1.1 Convolutions

If f, g are functions mapping $\mathbb{R}^n$ to $\mathbb{C}$, then we define the convolution of f and g to be:

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy,$$

provided the integral exists. This will happen if (for example) $f \in C^0_0(\mathbb{R}^n)$ and $f \in C^0(\mathbb{R}_n)$.

Lemma 1. *Suppose $f, g, h \in C^\infty_0(\mathbb{R}^n)$. Then:*

$$f \star g = g \star f, \qquad f \star (g \star h) = (f \star g) \star h.$$

and

$$\int_{\mathbb{R}^n} (f \star g)(x)dx = \int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(x)dx.$$

Proof. With the change of variables $y = x - z$, we have[1]

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-z)g(z)dz = (g \star f)(x)$$

Next, we calculate:

$$\begin{aligned}
[f \star (g \star h)](x) &= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} g(z)h(x-y-z)dz \right) dy \\
&= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} g(w-y)h(x-w)dw \right) dy \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)g(w-y)dy \right) h(x-w)dw \\
&= [(f \star g) \star h](x)
\end{aligned}$$

Above we have made the substitution $w = y + z$ to pass from the first to second line, and we have used the fact that $f, g, h \in C^\infty_0(\mathbb{R}^n)$ to invoke Fubini's theorem when passing

[1] If you're worried about a missing minus sign from the change of variables when n is odd, observe:

$$\int_{-\infty}^{\infty} k(x)dx = \int_{\infty}^{-\infty} k(-y)d(-y) = -\int_{\infty}^{-\infty} k(-y)dy = \int_{-\infty}^{\infty} k(-y)dy.$$

from the second to third line. Finally, we calculate:

$$\begin{aligned}\int_{\mathbb{R}^n} (f \star g)(x)dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)g(x-y)dy \right) dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)g(x-y)dx \right) dy \\ &= \int_{\mathbb{R}^n} \left(f(y) \int_{\mathbb{R}^n} g(z)dz \right) dy \\ &= \int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(z)dy.\end{aligned}$$

where again, the fact that $f, g \in C_0^\infty(\mathbb{R}^n)$ allows us to invoke Fubini. □

The assumption that the functions are smooth and compactly supported is certainly overkill in this theorem. It would be enough, for example, to consider functions in $C_0^0(\mathbb{R}^n)$, or even weaker spaces, provided we can justify the application of Fubini's theorem.

1.1.1 Differentiating convolutions

A remarkable property of the convolution is that the regularity of $f \star g$ is determined by the regularity of the *smoother* of f and g. This is a result of the following Lemma:

Theorem 2. *Suppose $f \in L^1_{loc.}(\mathbb{R}^n)$ and $g \in C_0^k(\mathbb{R}^n)$ for some $k \geq 0$. Then $f \star g \in C^k(\mathbb{R}^n)$ and*

$$D^\alpha(f \star g) = f \star D^\alpha g,$$

for any multiindex with $|\alpha| \leq k$.

Before we prove this, it's convenient to prove a technical Lemma which will streamline the proof.

Lemma 3. *a) Suppose $f \in C_0^0(\mathbb{R}^n)$ and $\{z_i\}_{i=1}^\infty \subset \mathbb{R}^n$ is a sequence with $z_i \to 0$ as $i \to \infty$. Then for any $x \in \mathbb{R}^n$:*

i) $\tau_{z_j} f(x) \to f(x)$ as $j \to \infty$.

ii) $\left|\tau_{z_j} f(x)\right| \leq (\sup_{\mathbb{R}^n} |f|) \mathbb{1}_{B_R(0)}(x)$, for some $R > 0$ and all j.

b) Suppose $f \in C_0^1(\mathbb{R}^n)$ and $\{h_j\}_{j=1}^\infty \subset \mathbb{R}$ is a sequence with $h_j \to 0$ as $j \to \infty$. Then for any $x \in \mathbb{R}^n$:

i) $\Delta_i^{h_j} f(x) \to D_i f(x)$ as $j \to \infty$.

ii) $\left|\Delta_i^{h_j} f(x)\right| \leq (\sup_{\mathbb{R}^n} |D_i f|) \mathbb{1}_{B_R(0)}(x)$, for some $R > 0$ and all j.

