*Unless otherwise stated, $U \subset \mathbb{R}^n$ is open and bounded with $C^1$ boundary, $U_T = (0,T) \times U$, $\Sigma_t = \{t\} \times U$ for $t \in [0,T]$ and $\partial^* U_T = [0,T] \times \partial U$.*

**Exercise 4.1.** (*) Suppose $L$ is a uniformly elliptic operator on $U$, with associated bilinear form $B$. Assume $a^{ij}, b^i, c \in C^{m+1}(U)$ and $f \in L^2(U) \cap H^m_{loc.}(U)$. Suppose $u \in H^1(U)$ satisfies:

$$B[u,v] = (f,v)_{L^2(U)}, \qquad \text{for all } v \in H^1_0(U).$$

i) Show that $u \in H^{m+2}_{loc.}$ and for any $V, W$ with $V \subset\subset W \subset\subset U$ we have the estimate:

$$||u||_{H^{m+2}(V)} \leq C\left(||f||_{H^m(W)} + ||u||_{L^2(W)}\right).$$

ii) Deduce that if $a^{ij}, b^i, c, f \in C^\infty(U)$, then $u \in C^\infty(U)$.

iii) Conclude that if $L$ is a uniformly elliptic operator with smooth coefficients on a bounded domain, the eigenfunctions of $L$ are smooth away from the boundary.

*[Hint: For part i), proceed by induction. The $m = 0$ case was covered in lectures. For the induction step, consider as a test function $v = -D_i \tilde{v}$ for some $\tilde{v} \in C^\infty_c(U)$ and integrate by parts.]*

**Exercise 4.2.** (*) Suppose $U$ has a smooth ($C^\infty$) boundary. Show that any $u \in H^1_0(U) \cap H^k(U)$ may be approximated in $H^k(U)$ by a sequence $u_m \in C^\infty(\overline{U})$ such that $u_m = 0$ on $\partial U$.

*[Hint: Note that you cannot simply cut u off near the boundary and mollify. Make sure you understand why!]*

**Exercise 4.3.** Let $n \leq 3$, and assume $\partial U$ is $C^2$. Consider the nonlinear boundary value problem:

$$-\Delta u + u + |u|^p = f, \quad \text{in } U, \qquad u = 0 \quad \text{on } \partial U,$$

For some $p > 1$.

i) Show that if $f \in L^2(U)$ satisfies $||f||_{L^2(U)} < \epsilon$ for some sufficiently small $\epsilon$, then a solution exists with $u \in H^2(U)$.

ii) (†) What happens for $n > 3$?

*[Hint: Show that the map which takes $w \in H^2(U)$ to the solution of*

$$-\Delta u + u = f - w^p, \quad \text{in } U, \qquad u = 0 \quad \text{on } \partial U,$$

*is a contraction map on $B_b := \left\{ u \in H^2(U) \middle| ||u||_{H^2(U)} \leq b \right\}$ for some $b > 0$, and invoke the contraction mapping principle]*

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Please send any corrections to cmw50@cam.ac.uk

Questions marked (∗) may be handed in, those marked (†) are optional and may be tough.

**Exercise 4.4.** Suppose $(u_m)_{m=1}^{\infty}$ is a sequence with $u_m \in L^2(U_T) \cap L^\infty((0,T);L^2(U))$ such that there exists $u \in L^2(U_T)$ with $u_m \rightharpoonup u$ in $L^2(U_T)$. Suppose further that:

$$||u_m||_{L^\infty((0,T);L^2(U))} \leq K.$$

for $K$ uniform in $m$. Show that in fact $u \in L^\infty((0,T);L^2(U))$.

