Examiner:

Below are four sample questions in a similar style to those you might expect in the Analysis of PDE exam. They've not been carefully checked, so may contain errors, and are not as uniform in difficulty as the questions on the final exam should be. While they cover a reasonable spread of topics from the course, no inference should be drawn about what topics are (un)likely to appear in the final exam. If you find any errors, please let me know (cmw50@cam.ac.uk).

The rubric for the final exam will be: "Answer **three** questions from **four**. Each will be equally weighted when marked."

Examiner:

1 Analysis of PDE

a) Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$. State what it means for f to be *real analytic.*

b) Show that a function $f : B_r \to \mathbb{R}$ is real analytic at 0 if and only if there exist constants $s > 0$, $C > 0$ and $\rho > 0$ such that:

$$\sup_{|x|<s} \left| D^\beta f(x) \right| \leqslant C \frac{|\beta|!}{\rho^{|\beta|}}.$$

Consider the Euler-Tricomi equation:

$$u_{xx} + x u_{yy} = 0, \tag{$\star$}$$

for an unknown function $u : \mathbb{R}^2 \to \mathbb{R}$.

c) In what region of the plane is $(\star)$:

   i) Elliptic

   ii) Hyperbolic

d) Find and sketch the characteristic surfaces in the region in which $(\star)$ is hyperbolic. *[Hint: since the equation is two dimensional, the surfaces will have dimension one, i.e. be curves]*

e) Consider the Cauchy problem of solving $(\star)$ subject to:

$$u(\xi, y) = u_0(y), \qquad u_x(\xi, y) = u_1(y)$$

where $\xi \in \mathbb{R}$ is a fixed constant. For which values of ξ would you expect a solution, suitably understood, to exist in a neighbourhood of the surface $\{x = \xi\}$ if:

   i) u_0, u_1 are real analytic?

   ii) $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$?

In each case, you should justify your answer.

2 Analysis of PDE

Suppose $U \subset \mathbb{R}^n$ is a bounded open set with C^∞ boundary, and let L be the second order differential operator:

$$Lu := -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + cu,$$

where $a^{ij}, b^i, c \in C^\infty(\overline{U})$.

a) State what it means for L to be *uniformly elliptic.* State, without justification, what it means for u to be a weak solution to the PDE problem:

$$(L+\gamma)u = f \text{ in } U, \qquad u = 0 \text{ on } \partial U. \tag{$\dagger$}$$

where $f \in L^2(U)$. Show that if L is uniformly elliptic, ($\dagger$) admits a unique weak solution provided $\gamma \geqslant \mu$, where $\mu \in \mathbb{R}$ is some constant.
[You may assume the Lax-Milgram theorem, provided it is clearly stated, but should directly establish any estimates you require to show that the hypotheses are satisfied.]

b) Let $f \in L^2(U)$. For each $\gamma > \mu$, denote by u_γ the unique solution of ($\dagger$).

i) Show that for any $\gamma > \mu$ there exists $w_\gamma \in H^1_0(U)$ such that:

$$\frac{u_\gamma - u_{\gamma'}}{\gamma - \gamma'} \to w_\gamma \quad \text{in } H^1(U) \quad \text{as } \gamma' \to \gamma.$$

ii) Show further that $w_\gamma \in H^4(U)$, and there exists a constant C depending on L, γ such that

$$\|w_\gamma\|_{H^4(U)} \leqslant C \|f\|_{L^2(U)}$$

[You may assume results from the course concerning elliptic regularity]

3 Analysis of PDE

Throughout this question, assume $n \geqslant 2$.

a) Suppose $f_1, \ldots, f_n \in C_c^\infty(\mathbb{R}^{n-1})$. For $i = 1, \ldots, n$ denote by $\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)$ the $(n-1)$-vector obtained by removing the i^{th} component of $x = (x_1, \ldots, x_n)$. Show that if $f(x) := f_1(\tilde{x}_1) f_2(\tilde{x}_2) \cdots f_n(\tilde{x}_n)$ then:

$$\|f\|_{L^1(\mathbb{R}^n)} \leqslant \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$$

b) Show that there exists a constant $C > 0$, depending only on n, such that:

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leqslant C \|Du\|_{L^1(\mathbb{R}^n)}$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$.

c) Show that if $q \neq \frac{n}{n-1}$, there is no constant C such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C \|Du\|_{L^1(\mathbb{R}^n)}$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$.

d) Show that if $1 \leqslant q \leqslant \frac{n}{n-1}$ there exists a constant $C > 0$, depending only on n, such that:

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C \left(\|Du\|_{L^1(\mathbb{R}^n)} + \|u\|_{L^1(\mathbb{R}^n)} \right)$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$. By constructing a counter-example, or otherwise, show that no such constant C can exist for $q > \frac{n}{n-1}$.
Hint: you may first wish to establish, using Hölder or otherwise, the interpolation inequality for L^p-norms. That is, if $0 < p_0 < p_1$ and $0 \leqslant \theta \leqslant 1$:

$$\|f\|_{L^{p_\theta}(\mathbb{R}^n)} \leqslant \|f\|^{1-\theta}_{L^{p_0}(\mathbb{R}^n)} \|f\|^{\theta}_{L^{p_1}(\mathbb{R}^n)}$$

holds for $f \in C_c^\infty(\mathbb{R}^n)$, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

4 Analysis of PDE

Let $U \subset \mathbb{R}^n$ be an open, bounded set in $\mathbb{R}^n$ with C^∞ boundary. Consider the Neumann problem for the Laplacian:

$$-\Delta u = f \text{ in } U, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \tag{$\triangle$}$$

where ν is the outwards directed normal to ∂U.

a) Define a weak solution to this problem for $f \in L^2(U)$, and show that with your definition if $u \in C^2(\overline{U})$ is a weak solution that it indeed solves ($\triangle$).

b) Show that ($\triangle$) admits a weak solution if, and only if:

$$\int_U f(x)dx = 0.$$

and comment on uniqueness.

c) Show that for any $u \in H^1(U)$ the Poincaré inequality:

$$\|u - \overline{u}\|_{L^2(U)} \leqslant \frac{1}{\sqrt{\lambda_1}} \|Du\|_{L^2(U)}$$

holds, where:

$$\overline{u} = \frac{1}{|U|} \int_U u(x)dx$$

is the mean of u and λ_1 is the smallest non-zero eigenvalue of the Neumann problem:

$$-\Delta u = \lambda u \text{ in } U, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U.$$

Show also that this choice of constant is optimal for the inequality.

You may assume the Lax-Milgram theorem and results concerning the spectrum of symmetric operators without proof.