Proof. a) i) Recall $\tau_{z_j} f(x) = f(x - z_j)$. Clearly since $z_j \to 0$, $f(x - z_j) \to f(x)$ as $j \to \infty$ by the continuity of f.

ii) Since $z_j \to 0$, there exists some $\rho > 0$ such that $z_j \in \overline{B_\rho(0)}$ for all j. Now

$$\text{supp } \tau_{z_j} f = \text{supp } f + z_j \subset \text{supp } f + \overline{B_\rho(0)}.$$

Since the sum of two bounded set is bounded, we conclude that there exists $R > 0$ such that supp $\tau_{z_j} f \subset B_R(0)$. Thus $\tau_{z_j} f = \tau_{z_j} f \mathbb{1}_{B_R(0)}$ and we estimate:

$$\left|\tau_{z_j} f(x)\right| = \left|\tau_{z_j} f(x)\right| \mathbb{1}_{B_R(0)}(x) \leq \sup_{\mathbb{R}^n} |f| \, \mathbb{1}_{B_R(0)}(x).$$

b) Suppose $f \in C^1_0(\mathbb{R}^n)$ and $\{h_j\}_{j=1}^\infty \subset \mathbb{R}$ is a sequence with $h_j \to 0$ as $j \to \infty$. Then for any $x \in \mathbb{R}^n$:

i) From the definition of the difference quotient and of the partial derivative:

$$\Delta_i^{h_j} f(x) = \frac{f(x + h_j e_i) - f(x)}{h} \to D_i f(x), \qquad \text{as } j \to \infty.$$

ii) Since $h_j \to 0$, there is some $k > 0$ such that $|h_j| \leq k$ for all j. We have:

$$\begin{aligned} \text{supp } \Delta_i^{h_j} f &\subset \text{supp } \tau_{-he_i} f \cup \text{supp } f = (\text{supp } f - h_j e_i) \cup \text{supp } f \\ &\subset \left(\text{supp } f + \overline{B_\rho(0)}\right) \cup \text{supp } f \\ &\subset B_R(0) \end{aligned}$$

for some $R > 0$ since the union of two bounded sets is bounded. Thus $\Delta_i^{h_j} f = \Delta_i^{h_j} f \mathbb{1}_{B_R(0)}$. We also observe that by the mean value theorem, for any $h \in \mathbb{R}$, there exists $s \in \mathbb{R}$ with $|s| < |h|$ such that

$$\frac{f(x + h_j e_i) - f(x)}{h} = D_i f(x + s e_i)$$

thus

$$\left|\Delta_i^{h_j} f(x)\right| \leq \sup_{\mathbb{R}^n} |D_i f| \, .$$

Putting these two facts together, we readily find:

$$\left|\Delta_i^{h_j} f(x)\right| = \left|\Delta_i^{h_j} f(x)\right| \mathbb{1}_{B_R(0)}(x) \leq \sup_{\mathbb{R}^n} |D_i f| \, \mathbb{1}_{B_R(0)}(x).$$

□

Now, with this technical result in hand we can attack the proof our original theorem.

