Example Sheet 1 Part III: Analysis of PDE
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Exercise 1.1. Classify the example PDEs given in lectures into linear / semilinear / quasilinear / fully nonlinear.

Exercise 1.2. Find three examples of well known PDE in mathematics or physics (other than the examples given in lectures). Write a sentence or two about the importance of each and classify them into linear / semilinear / quasilinear / fully nonlinear.

Exercise 1.3. Fill in the gaps of the proof of the Picard-Lindelöf theorem from lectures.

Exercise 1.4. By induction on $m \in \mathbb{N}$, show that if $x \in \mathbb{R}^m$ and $j \in \mathbb{N}$ then:

$$(x_1 + \ldots + x_m)^j = \sum_{|\alpha|=j} \left(\begin{array}{c} |\alpha| \\ \alpha \end{array} \right) x^\alpha,$$

where the multinomial coefficient is defined by:

$$\left(\begin{array}{c} |\alpha| \\ \alpha \end{array} \right) = \frac{|\alpha|!}{\alpha!} = \frac{|\alpha|!}{\alpha_1!\alpha_2!, \cdots \alpha_m!}.$$

[Hint: Note that the case $m = 1$ is trivial and $m = 2$ is the binomial theorem. Work by induction using $(x_1 + \ldots + x_{m-1} + x_m)^j = (x_1 + \ldots + (x_{m-1} + x_m))^j$]

Exercise 1.5 (*)**.** Let $x \in \mathbb{R}^n$ and suppose that $f(x) = \sum_\alpha f_\alpha x^\alpha$ and $g(x) = \sum_\alpha g_\alpha x^\alpha$ are two formal power series.

a) Show that if $g \gg f$ then $D^\beta g \gg D^\beta f$ for any multiindex β, where we differentiate each formal series term by term.

b) Suppose that $g \gg f$ and g converges for $|x| < r$. Show that for any $s < r$:

$$\sup_{|x|\leq s} |f(x)| \leq \sup_{|x|\leq s} g(x)$$

c) By making an appropriate choice of majorant g, show that if f is real analytic at $x = 0$ then there exist constants $s > 0$, $C > 0$ and $\rho > 0$ such that:

$$\sup_{|x|\leq s} \left| D^\beta f(x) \right| \leq C \frac{|\beta|!}{\rho^{|\beta|}} \tag{1}$$

d) Conversely, suppose $f : B_r(0) \to \mathbb{R}$ is a smooth function such that (2) holds for some $s > 0$, $C > 0$ and $\rho > 0$. Show that f is real analytic at 0 (you may assume the multivariable Taylor theorem).

Please send any corrections to cmw50@cam.ac.uk
Questions marked $(*)$ may be handed in, those marked $(\dagger)$ are optional.

e) Show that if $f : B_r(0) \to \mathbb{R}$ is real analytic at 0, there exists s with $0 < s < r$ such that f is real analytic at x_0 for all $x_0 \in B_s(0)$.

f) Suppose $U \subset \mathbb{R}^n$ is an open and connected set, and that $f : U \to \mathbb{R}$ is analytic on U. Show that if there exists a point $x \in U$ such that $D^\beta f(x) = 0$ for all β, then $f = 0$ on U.

g) (†) *If you are familiar with the Fourier transform.*
Suppose $f \in L^2(\mathbb{R})$ and let $\hat{f} \in L^2(\mathbb{R})$ be its Fourier transform. Show that if $e^{b|k|}\hat{f}(k) \in L^2(\mathbb{R})$ for some $b > 0$ then f is real analytic at each point in $\mathbb{R}$.