**Exercise 4.5.** Let $f \in L^2(U_T)$ and $\psi \in L^2(U)$. Consider the parabolic initial boundary value problem:

$$\begin{cases} u_t + Lu = f & \text{in } U_T, \\ u = \psi & \text{on } \Sigma_0, \\ u = 0 & \text{on } \partial^* U_T. \end{cases} \qquad (\diamond)$$

where:

$$Lu := -\sum_{i,j=1}^n \left(a^{ij}(t,x)u_{x_i}\right)_{x_j} + \sum_{i=1}^n b^i(t,x)u_{x_i} + c(t,x)u$$

with $a^{ij} = a^{ji}, b^i, c \in C^1(\overline{U_T})$ satisfies the uniform ellipticity condition:

$$\sum_{i,j=1}^n a^{ij}(t,x)\xi_i\xi_j \geq \theta|\xi|^2$$

for some $\theta > 0$ and all $(t,x) \in U_T$, $\xi \in \mathbb{R}^n$. We say $u \in L^2\left((0,T);H_0^1(U)\right)$ is a weak solution of $(\diamond)$ if:

$$\int_{U_T} \left\{ -uv_t + \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right\} dxdt = \int_{\Sigma_0} \psi v dx + \int_{U_T} fv dxdt$$

holds for all $v \in H^1(U_T)$ such that $v = 0$ on $\Sigma_T \cup \partial^* U_T$.

i) Show that if $u \in C^2(\overline{U_T})$ then in fact $u$ solves $(\diamond)$ in the classical sense.

ii) Show that a weak solution, if it exists, is unique.

iii) Establish that for any $\psi \in L^2(U)$, $f \in L^2(U_T)$ there exists a weak solution satisfying:

$$\int_{U_T} \left(u^2 + |Du|^2\right) dxdt \leq C\left(||\psi||^2_{L^2(U)} + ||f||^2_{L^2(U_T)}\right)$$

for some $C > 0$, and verify that in fact $u \in L^\infty((0,T);L^2(U))$.

iv) Show that if $\psi \in H_0^1(U)$ then in fact

$$u \in L^2((0,T);H^2(U)) \cap L^\infty((0,T);H_0^1(U)) \cap H^1((0,T);L^2(U))$$

and establish estimates for the norm of $u$ in each of these spaces.

v) Let $S_T = (0,T) \times \mathbb{R}^n$. Suppose that $a^{ij}, b^i, c \in C^1(\overline{S_T})$ and that the uniform ellipticity condition holds in $S_T$.

a) Let $f \in L^2(S_T)$ and $\psi \in L^2(\mathbb{R}^n)$. Define a weak solution $u \in L^2((0,T); H^1(\mathbb{R}^n))$ to the Cauchy problem

$$\begin{cases} u_t + Lu = f & \text{in } (0,T) \times \mathbb{R}^n, \\ u = \psi & \text{on } \{0\} \times \mathbb{R}^n. \end{cases} \qquad (\star)$$

and show that if additionally $u \in C^2(\overline{(0,T) \times \mathbb{R}^n})$, then a weak solution according to your definition solves $(\star)$ in a classical sense.

b) By solving the finite problem on a suitable sequence of domains, show that if $\psi \in L^2(\mathbb{R}^n)$, $f \in L^2(S_T)$, then there exists a unique weak solution to $(\star)$.

c) Show that if $\psi \in H^1(\mathbb{R}^n)$ then

$$u \in L^2((0,T); H^2(\mathbb{R}^n)) \cap L^\infty((0,T); H^1_0(\mathbb{R}^n)) \cap H^1((0,T); L^2(\mathbb{R}^n))$$

vi) (†) Suppose $b^i \equiv 0$. By making the obvious changes to the function spaces required to consider complex valued $u$, establish similar results for the initial-boundary value problem for the Schrödinger-type equation:

$$\begin{cases} iu_t + Lu = f & \text{in } U_T, \\ u = \psi & \text{on } \Sigma_0, \\ u = 0 & \text{on } \partial^* U_T. \end{cases}$$

and extend your conclusions to the Cauchy problem.