Proof of Theorem 2. 1. First we establish the result for $k = 0$. We need to show that if $f \in L^1_{loc.}(\mathbb{R}^n)$ and $g \in C^0_0(\mathbb{R}^n)$ then $f \star g$ is continuous. To show this, it suffices to show that $f \star g(x - z_j) \to f \star g(x)$ for any sequence $\{z_j\}_{j=1}^\infty$ with $z_j \to 0$. Now, note that

$$f \star g(x - z_j) = \int_{\mathbb{R}^n} f(y) g(x - z_j - y) dy = \int_{\mathbb{R}^n} f(y) \tau_{z_j} g(x - y) dy.$$

Now, sending $j \to \infty$, we are done, so long as we can justify interchanging the limit and the integral. Note that for any fixed x and all j:

$$\left|f(y)\tau_{z_j} g(x-y)\right| \leq \sup_{\mathbb{R}^n} |g| \, \mathbb{1}_{B_R(0)}(x-y) \, |f(y)|$$

for some R by the previous Lemma. Since $f \in L^1_{loc.}(\mathbb{R}^n)$ the right hand side is integrable, and so by the dominated convergence theorem:

$$\lim_{j\to\infty} f \star g(x - z_j) = \int_{\mathbb{R}^n} \lim_{j\to\infty} f(y)\tau_{z_j} g(x-y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy = f \star g(x).$$

2. Now suppose that $f \in L^1_{loc.}(\mathbb{R}^n)$ and $g \in C^1_0(\mathbb{R}^n)$. Clearly $f \star D_i g$ is continuous by the previous argument. To show $f \star g \in C^1(\mathbb{R}^n)$, it suffices to show that for any $x \in \mathbb{R}^n$ and any sequence $\{h_j\}_{j=1}^\infty \subset \mathbb{R}$ with $h_j \to 0$ we have:

$$\lim_{j\to\infty} \Delta_i^{h_j} f \star g(x) = f \star D_i g(x).$$

Note that

$$\begin{aligned}
\Delta_i^{h_j} f \star g(x) &= \frac{f \star g(x + h_j e_i) - f \star g(x)}{h} \\
&= \int_{\mathbb{R}^n} f(y) \left(\frac{g(x + h_j e_i - y) - g(x-y)}{h} \right) dy \\
&= \int_{\mathbb{R}^n} f(y) \Delta_i^{h_j} g(x-y) dy
\end{aligned}$$

so that again we are done provided we can send $j \to \infty$ and interchange the limit and the integral. An argument precisely analogous to the previous case allows us to invoke the DCT and deduce that:

$$\lim_{j\to\infty} \Delta_i^{h_j} f \star g(x) = \int_{\mathbb{R}^n} \lim_{j\to\infty} f(y) \Delta_i^{h_j} g(x-y)dy = f \star D_i g(x).$$

3. The case where $f \in C^k_0(\mathbb{R}^n)$ with $k > 1$ now follows by a simple induction. □

Exercise 1.1. Show that Theorem 2 holds under the alternative hypotheses:

a) $f \in L^1(\mathbb{R}^n)$, $g \in C^k(\mathbb{R}^n)$ with $\sup_{\mathbb{R}^n} |D^\alpha g| < \infty$ for all $|\alpha| \leq k$.

b) $f \in L^1(\mathbb{R}^n)$ with $\operatorname{supp} f$ compact, $g \in C^k(\mathbb{R}^n)$.

We have shown that when two functions are convolved, loosely speaking the resulting function is at least as regular as the *better* of the two original functions. It is also important to know how convolution modifies the support of a function.

Lemma 4. *Suppose $f \in L^1_{loc.}(\mathbb{R}^n)$ and $g \in C^k_0(\mathbb{R}^n)$ for some $k \geq 0$. Then*[2]

$$\textit{supp}\ (f \star g) \subset \textit{supp}\ f + \textit{supp}\ g.$$

Proof. Recall:

$$f \star g(x) = \int_{\mathbb{R}^n} f(y) g(x-y) dy.$$

Clearly, if $f \star g(x) \neq 0$, then there must exist $y \in \mathbb{R}^n$ such that $y \in \text{supp}\ f$ and $x - y = z \in \text{supp}\ g$. Thus $x = y + z$ with $y \in \text{supp}\ f$ and $z \in \text{supp}\ g$. This tells us that:

$$\{x \in \mathbb{R}^n : f \star g(x) \neq 0\} \subset \text{supp}\ f + \text{supp}\ g.$$