[Hint: Write $f^{(l)}(x)$ using the Fourier inversion formula and estimate this with the Cauchy-Schwartz inequality]

Exercise 1.6. Consider the following transport equation in two dimensions:

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0. \tag{2}$$

i) Find the characteristic surfaces (in this case they will in fact be curves).

ii) Show that along a characteristic curve u is constant, and hence solve (3) subject to $u(x, 0) = f(x)$ for $x \geq 0$, where $f : [0, \infty) \to \mathbb{R}$ is given.

iii) Show directly from your solution that if f is real analytic at $x > 0$ then u is analytic in a neighbourhood of $(x, 0)$.

iv) (†) This approach to solving first order equations is called the *method of characteristics.* Write a brief account of this method for first order quasilinear equations in 2 dimensions (you may wish to look for example at F. John, Ch 1).

Exercise 1.7 (*). i) Show that $\{t = 0\}$ is a characteristic surface of the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

ii) Suppose that u is a smooth solution to the heat equation in a neighbourhood of $\{t = 0\}$ and that $u(0, x) = u_0(x)$. Show that all derivatives of u at $(t, x) = (0, 0)$ can be expressed in terms of $u_0(x)$.

iii) Take $u_0(x) = \frac{1}{1+x^2}$. Show that the formal Taylor series for u about $(t, x) = (0, 0)$ obtained in part *ii)* does not converge on any neighbourhood of the origin.
[This example is due to Kovalevskaya.]

Exercise 1.8. Consider Laplace's equation in two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

Construct a sequence of real analytic functions $(u_k)_{k=1}^{\infty}$ with $u_k : \mathbb{R}^2 \to \mathbb{R}$ solving (4) such that for any $l \in \mathbb{N}$:

$$\sup_x \left|(\partial_x)^l u_k(x,0)\right| + \sup_x \left|(\partial_x)^l \partial_y u_k(x,0)\right| \to 0$$

as $k \to \infty$, but such that for any $\epsilon > 0$

$$\sup_x |u_k(x,\epsilon)| \to \infty.$$

What does this mean for the well-posedness of the Cauchy problem for Laplace's equation?

[Hint: First solve the Cauchy problem for the initial surface $\{y = 0\}$ with data $u(x,0) = \cos kx$, $u_y(x,0) = 0$ by seeking a solution of the form $u(x,y) = X(x)Y(y)$.]

Exercise 1.9. Consider the general second order linear PDE in two dimensions:

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + f = 0$$

where a, b, c, d, e, f are functions of x, y. Give a criterion for this equation to be elliptic at (x, y).

Exercise 1.10. Consider the wave equation in $1+1$ dimensions:

$$-\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{4}$$

i) Find all the characteristic surfaces.

ii) Show that a general solution has the form:

$$u(t,x) = f(t-x) + g(t+x)$$

iii) Find explicitly the solution to the Cauchy problem with data given on $\{t = 0\}$:

$$u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x).$$

iv) Show that[1]

$$\sup_x |\partial_x u(x,t)| + \sup_x |\partial_t u(x,t)| \lesssim \sup_x \left|u_0'(x)\right| + \sup_x |u_1(x)|.$$

Exercise 1.11. Consider the wave equation:

$$-\frac{\partial^2 u}{\partial t^2} + \Delta u = 0, \tag{5}$$

for $u : \mathbb{R}^{1+3} \to \mathbb{R}$. Suppose that $\Sigma = \{\phi(x,y,z) = t\}$ is a hypersurface.

[1] We write $A \lesssim B$ to mean that there exists a universal constant C such that $A \leq CB$.

i) Show that Σ is everywhere characteristic if and only if ϕ obeys the eikonal equation:

$$|\nabla\phi|^2 = 1.$$

ii) Find all planes in $\mathbb{R}^{1+3}$ which are everywhere characteristic.

iii) Suppose $u_0, u_1 : \mathbb{R}^3 \to \mathbb{R}$ are everywhere real analytic. By explicitly casting the problem as a first order system, show that in a neighbourhood of $\{t = 0\}$ there exists a unique real analytic solution to (6) satisfying $u|_t = u_0,\ \partial_t u|_t = u_1$.