle \langle \circ = \delta \rangle \langle = 0$$

$$\text{but } \overline{\supset} \circ \underline{\supset} = \overline{\supset} = \underline{\supset}$$

- equipped w/ a bilinear form

$$\langle \cdot, \cdot \rangle : S(n, m) \times S(n, m) \rightarrow \mathbb{k}$$

$$(\overline{\supset}, \underline{\supset}) \mapsto \underline{\supset} = 1 \cdot =$$

$$(\underline{\supset}, \underline{\supset}) \mapsto \underline{0} = \delta \cdot =$$

$$(\underline{\supset}, \overline{\supset}) \mapsto \underline{\supset} = 0 \cdot = + \infty$$

Thm (Graham-Lehrer '96)

Let $\text{rad}(m) = \{ x \in S(n, m) \mid \langle x, y \rangle = 0 \ \forall y \in S(n, m) \}$.
and set $\Lambda_0 = \{ m \in \Lambda \mid \text{rad}(m) \neq S(n, m) \}$. Then
the set $\{ L(n, m) := S(n, m) / \text{rad}(m) \mid m \in \Lambda_0 \}$ is a
complete set of non-isomorphic simple modules.

§2 q-numbers and Combinatorics

Def For $n \in \mathbb{N}$, define the q-number $[n] \in \mathbb{Z}[\delta]$ by

- $[0] = 0$
- $[1] = 1$
- $\delta[n] = [n-1] + [n+1]$
- $[-n] := -[n]$

Ex $[2] = \delta$, $[3] = \delta^2 - 1$, $[4] = \delta^3 - 2\delta$, ...

For the rest of the talk: fix k a field of char p and $\delta \in k$. Then $[n]$ can be seen as $\in k$. Let $l \in \mathbb{Z}_{>0}$ minimal s.t. $[l] = 0$ (if it exists, o/w $l = \infty$).

Def The pair (l, p) is the mixed characteristic of (k, δ) .

For $n \in \mathbb{N}$,

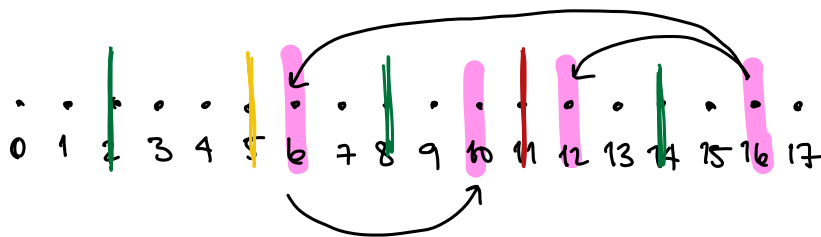
- (l, p) -adic expansion of $n+1$

$$\begin{aligned} n+1 &= n_0 + n_1 l + n_2 l p + \dots + n_k l p^{k-1} \\ &= \sum_{i \geq 0} n_i p^{(i)} \quad \text{where } p^{(i)} = l p^{i-1}, \quad p^{(0)} = 1. \\ &= [n_k, n_{k-1}, \dots, n_0]_{l, p} \end{aligned}$$

$$\text{supp}(n) = \{ n_k p^{(k)} \pm n_{k-1} p^{(k-1)} \pm \dots \pm n_0 p^{(0)} - 1 \}$$

Ex $l=3, p=2, n=16$. $n+1=17=1 \cdot 12 + 0 \cdot 6 + 1 \cdot 3 + 2$

$$\text{supp}(n) = \{ 12 \pm 3 \pm 2 - 1 \} = \{ 16, 12, 10, 6 \}$$



$$\left. \begin{aligned} 16[\{1\}] &= 12 \\ 16[\{3\}] &= 6 \\ 16[\{1, 2, 3\}] &= 10 \end{aligned} \right\} \begin{array}{l} \text{downward reflection} \\ \text{through down-admissible sets} \end{array}$$

$$\text{Then } \text{supp}(n) = \{ n[S] \mid S \text{ down-adm.} \}$$

Similarly, up-admissible sets S for n and upward reflections $n(S)$.

$$\text{Defined so that } n[S](S) = n(S)[S] = n.$$

Thm • If $l = \infty$, $\Pi_n(S)$ is semisimple.

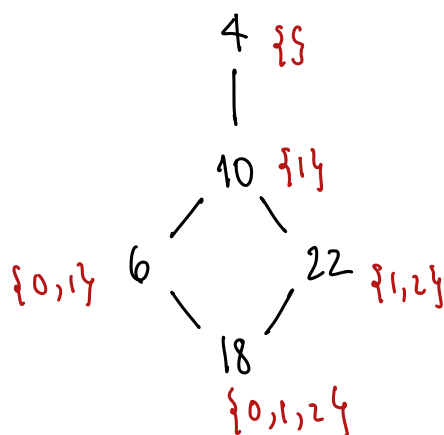
- (Spencer 2023) If $l < \infty$, then the simple composition factors (with mult. 1) of $S(n, m)$ are $\{L(n, m(S)) \mid m(S) \leq n, S \text{ up-adm.}\}$

Question : what is there submodule structure?

Ex $n = 22, m = 4$. Then

$$\{L(n, m(S))\} = \{L(n, 4), L(n, 6), L(n, 10), L(n, 18), L(n, 22)\}$$

$\{1\}$
 $\{0, 1\}$
 $\{1\}$
 $\{0, 1, 2\}$
 $\{1, 2\}$



$$0 \leq 18 \leq \begin{smallmatrix} 22 \\ 18 \end{smallmatrix} \leq \begin{smallmatrix} 6 & 22 \\ 18 \end{smallmatrix} \leq \begin{smallmatrix} 10 & 22 \\ 6 & 18 \end{smallmatrix} \leq \begin{smallmatrix} 4 & 10 & 22 \\ 6 & 18 \end{smallmatrix}$$

Thm (MSS) The lattice of submodules of $S(n, m)$ is given by the lattice of the poset $\{S \mid S \text{ up-adm.}, m(S) \leq n\}$ with reverse-inclusion order.

