Unless otherwise stated, $U \subset \mathbb{R}^n$ is open and bounded with C^1 boundary, and L is the operator:

$$Lu := -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$$

for $a^{ij} = a^{ji}, b^i, c \in L^\infty(U)$, and a^{ij} satisfying the uniform ellipticity condition:

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $\xi \in \mathbb{R}^n$, almost every $x \in U$ and some $\theta > 0$.

Exercise 3.1. For $\gamma > 0$ sufficiently large let $L_\gamma^{-1} : L^2(U) \to H_0^1(U)$ be the bounded linear operator which maps $f \in L^2(U)$ to the weak solution $u \in H_0^1(U)$ of the boundary value problem:

$$Lu + \gamma u = f \quad \text{in } U, \qquad u = 0 \quad \text{on } \partial U.$$

Show that $\operatorname{Ran} L_\gamma^{-1}$ is dense in $H_0^1(U)$.

[Hint: consider $f = L_\gamma \phi$ for $\phi \in C_c^\infty(U)$, and approximate the functions $\sum_{j=1}^n a^{ij}\phi_{x_j}$ in $L^2(U)$.]

Exercise 3.2. Suppose that L is positive and formally self-adjoint. Let $\{w_k\}_{k=1}^\infty$ be an orthonormal basis for $L^2(U)$ consisting of eigenfunctions $w_k \in H_0^1(U)$ solving

$$Lw_k = \lambda_k w_k \quad \text{in } U, \qquad w_k = 0 \quad \text{on } \partial U,$$

in a weak sense, where $0 < \lambda_1 \leq \lambda_2 \leq \ldots$.

i) If B is the bilinear form on $H_0^1(U)$ corresponding to L, show that B can be thought of as a new inner product on $H_0^1(U)$, defining an equivalent norm to $(\cdot,\cdot)_{H^1(U)}$.

ii) Show that:

$$B[w_k, w_l] = \lambda_k \delta_{kl}.$$

iii) Deduce that $\left\{\lambda_k^{-\frac{1}{2}} w_k\right\}_{k=1}^\infty$ is a basis for $H_0^1(U)$ which is orthonormal with respect to the inner product defined by B.

iv) Show that:

$$\lambda_1 = \min_{u \in H_0^1(U), ||u||_{L^2(U)}=1} B[u,u].$$

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Questions marked ($*$) may be handed in, those marked (†) are optional and may be tough.

v) (†) Show that:

$$\lambda_k = \max_{V \in \mathcal{V}_{k-1}} \left(\min_{u \in V, ||u||_{L^2(U)}=1} B[u,u] \right)$$

where $V \in \mathcal{V}_k$ if

$$V = \{u \in H_0^1(U) | (u, v_i)_{L^2(U)} = 0, i = 1, \dots, k\}$$

for some k distinct vectors $v_i \in H_0^1(U)$, $i = 1, \dots, k$. By convention $\mathcal{V}_0 = \{H_0^1(U)\}$. In other words, $\mathcal{V}_k$ is the set of codimension k subspaces of $H_0^1(U)$. This is the Courant minimax principle.

Exercise 3.3. Suppose that the boundary value problem:

$$Lu = f \quad \text{in } U, \qquad u = 0 \quad \text{on } \partial U. \tag{$\triangle$}$$

admits a weak solution $u \in H_0^1(U)$ for every $f \in L^2(U)$. Show that there exists a constant C such that:

$$||u||_{L^2(U)} \leq C\, ||f||_{L^2(U)}$$

for all $f \in L^2(U)$ and $u \in H_0^1(U)$ solving ($\triangle$) in the weak sense.

[Hint: Assume the result is not true, so that a sequence of $u_k \in H_0^1(U)$ and $f_k \in L^2(U)$ solving ($\triangle$) exist with $||u_k||_{L^2(U)} \geq k\, ||f_k||_{L^2(U)}$. Derive a contradiction.]