Since supp f is closed and supp g is compact, we know that supp f + supp g is closed, thus

$$\text{supp}\ f \star g = \overline{\{x \in \mathbb{R}^n : f \star g(x) \neq 0\}} \subset \text{supp}\ f + \text{supp}\ g,$$

which is the result we require. □

Exercise 1.2. Show that if $f \in C^k_0(\mathbb{R}^n)$ and $g \in C^l_0(\mathbb{R}^n)$ then $f \star g \in C^{k+l}_0(\mathbb{R}^n)$. Conclude that $\mathcal{D}(\mathbb{R}^n)$ is closed under convolution.

1.1.2 Approximation of the identity

An important use of the convolution is to construct smooth approximations to functions in various function spaces. The following theorem is very useful in constructing approximations:

Theorem 5. *Suppose $\phi \in C^\infty_0(\mathbb{R}^n)$ satisfies:*

i) $\phi \geq 0$

ii) *supp* $\phi \subset B_1(0)$

iii) $\int_{\mathbb{R}^n} \phi(x) dx = 1$

Such a ϕ exists, for example we can take $\phi(x) = C \exp\left(\frac{1}{4|x|^2-1}\right)$ for $|x| < 1/2$ and $\phi(x) = 0$ otherwise, with C chosen suitably. Define:

$$\phi_\epsilon(y) = \frac{1}{\epsilon^n} \phi\left(\frac{y}{\epsilon}\right).$$

Then:

[2] Strictly speaking, we haven't defined the support of a measurable function. We can do this in several ways, but the simplest is to define:

$$\text{supp}\ f = \bigcap \{E \subset \mathbb{R}^n : E \text{ is closed, and } f = 0 \text{ a.e. on } E^c\}.$$

In other words supp f is the smallest closed set such that f vanishes almost everywhere on its complement.

a) If $f \in C_0^k(\mathbb{R}^n)$*, then* $\phi_\epsilon \star f$ *is smooth, and*

$$D^\alpha (\phi_\epsilon \star f) \to D^\alpha f$$

uniformly on $\mathbb{R}^n$ *for any multi-index with* $|\alpha| \leq k$*.*

b) If $g \in L^p(\mathbb{R}^n)$ *with* $1 \leq p < \infty$*, then* $\phi_\epsilon \star g$ *is smooth, and*

$$\phi_\epsilon \star g \to g \qquad \text{in } L^p(\mathbb{R}^n).$$

c) Suppose $f \in C^k(\mathbb{R}^n)$ *with* $\sup_{\mathbb{R}^n} |D^\alpha f| < \infty$ *for* $|\alpha| \leq k$*, and suppose* $g \in L^1(\mathbb{R}^n)$ *with* $g \geq 0$*,* $\int_{\mathbb{R}^n} g(x)dx = 1$*. Set* $g_\epsilon(y) = \epsilon^{-n} g\left(\epsilon^{-1} y\right)$*. Then* $f \star g_\epsilon \in C^k(\mathbb{R}^n)$*, and*

$$D^\alpha (f \star g_\epsilon)(x) \to D^\alpha f(x)$$

for any $x \in \mathbb{R}^n$ *for any multi-index with* $|\alpha| \leq k$*.*

Proof. a) Note that the rescaling of ϕ to produce ϕ_ϵ is such that a change of variables gives:

$$\int_{\mathbb{R}^n} \phi_\epsilon(y) dy = 1.$$

By Theorem 2, we have that $D^\alpha(\phi_\epsilon \star f) = \phi_\epsilon \star D^\alpha f$ for any $|\alpha| \leq k$. Using these two facts, we calculate:

$$\begin{aligned} D^\alpha(\phi_\epsilon \star f)(x) - D^\alpha f(x) &= \int_{\mathbb{R}^n} \phi_\epsilon(y) D^\alpha f(x-y) dy - D^\alpha f(x) \int_{\mathbb{R}^n} \phi_\epsilon(y) dy \\ &= \int_{\mathbb{R}^n} \phi_\epsilon(y) \left[D^\alpha f(x-y) - D^\alpha f(x) \right] dy \\ &= \int_{B_1(0)} \phi(z) \left[D^\alpha f(x - \epsilon z) - D^\alpha f(x) \right] dz \end{aligned}$$

where in the last line we made the substitution $y = \epsilon z$, and noted that ϕ has support in $B_1(0)$, so we can restrict the range of integration. Now, since $\phi \geq 0$, we can estimate:

$$\begin{aligned} |D^\alpha(\phi_\epsilon \star f)(x) - D^\alpha f(x)| &\leq \int_{B_1(0)} \phi(z) \left| D^\alpha f(x - \epsilon z) - D^\alpha f(x) \right| dz \\ &\leq \sup_{z \in B_1(0)} \left| D^\alpha f(x - \epsilon z) - D^\alpha f(x) \right| \times \int_{B_1(0)} \phi(z) dx \\ &= \sup_{z \in B_1(0)} \left| D^\alpha f(x - \epsilon z) - D^\alpha f(x) \right| \end{aligned}$$

since $\int_{\mathbb{R}^n} \phi = 1$. Now, since $D^\alpha f$ is continuous and of compact support, it is uniformly continuous on $\mathbb{R}^n$. Fix $\tilde{\epsilon} > 0$. There exists δ such that for any $v, w \in \mathbb{R}^n$ with $|x - y| < \delta$, we have

$$|D^\alpha f(v) - D^\alpha f(w)| < \tilde{\epsilon}$$

For any $x \in \mathbb{R}^n$, taking $\epsilon < \delta$, and $v = x + \epsilon z$, $w = x$ with $z \in B_1(0)$ we have $|v - w| < \delta$, so:

$$|D^\alpha f(x - \epsilon z) - D^\alpha f(x)| < \tilde{\epsilon}$$

holds for any $x \in \mathbb{R}^n, z \in B_1(0)$. We have therefore shown that for any $\tilde{\epsilon} > 0$, there exists δ such that for any $\epsilon < \delta$ we have:

$$\sup_{x \in \mathbb{R}^n} |D^\alpha(\phi_\epsilon \star f)(x) - D^\alpha f(x)| < \tilde{\epsilon}.$$

This is the statement of uniform convergence on $\mathbb{R}^n$.

b) For this proof, we shall require certain facts from Measure Theory. First we require Minkowski's Integral Identity (see Exercise 1.3). This states[3] that for $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ a measurable function, we have the estimate:

$$\left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} F(x,y)dx\right|^p dy\right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x,y)|^p \, dy\right]^{\frac{1}{p}} dx$$

Now, following the calculation in the previous proof, we readily have that:

$$|(\phi_\epsilon \star g)(x) - g(x)| \leq \int_{\mathbb{R}^n} \phi(z) \, |g(x - \epsilon z) - g(x)| \, dz$$

Integrating and applying Minkowski's integral inequality, we have:

$$\begin{aligned}
||\phi_\epsilon \star g - g||_{L^p(\mathbb{R}^n)} &= \left[\int_{\mathbb{R}^n} |(\phi_\epsilon \star g)(x) - g(x)|^p \, dx\right]^{\frac{1}{p}} \\
&\leq \left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \phi(z) \, |g(x - \epsilon z) - g(x)| \, dz\right|^p dx\right]^{\frac{1}{p}} \\
&\leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \phi(z)^p \, |g(x - \epsilon z) - g(x)|^p \, dx\right]^{\frac{1}{p}} dz \\
&= \int_{\mathbb{R}^n} \phi(z) \, ||\tau_{\epsilon z} g - g||_{L^p(\mathbb{R}^n)} \, dz \qquad (1.1)
\end{aligned}$$