**Exercise 4.6.** Let $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$, $S_{*,T} := \mathbb{R}^3_* \times (-T,T)$ and $|x| = r$. You may assume the result that if $u = u(r,t)$ is radial, we have

$$\Delta u(|x|,t) = \Delta u(r,t) = \frac{\partial^2 u}{\partial r^2}(r,t) + \frac{2}{r}\frac{\partial u}{\partial r}(r,t)$$

i) Suppose $f, g \in C^2_c(\mathbb{R})$. Show that

$$u(x,t) = \frac{f(r+t)}{r} + \frac{g(r-t)}{r}$$

is a solution of the wave equation on $S_{*,T}$ which vanishes for large $|x|$.

ii) Show that if $f \in C^3_c(\mathbb{R})$ is an odd function (i.e. $f(s) = -f(-s)$ for all $s$) then

$$u(x,t) = \frac{f(r+t) + f(r-t)}{2r}$$

extends as a $C^2$ function which solves the wave equation on $S_T$, with

$$u(0,t) = f'(t).$$

iii) By considering a suitable sequence of functions $f$, or otherwise, deduce that there exists no constant $C$ independent of $u$ such that the estimate

$$\sup_{S_T} (|Du| + |u_t|) \leq C \sup_{\Sigma_0} (|Du| + |u_t|)$$

holds for all solutions $u \in C^2(S_T)$ of the wave equation which vanish for large $|x|$.

**Exercise 4.7.** Let $S_T = (0, T) \times \mathbb{R}^3$ and $\Sigma_t = \{t\} \times \mathbb{R}^3$. Consider the semilinear Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u = \lambda u^3 & \text{in } S_T, \\ u = \psi & \text{on } \Sigma_0, \\ u_t = \psi' & \text{on } \Sigma_0, \end{cases} \tag{$\flat$}$$

where $\psi \in H^1(\mathbb{R}^3)$, $\psi' \in L^2(\mathbb{R}^3)$ and $\lambda \in \mathbb{R}$ is a constant. Let $X := L^\infty([0, T], H^1(\mathbb{R}^3)) \cap W^{1,\infty}([0, T], L^2(\mathbb{R}^3))$, and equip this space with the norm:

$$||f||_X := \sup_{t \in (0,T)} \left( ||f||_{H^1(\Sigma_t)} + ||f_t||_{L^2(\Sigma_t)} \right)$$

i) Suppose $w \in X$ satisfies:

$$||w||_X \leq 8 \left( ||\psi||_{H^1(\mathbb{R}^3)} + ||\psi'||_{L^2(\mathbb{R}^3)} \right) := b$$

Show that if $u$ is the unique weak solution to the linear Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u = w^3 & \text{in } S_T, \\ u = \psi & \text{on } \Sigma_0, \\ u_t = \psi' & \text{on } \Sigma_0, \end{cases}$$

then $u$ satisfies:

$$||u||_X \leq \left[ CTb^3 + \frac{b}{2} \right] (1 + T)$$

for some $C > 0$ independent of $u, T$.

*[Hint: Recall the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$]*

ii) Let $A : X \to X$ be the map $w \mapsto u$. Show that for $T$ sufficiently small, $A$ maps $B_b$ to itself, where:

$$B_b = \{u \in X : ||u||_X \leq b\}$$

is the closed ball of radius $b$ in $X$.

iii) Suppose that $w, v \in B_b$. Show that:

$$||Aw - Av||_X \leq C_b T (1 + T) \, ||w - v||_X$$

for some constant $C$ depending on $b$ but not $T$. Deduce that ($\flat$) admits a unique solution $u \in X$, provided $T < T_0$ for some $T_0 = T_0(\psi, \psi')$.

iv) Show that for $\lambda > 0$ a solution exists to the equation $u_{tt} - \Delta u = \lambda u^3$ of the form $u(t, x) = \alpha(t - \tau)^\beta$, where $\alpha, \beta$ are to be determined. Deduce that there exist solutions to ($\flat$) arising from initial data with finite energy which become singular in finite time.

v) ($\dagger$, *for those with some knowledge of relativity*) By a conformal transformation mapping the Minkowski space into a finite region of the Einstein static universe $S^3 \times \mathbb{R}$, show that solutions to ($\flat$) arising from sufficiently small[5] data exist for all time.

[5] in a sense that you should determine