Exercise 3.4 (*)**.** Let $U \subset \mathbb{R}^n$ be an open, not necessarily bounded set and assume $V \subset\subset U$. Suppose $u \in L^2(U)$ and define the i^{th} difference quotient:

$$\Delta_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}$$

for $i = 1, \dots, n$; $x \in V$ and $0 < |h| < \text{dist}(V, \partial U)$. Let $\Delta^h u = (\Delta_1^h u, \dots, \Delta_n^h u)$.

i) a) Suppose $u \in C^\infty(U)$. Show that for $x \in V$, $0 < |h| < \frac{1}{2}\text{dist}(V, \partial U)$:

$$|u(x + he_i) - u(x)| \leq |h| \int_0^1 |Du(x + the_i)| dt$$

b) Deduce that:

$$\int_V \left|\Delta^h u(x)\right|^2 dx \leq \sum_{i=1}^n \int_V \int_0^1 |Du(x + the_i)|^2 dt dx$$

c) Conclude that:

$$\left|\left|\Delta^h u\right|\right|_{L^2(V)} \leq \sqrt{n}\, ||Du||_{L^2(U)}$$

for all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial U)$. Deduce that this inequality holds for all $u \in H^1(U)$.

ii) Now suppose that $u \in L^2(U)$ satisfies:

$$\left|\left|\Delta^h u\right|\right|_{L^2(V)} \leq C$$

for all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial U)$ and some constant C.

a) Show that if $\phi \in C_c^\infty(V)$ and h is sufficiently small then:

$$\int_V u(x) \left[\Delta_i^h \phi(x)\right] dx = -\int_V \left[\Delta_i^{-h} u(x)\right] \phi(x) dx.$$

b) Show there exists a sequence $h_k \to 0$ and $v_i \in L^2(V)$ with $||v_i||_{L^2(V)} \leq C$ such that

$$\Delta_i^{-h_k} u \rightharpoonup v_i$$

in $L^2(V)$ as $k \to \infty$.

c) Deduce that $D_i u = v_i$ in the weak sense, and hence $u \in H^1(V)$.

Exercise 3.5. Let $I = (a, b)$ for some $a < b$.

i) Let $x \in I$. Show that if $u \in C_c^\infty(I)$ then:

$$|u(x)|^2 \leq 2\,||u||_{L^2(I)} \left|\left|u'\right|\right|_{L^2(I)}.$$

Deduce that there exists a bounded linear map $\delta_x : H_0^1(I) \to \mathbb{R}$ such that $\delta_x[u] = u(x)$ for $u \in C^0(I) \cap H_0^1(I)$, and find explicitly $v_x \in H_0^1(I)$ such that $\delta_x[u] = (v_x, u)_{H^1(I)}$ for all $u \in H_0^1(I)$, where we define as usual:

$$(u_1, u_2)_{H^1(I)} = \int_a^b \left(u_1' u_2' + u_1 u_2\right) dx$$

ii) Consider the Sturm-Liouville operator:

$$Lu := -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u,$$

where $p, q \in L^\infty(I)$ with $p(x) \geq c > 0$ and $q(x) \geq 0$ for almost every x.

a) Define a weak solution to the boundary value problem:

$$Lu = f \quad \text{in } I, \qquad u(a) = u(b) = 0,$$

for $f \in \left(H_0^1(I)\right)^*$ and show that there exists a unique weak solution to:

$$Lu = \delta_x \quad \text{in } I, \qquad u(a) = u(b) = 0.$$

b) Let $u, \tilde{u} \in H^1_0(I)$ be two non-trivial weak solutions to the eigenvalue problem:

$$Lu = \lambda u \quad \text{in } I, \qquad u(a) = u(b) = 0, \tag{$\star$}$$

for some $\lambda \in \mathbb{R}$. Show that the Wronskian:

$$W(x) = p(x)\left[u'(x)\tilde{u}(x) - u(x)\tilde{u}'(x)\right]$$

admits a weak derivative which vanishes identically on I. (†) Conclude that there exists $\alpha \in \mathbb{R}$ such that $u(x) = \alpha\tilde{u}(x)$.