To establish our result it will suffice to set $\epsilon = \epsilon_j$, where $\{\epsilon_j\}_{j=1}^\infty \subset \mathbb{R}$ is any sequence with $\epsilon_j \to 0$, and show that $\left|\left|\phi_{\epsilon_j} \star g - g\right|\right|_{L^p(\mathbb{R}^n)} \to 0$. Note that since $\left|\left|\tau_{\epsilon_j z} g\right|\right|_{L^p(\mathbb{R}^n)} = ||g||_{L^p(\mathbb{R}^n)}$ we have:

$$\phi(z) \left|\left|\tau_{\epsilon_j z} g - g\right|\right|_{L^p(\mathbb{R}^n)} \leq 2\phi(z) \, ||g||_{L^p(\mathbb{R}^n)}$$

so the integrand is dominated uniformly in j by an integrable function. Now we claim (another Measure Theoretic / Functional Analysis fact, see Lemma 7) that as y varies, $\tau_y : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a continuous family of bounded linear operators. This means that for each $z \in \mathbb{R}^n$ we have:

$$\lim_{j \to \infty} \left|\left|\tau_{\epsilon_j z} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

Thus we can apply the Dominated Convergence Theorem to the integral on the right hand side of 1.1, and conclude that

$$\lim_{j \to \infty} \left|\left|\phi_{\epsilon_j} \star g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

[3] There is more general statement for a map $F : X \times Y \to \mathbb{C}$, which is measurable with respect to the product measure $\mu \times \nu$ where (X, μ) and (Y, ν) are measure spaces.

c) Again, by Theorem 2, we have that $D^\alpha(f \star g_\epsilon) = D^\alpha f \star g_\epsilon$ for any $|\alpha| \leq k$. By a change of variables, we calculate:

$$D^\alpha(f \star g_\epsilon)(x) = \int_{\mathbb{R}^n} g_\epsilon(y) D^\alpha f(x-y)dy = \int_{\mathbb{R}^n} g(y) D^\alpha f(x - \epsilon z)dz$$

Now, clearly for each fixed $x \in \mathbb{R}^n$:

$$g(z)D^\alpha f(x - \epsilon z) \to g(z) D^\alpha f(x)$$

for $z \in \mathbb{R}^n$ as $\epsilon \to 0$. Furthermore,

$$|g(z)D^\alpha f(x - \epsilon z)| \leq g(z) \sup_{\mathbb{R}^n} |D^\alpha f|$$

which is an integrable function of z, so by the Dominated convergence theorem, we conclude:

$$D^\alpha(f \star g_\epsilon)(x) \to D^\alpha f(x) \int_{\mathbb{R}^n} g(z)dz = D^\alpha f(x)$$

as $\epsilon \to 0$.

□

The final application of convolutions is to the construction of cut-off functions. These are often extremely useful for localising a problem to a particular region of interest for some reason or other.

Lemma 6. *Suppose $\Omega \subset \mathbb{R}^n$ is open, and $K \subset \Omega$ is compact. Then there exists $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ in a neighbourhood of K.*

Proof. Since K is compact and $\partial\Omega$ is closed, there exists $\epsilon > 0$ such that $d(K, \partial\Omega) > 4\epsilon$. We define $K_\epsilon = K + \overline{B_{2\epsilon}(0)}$. As the sum of two compact sets, K_ϵ is compact. Moreover, $K_\epsilon \subset \Omega$. Suppose ϕ_ϵ is as in Theorem 5. Consider:

$$\chi := \phi_\epsilon \star \mathbb{1}_{K_\epsilon}.$$

We have by Theorem 2 that $\chi \in C^\infty(\mathbb{R}^n)$ and from Lemma 4 we deduce:

$$\text{supp } \chi = K_\epsilon + \text{supp } \phi_\epsilon \subset K + \overline{B_{2\epsilon}(0)} + \overline{B_\epsilon(0)} = K + \overline{B_{3\epsilon}(0)} \subset \Omega.$$

