In all the exercises, $U \subset \mathbb{R}^n$ denotes an open set, $k = 0, 1, \ldots,$ and $1 \leq p < \infty$ unless otherwise stated.

Exercise 2.1. Show that the Hölder space $C^{0,\gamma}(\overline{U})$ defined in lectures is a Banach space. Show that if U is convex then $C^1(\overline{U}) \subset C^{0,\gamma}(\overline{U})$ for any $0 < \gamma \leq 1$.

[Hint: use the mean value theorem]

Exercise 2.2. Show there exists a function $u \in W^{1,p}(U)$ which is unbounded on any open set.
[Hint: consider a suitable sum of shifted and scaled versions of the unbounded function given in lectures]

Exercise 2.3. Suppose $u, v \in W^{k,p}(U)$ and $|\alpha| \leq k$. Establish the following results:

i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta (D^\alpha u) = D^\alpha \left(D^\beta u\right)$ for all β with $|\alpha| + |\beta| \leq k$.

ii) For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha (\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ for all $|\alpha| \leq k$.

iii) If $V \subset U$ is open, then $u|_V \in W^{k,p}(V)$.

iv) If $\zeta \in C^\infty(\overline{U})$, then $\zeta u \in W^{k,p}(U)$.

Exercise 2.4 (*)**.** Suppose U is connected and $u \in W^{1,p}(U)$ satisfies $Du = 0$ almost everywhere in U. Show that u is constant almost everywhere in U.

[Hint: show that on any $V \subset\subset U$ the mollification $\eta_\epsilon \star u$ is constant for ϵ sufficiently small, and take a limit $\epsilon \to 0$.]

Exercise 2.5. Suppose $u \in W^{k,p}(\mathbb{R}^n)$. Show that there exists a sequence $(u_m)_{m=1}^\infty$ such that $u_m \in C_c^\infty(\mathbb{R}^n)$ and $u_m \to u$ in $W^{k,p}(\mathbb{R}^n)$.

[Hint: First establish that for any ϵ there exists a compactly supported function χ such that $||\chi u - u||_{W^{k,p}(\mathbb{R}^n)} < \epsilon$]

Exercise 2.6 (*)**.** Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \leq p < \infty$.

i) Suppose $F : \mathbb{R} \to \mathbb{R}$ is C^1 with F' bounded. Show that $v := F(u) \in W^{1,p}(U)$ and $v_{x_i} = F'(u) u_{x_i}$ for $i = 1, \ldots, n$.

ii) Show that $|u| \in W^{1,p}(U)$, with $D_i|u| = \operatorname{sgn} u \; u_{x_i}$.

[Hint: Consider the functions $F_\epsilon(x) = \sqrt{x^2 + \epsilon^2} - \epsilon$]

Please send any corrections to cmw50@cam.ac.uk
Questions marked $(*)$ may be handed in, those marked (†) are optional.

Exercise 2.7. Suppose $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism[2] with inverse $\Phi : \mathbb{R}^n \to \mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be open and bounded, with Lipshitz boundary, and suppose $u \in C^\infty(\overline{U})$.

i) Show that there exists a constant C, independent of u, such that:

$$||u \circ \Phi||_{W^{1,p}(\Psi(U))} \leq C \, ||u||_{W^{1,p}(U)} .$$

ii) Let $1 \leq p < \infty$. By considering a suitable approximating sequence, or otherwise, show that if $u \in W^{1,p}(U)$ then $u \circ \Phi \in W^{1,p}(\Psi(U))$

iii) (†) Show that a similar result holds for $W^{k,p}$ provided Ψ is C^k.

Exercise 2.8. Finish the proof of the trace theorem from lectures: by choosing a partition of unity subordinate to a suitable collection of open sets, show that the problem can be reduced to the case of a straight boundary.

Exercise 2.9. Suppose $1 \leq q < \infty$ and $1 \leq p < n$. Suppose that the estimate

$$||u||_{L^q(\mathbb{R}^n)} \leq C \, ||Du||_{L^p(\mathbb{R}^n)}$$

holds for all $u \in C^\infty_c(\mathbb{R}^n)$, where $C > 0$ is a constant depending only on n, p. Show that p, q, n must satisfy:

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

[Hint: consider a family of functions $u_\lambda(x) := u(\lambda x)$ for some fixed $u \in C^\infty_c(\mathbb{R}^n)$ and $\lambda > 0$.]