§3 Schur-Weyl duality

Let U_q denote Lusztig's quantum group of sl_2 (q -deformation of $U(sl_2)$) and V its 2-dim. fundamental repn.

Schur-Weyl duality

$$U_q \hookrightarrow V^{\otimes n} \hookrightarrow \mathcal{H}_n(q) \quad \begin{array}{l} \text{Hopf alg.} \\ \text{coproduct} \end{array} \quad \begin{array}{l} R\text{-matrices} \end{array} \quad \begin{array}{l} \text{Hecke algebra} \\ \text{of } S_n \end{array}$$

& both actions commute. Even better: taking $\mathcal{H}_n(q+q^{-1}) = \mathcal{H}_n(q) / \langle e \rangle$, we have

$$\mathcal{H}_n \simeq \text{End}_{U_q}(V^{\otimes n})$$

* $\text{End}_{U_q}(V^{\otimes n})$ has its own (different) cellular structure (Andersen-Stroppel-Tubbenhauer 2015)

* Cell modules are

$$C(lm) := \text{Hom}_{U_q}(\Delta(lm), V^{\otimes n})$$

↑ Weyl module

* $V^{\otimes n}$ decomposes as a sum of tilting modules

Eg $l=3, p=2$. $V^{\otimes 10} \simeq T(10) \oplus 9T(8) \oplus 34T(6) \oplus 40T(4) \oplus 8T(2) \oplus T(0)$

$$S(10, 4) = \begin{array}{c} 4 \\ | \\ 10 \\ | \\ 6 \end{array} \quad \begin{array}{l} (\dim 40) \\ (\dim 1) \\ (\dim 34) \end{array}$$

This example motivates:

$$\left\{ \begin{array}{l} \text{composition factor} \\ L(m(S)) \text{ in } C(m) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{choice of} \\ f: \Delta(m) \rightarrow T(m(S)) \hookrightarrow V^{\otimes n} \end{array} \right\}$$

New objective :

understand morphisms between tilting modules and their compositions.

↳ (Sutton-Tubbenhauer-Wedrich-Zhu, 2023)

up/down morphisms between

(d, p) -Jones-Wenzl idempotents

(Martin-Spencer)

§ 4 Jantzen filtrations

General framework:

R a PID, $\mathfrak{p}$ a maximal ideal, $v: R^\times \rightarrow \mathbb{N}$ the $\mathfrak{p}$ -adic valuation, $\mathbb{F} = R/\mathfrak{p}$.

M a free R -module of finite rank.

$\langle \cdot, \cdot \rangle: M \times M \rightarrow R$ bilinear form.

For $k \geq 1$,

$$M(k) := \{ m \in M \mid \langle m, M \rangle \in \mathfrak{p}^k \}$$

or $v(\langle m, m' \rangle) \geq k \ \forall m' \in M$

and let $\overline{M(k)} = M(k) \otimes_R \mathbb{F}$.

Def Jantzen filtration of M is

$$\overline{M} = \overline{M(0)} \supseteq \overline{M(1)} \supseteq \dots \supseteq \overline{M(n)} = 0$$

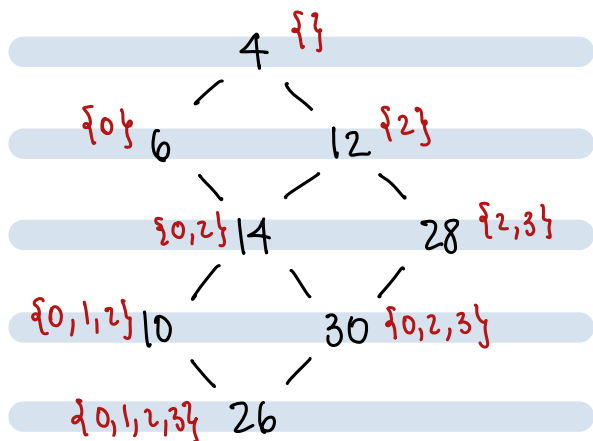
and the k^{th} layer is $\overline{M(k)} / \overline{M(k+1)}$.

Thm (Jantzen) Let G be the Gram matrix of $\langle \cdot, \cdot \rangle$.

Then $v(\det G) = \sum_{k \geq 1} \dim_{\mathbb{F}} \overline{M(k)}$.

In the case $\delta = 2, \ell = p$: $R = \mathbb{Z}$, $\mathfrak{p} = \langle p \rangle$. Jantzen layers are the "layers in the Alperin diagram"

eg in $TL_{30}(2)$, $l=p=2$, consider $S(30,4)$:



Prop A composition fac.

$L(n, m(S))$ of $S(n, m)$ falls in the k^{th} layer iff $|S|=k$.

What about the mixed case?

Problems: $\mathbb{Z}[S]$ not a PID, what are the analogues of p and v ?

Define $v_{l,p}: \mathbb{Z}[S] \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$v_l(f) := \max \{ r \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \text{ s.t. } [l]^r \mid f \}$$

$$v_{l,p}(f) := v_l(f) + v_p\left(\pi\left(\frac{f}{[l]^{v_l(f)}}\right)\right),$$

where $\pi: \mathbb{Z}[S] \rightarrow \mathbb{Z}[S]/(m_S)$, m_S min. poly. for S .

Then $v_{l,p}$ is a ring filtration, not a valuation.

The prop holds, but not Jantzen's theorem.