Thus $\chi \in C_0^\infty(\Omega)$. Now, suppose $x \in K + B_\epsilon(0)$. Then $x + B_\epsilon(0) \subset K_\epsilon$ and so:

$$\begin{aligned} \chi(x) &= \int_{\mathbb{R}^n} \phi_\epsilon(y) \mathbb{1}_{K_\epsilon}(x-y)dy \\ &= \int_{B_\epsilon(0)} \phi_\epsilon(y) \mathbb{1}_{K_\epsilon}(x-y)dy \\ &= \phi_\epsilon(y)dy = 1. \end{aligned}$$

Thus $\chi(x) = 1$ for $x \in K + B_\epsilon(0)$, which is a neighbourhood of K. □

The following exercise and Lemma are included for completeness, as they establish results required for the proof of Theorem 5.

Exercise 1.3 (*)**.** Suppose that $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a positive integrable simple function,

a) Show that Minkowski's integral inequality holds for the case $p = 1$:

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x,y)dx \right| dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x,y)| \, dydx$$

b) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hint: set $t = p^{-1}$, *consider the function* $\log\left[ta^p + (1-t)b^q\right]$ *and use the concavity of the logarithm*

c) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $||f||_p = 1$ and $||g||_q = 1$ then

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq 1.$$

Deduce Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \ \leq ||f||_p \, ||g||_q \, , \qquad \text{for all } f \in L^p(\mathbb{R}^n), \quad g, \in L^q(\mathbb{R}^n).$$

d) Set $G(y) = \left(\int_{\mathbb{R}^n} F(x,y)dx\right)^{p-1}$

i) Show that if $q = \frac{p}{p-1}$:

$$||G||_{L^q(\mathbb{R}^n)} = \left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p-1}$$

ii) Show that:

$$\left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G(y)F(x,y)dy \right) dx$$

iii) Applying Hölder's inequality, deduce:

$$\left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p} \leq ||G||_{L^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} ||F(x,\cdot)||_{L^p(\mathbb{R}^n)} \, dx$$

e) Deduce that Minkowski's integral inequality

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x,y)dx \right|^p dy \right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x,y)|^p \, dy \right]^{\frac{1}{p}} dx$$

holds for any measurable function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, where $1 \leq p < \infty$.

Lemma 7. *Suppose $p \in [1, \infty)$ and $g \in L^p(\mathbb{R}^n)$. Let $\{z_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ be a sequence of points such that $z_j \to 0$ as $j \to \infty$. Then:*

$$\left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} \to 0.$$

Proof. 1. First, suppose $g = \mathbb{1}_Q$, where $Q = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$ is a n-box, with side-lengths $I_m = b_m - a_m$ for $m = 1, \ldots, n$. Now, since when a box is translated by a vector z_j each side is translated by a distance of at most $|z_j|$, and has area at most $I_{max.}^{n-1}$, where I_{max} is the longest side-length we can crudely estimate

$$\left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} \leq 2n\,|z_j|\, I_{\max}^{n-1}.$$

Note that this estimate requires $p < \infty$: it does not hold for $p = \infty$. We conclude that:

$$\lim_{j\to\infty} \left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

2. Now suppose $g = \mathbb{1}_A$, where A is a measurable set of finite measure. Fix $\epsilon > 0$. By the Borel regularity of Lebesgue measure, there exists a compact $K \subset A$ and an open $U \supset A$ such that $|U \setminus K| < \epsilon$. Since U is open, we can write U as a collection of open n-boxes:

$$U = \bigcup_{\alpha \in \mathscr{A}} Q_\alpha$$

Since K is compact, it is covered by a finite subset of these:

$$K \subset \bigcup_{i=1}^{N} Q_i := B.$$

Now, note that $K \subset B \subset U$, so the symmetric difference $A\Delta B \subset U \setminus K$. Thus[4] $||\mathbb{1}_A - \mathbb{1}_B||_{L^p(\mathbb{R}^n)} = |A\Delta B| < \epsilon$. By the paragraph 1 above, we know that there exists J such that for all $j \geq J$ we have:

$$\left|\left|\tau_{z_j} \mathbb{1}_B - \mathbb{1}_B\right|\right|_{L^p(\mathbb{R}^n)} < \epsilon$$