Exercise 2.10. Let U be bounded with C^1 boundary. Suppose

$$Lu := -\sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x) u$$

where $a^{ij}, b^i, c \in C^\infty(\overline{U})$ and $a^{ij} = a^{ji}$ satisfy the uniform ellipticity condition.

i) Find a weak formulation for the Neumann problem:

$$\begin{aligned} (L+\lambda)u &= f \qquad \text{in } U, \\ \textstyle\sum_{i,j=1}^n a^{ij}(x) u_{x_i} \nu_j &= 0 \qquad \text{on } \partial U, \end{aligned} \tag{$\star$}$$

where ν is the outward unit normal to ∂U. In particular you should demonstrate that $u \in C^2(\overline{U})$ is a classical solution of ($\star$) if and only if it satisfies your criterion to be a weak solution.

ii) Show that if $\lambda > 0$ is sufficiently large and $f \in L^2(U)$, ($\star$) admits a unique weak solution $u \in H^1(U)$.

[2] i.e., both Ψ and its inverse Φ are continuously differentiable

Exercise 2.11 (†, for those familiar with Fourier analysis)**.** For $0 \leq s < \infty$ define $\hat{H}^s(\mathbb{R}^n)$ to be the set of functions $f \in L^2(\mathbb{R}^n)$ such that:

$$||f||^2_{\hat{H}^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1+|\xi|^2)^s \left|\hat{f}(\xi)\right|^2 d\xi < \infty,$$

where $\hat{f} \in L^2(\mathbb{R}^n)$ is Fourier-Plancherel transform of f.

i) Show that $\hat{H}^s(\mathbb{R}^n)$ is a Hilbert space.

ii) Show that if $k = 0, 1, 2, \dots$ there exists a constant $C_k > 0$ such that:

$$\frac{1}{C_k} ||f||_{\hat{H}^k(\mathbb{R}^n)} \leq ||f||_{H^k(\mathbb{R}^n)} \leq C_k ||f||_{\hat{H}^k(\mathbb{R}^n)}.$$

for all[3] $f \in \mathscr{S}(\mathbb{R}^n)$. Deduce that $H^k(\mathbb{R}^n) = \hat{H}^k(\mathbb{R}^n)$.

iii) Assume $s > \frac{1}{2}$ and suppose $u \in \mathscr{S}(\mathbb{R}^n)$. Define $Tu \in C^\infty(\mathbb{R}^{n-1})$ by:

$$Tu(x') = u(x', 0), \qquad x' \in \mathbb{R}^{n-1}.$$

a) Show that

$$\widehat{Tu}(\xi') = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\xi', \xi_n) d\xi_n.$$

b) Deduce that then:

$$\left|\widehat{Tu}(\xi')\right|^2 \leq \frac{1}{2\pi} \left(\int_{\mathbb{R}} (1+|\xi|^2)^s \left|\hat{u}(\xi', \xi_n)\right|^2 d\xi_n \right) \left(\int_{\mathbb{R}} \frac{d\xi_n}{\left(1+|\xi|^2\right)^s} \right)$$

c) By changing variables in the second integral above to $\xi_n = t\sqrt{1+|\xi'|^2}$, conclude that there exists a constant $C(s)$ such that:

$$||Tu||_{\hat{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C(s) ||u||_{\hat{H}^s(\mathbb{R}^n)}.$$

d) Conclude that T extends to a bounded linear operator $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.

e) Suppose $v \in \mathscr{S}(\mathbb{R}^{n-1})$ and let $\phi \in C^\infty_c(\mathbb{R})$ satisfy $\int_{\mathbb{R}} \phi(t)dt = \sqrt{2\pi}$. Define u through its Fourier transform by:

$$\hat{u}(\xi', \xi_n) = \frac{\hat{v}(\xi')}{\sqrt{1+|\xi'|^2}} \phi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}} \right).$$

Show that there exists a constant $C > 0$ such that:

$$||u||_{\hat{H}^s(\mathbb{R}^n)} \leq C ||v||_{\hat{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})}$$

and that $Tu = v$.

[3] Recall u belongs to the *Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ if $u \in C^\infty(\mathbb{R}^n)$ and $\sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta u(x)\right| < \infty$ for all multiindices α, β.

f) Conclude that $T : \hat{H}^s(\mathbb{R}^n) \to \hat{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is surjective.

Exercise 2.12 (†, hard)**.** Suppose that for some $c \geq 0$ the inequality:

$$||u||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq c\,||Du||_{L^1(\mathbb{R}^n)}$$

holds all $u \in C_c^\infty(\mathbb{R}^n)$. Show that if U is a bounded set with ∂U of class C^1 then the isoperimetric inequality holds:

$$|U|^{\frac{n-1}{n}} \leq c|\partial U|,$$

with the same constant c. Deduce that $c \geq n^{-1}|B|^{-\frac{1}{n}}$, with B the unit ball in $\mathbb{R}^n$.