[Hint: Consider using $\phi\tilde{u}$ as a test function in the weak formulation of ($\star$) for some $\phi \in C^\infty_c(I)$.]

c) Show that there exists an orthonormal basis for $L^2(I)$ consisting of eigenfunctions $u_n \in H^1_0(I)$ which are weak solutions to ($\star$) with $\lambda = \lambda_n$, where $0 < \lambda_1 < \lambda_2 < \ldots$.

d) (†) Show that if $p, q \in C^\infty(\overline{I})$, then $u_n \in C^\infty(\overline{I})$.

iii) (†) Fix $y \in (a, b)$ Consider the modified Sturm-Liouville operator with a 'delta function potential[4]':

$$Lu := -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + a\delta_y u,$$

where $a \in \mathbb{R}$. Define a weak solution to the boundary value problem:

$$Lu = f \quad \text{in } I, \qquad u(a) = u(b) = 0, \tag{$\diamond$}$$

and show that either a weak solution to ($\diamond$) exists for every $f \in L^2(I)$ or else a solution to the homogeneous problem exists.

Exercise 3.6 (†)**.** Let $n \geq 3$. For $u \in C^\infty_c(\mathbb{R}^n)$ define:

$$||u||_{\dot{H}^1(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |Du|^2 dx\right)^{\frac{1}{2}}.$$

i) Show that $||\cdot||_{\dot{H}^1(\mathbb{R}^n)}$ defines a norm on $C^\infty_c(\mathbb{R}^n)$. We denote by $\dot{H}^1(\mathbb{R}^n)$ the completion of $C^\infty_c(\mathbb{R}^n)$ in this norm.

ii) Show that $H^1(\mathbb{R}^n) \subset \dot{H}^1(\mathbb{R}^n)$, and that the inclusion is proper.

iii) Show that there exists a constant C such that:

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq C\, ||u||^2_{\dot{H}^1(\mathbb{R}^n)}$$

for all $u \in \dot{H}^1(\mathbb{R}^n)$. Estimates such as this known as Hardy inequalities.

[Hint: Write $\frac{1}{|x|^2} = \frac{1}{n-2}\text{div}\left(\frac{x}{|x|^2}\right)$ and integrate by parts]

[4] such operators appear in physics, for example in quantum mechanics.

iv) Show that for any f such that $|x|f \in L^2(\mathbb{R}^n)$ there exists a unique weak solution $u \in \dot{H}^1(\mathbb{R}^n)$ to Poisson's equation:

$$-\Delta u = f \text{ on } \mathbb{R}^n, \qquad u \to 0 \text{ as } |x| \to \infty.$$

You should carefully define what is meant by a weak solution in this context.

Exercise 3.7 (†)**.** Repeat Exercise 3.4 replacing $L^2(U)$ with $L^p(U)$ everywhere, for some $1 < p < \infty$. For part *ii*) you will first need to prove that if X is a reflexive Banach space then every bounded sequence in X admits a weakly convergent subsequence. What goes wrong in the case $p = 1$?

Exercise 3.8 (†)**.** Let $U \subset \mathbb{R}^n$ be open and suppose $1 \leq p < \infty$. Let $L^p_k(U)$ be the set of vectors $v = (v^\alpha)_{|\alpha| \leq k}$ whose components are indexed by multiindices of order at most k, such that each $v^\alpha \in L^p(U)$. Define:

$$||v||_{L^p_k(U)} := \left(\sum_{|\alpha| \leq k} ||v^\alpha||^p_{L^p(U)} \right)^{\frac{1}{p}}.$$

i) Show that $L^p_k(U)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.

ii) Show that the map $\iota : W^{k,p}(U) \to L^p_k(U)$ given by:

$$\iota : u \mapsto (D^\alpha u)_{|\alpha| \leq k}$$

is a bounded linear map satisfying:

$$||u||_{W^{k,p}(U)} = ||\iota(u)||_{L^p_k(U)}, \qquad \forall u \in W^{k,p}(U)$$

i.e. an isometry. Deduce that ι is injective.

iii) Show that $\iota(W^{k,p}(U))$ is a closed subspace of $L^p_k(U)$ and deduce that $W^{k,p}(U)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.