Therefore:

$$\begin{aligned}
\left|\left|\tau_{z_j} \mathbb{1}_A - \mathbb{1}_A\right|\right|_{L^p(\mathbb{R}^n)} &= \left|\left|\tau_{z_j} \mathbb{1}_A - \tau_{z_j} \mathbb{1}_B + \tau_{z_j} \mathbb{1}_B - \mathbb{1}_B + \mathbb{1}_B - \mathbb{1}_A\right|\right|_{L^p(\mathbb{R}^n)} \\
&\leq \left|\left|\tau_{z_j} \mathbb{1}_A - \tau_{z_j} \mathbb{1}_B\right|\right|_{L^p(\mathbb{R}^n)} + \left|\left|\tau_{z_j} \mathbb{1}_B - \mathbb{1}_B\right|\right|_{L^p(\mathbb{R}^n)} + ||\mathbb{1}_B - \mathbb{1}_A||_{L^p(\mathbb{R}^n)} \\
&= 2\,||\mathbb{1}_A - \mathbb{1}_B||_{L^p(\mathbb{R}^n)} + \left|\left|\tau_{z_j} \mathbb{1}_B - \mathbb{1}_B\right|\right|_{L^p(\mathbb{R}^n)} \\
&< 3\epsilon
\end{aligned}$$

for all $j \geq J$. Thus

$$\lim_{j\to\infty} \left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

[4] This is another point at which $p \neq \infty$ is crucial.

3. Now suppose g is a simple function, i.e. $g = \sum_{i=1}^{N} g_i \mathbb{1}_{A_i}$ for $g_i \in \mathbb{C}$ and A_i measurable sets of finite measure. Then we have:

$$\left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} \leq \sum_{i=1}^{N} |g_i| \left|\left|\tau_{z_j} \mathbb{1}_{A_i} - \mathbb{1}_{A_i}\right|\right|_{L^p(\mathbb{R}^n)}$$

so as $j \to \infty$ we have:

$$\lim_{j\to\infty} \left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

4. Now suppose that $g \in L^p(\mathbb{R}^n)$. Fix $\epsilon > 0$. Recall that there exists a simple function $\tilde{g}$ such that $||g - \tilde{g}||_{L^p(\mathbb{R}^n)} < \epsilon$. By the previous part, we can find J such that $\left|\left|\tau_{z_j}\tilde{g} - \tilde{g}\right|\right|_{L^p(\mathbb{R}^n)} < \epsilon$ for all $j \geq J$. Now:

$$\begin{aligned}
\left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} &= \left|\left|\tau_{z_j} g - \tau_{z_j}\tilde{g} + \tau_{z_j}\tilde{g} - \tilde{g} + \tilde{g} - g\right|\right|_{L^p(\mathbb{R}^n)} \\
&\leq \left|\left|\tau_{z_j} g - \tau_{z_j}\tilde{g}\right|\right|_{L^p(\mathbb{R}^n)} + \left|\left|\tau_{z_j}\tilde{g} - \tilde{g}\right|\right|_{L^p(\mathbb{R}^n)} + ||\tilde{g} - g||_{L^p(\mathbb{R}^n)} \\
&= 2\,||g - \tilde{g}||_{L^p(\mathbb{R}^n)} + \left|\left|\tau_{z_j}\tilde{g} - g\right|\right|_{L^p(\mathbb{R}^n)} \\
&< 3\epsilon
\end{aligned}$$

Thus, we conclude that

$$\lim_{j\to\infty} \left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

and we're done. □