We consider a PDE problem that generalises in a fairly straightforward fashion the Cauchy problem for ODEs. As coordinates on $\mathbb{R}^n$ we take $x = (x', t)$, where $x' = (x^1, \ldots, x^{n-1}) \in \mathbb{R}^{n-1}$ and $t = x^n$. We set $B_r^n = \{t^2 + |x'|^2 < r^2\}$ and $B_r^{n-1} = \{|x'| < r, t = 0\}$.

We consider a system of equations for an unknown $\underline{u}(x) \in \mathbb{R}^m$. More concretely we seek a solution to:

$$\underline{u}_t = \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x') \cdot \underline{u}_{x_j} + \underline{c}(\underline{u}, x'), \qquad \text{on } B_r^n, \tag{1}$$

$$\underline{u} = 0, \qquad \text{on } B_r^{n-1}. \tag{2}$$

Problems of this type are known as Cauchy problems. Here we assume that we are given the real analytic functions:

$$\underline{\underline{B}}_j : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathrm{Mat}(m \times m), \qquad j = 1, \ldots, n-1$$
$$\underline{c} : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}^m.$$

Note that we assume that $\underline{\underline{B}}_j, \underline{c}$ don't depend on t. We can always arrange this by introducing a new unknown u^{m+1} satisfying $u_t^{m+1} = 1$ in B_r^n and $u^{m+1} = 0$ on B_r^{n-1} and extending the system of equations accordingly.

We will write $\underline{\underline{B}}_j = ((b_j^{kl}))$ and $\underline{c} = (c^k)$ so that in components (1) reads:

$$u_t^k = \sum_{j=1}^{n-1} \sum_{l=1}^{m} b_j^{kl}(\underline{u}, x') u_{x_j}^l + c^k(\underline{u}, x'), \qquad k = 1, \ldots, m. \tag{3}$$

Example 1. Take $n = m = 2$ and write $\underline{u} = (f, g)$.

a) We can consider the system

$$\left.\begin{aligned} f_t &= g_x \\ g_t &= f_x + F \end{aligned}\right\} \quad \text{in } B_r^n$$
$$f = g = 0 \qquad \text{on } B_r^{n-1}$$

which is of the form (1) for

$$\underline{\underline{B}}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \underline{c} = \begin{pmatrix} 0 \\ F \end{pmatrix}.$$

Eliminating g we find that this system implies $f_{tt} - f_{xx} = F_x$ with $f = f_t = 0$. This is the inhomogeneous wave equation in 1+1 dimensions.

b) Alternatively, we can consider the system

$$\left.\begin{aligned} f_t &= g_x \\ g_t &= -f_x + F \end{aligned}\right\} \quad \text{in } B_r^n$$
$$f = g = 0 \qquad \text{on } B_r^{n-1}$$

which is of the form (1) for

$$\underline{\underline{B}}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \underline{c} = \begin{pmatrix} 0 \\ F \end{pmatrix}.$$

Eliminating g we find that this system implies $f_{tt} + f_{xx} = F_x$ with $f = f_t = 0$. This is the inhomogeneous Laplace (or Poisson) equation in 2 dimensions.

The main result we shall show is that a solution to problem (1) subject to (2) admits a unique real analytic solution, at least for sufficiently small $r > 0$.

Theorem 1 (Cauchy–Kovalevskaya)**.** *Assume* $\{\underline{\underline{B}}_j\}_{j=1}^{n-1}$ *and* $\underline{c}$ *are analytic functions. Then for sufficiently small* $r > 0$ *there exists a unique real analytic function*

$$\underline{u} = \sum_\alpha \underline{u}_\alpha x^\alpha \tag{4}$$

solving the problem (1) subject to (2).

Proof (*). 1. Our strategy will be to compute the coefficients

$$\underline{u}_\alpha = \frac{D^\alpha \underline{u}(0)}{\alpha!}$$

in terms of $\underline{\underline{B}}_j, \underline{c}$ and show that the power series (4) we obtain converges on B_r^n for r sufficiently small.

2. As $\underline{\underline{B}}_j, \underline{c}$, we can write

$$\underline{\underline{B}}_j(z, x') = \sum_{\gamma,\delta} \underline{\underline{B}}_{j,\gamma,\delta} z^\gamma x^\delta, \qquad j = 1, \dots, n-1,$$

and

$$\underline{c}(z, x') = \sum_{\gamma,\delta} \underline{c}_{\gamma,\delta} z^\gamma x^\delta,$$

where these power series converge if $|z| + |x'| < s$ for some small $s > 0$. Here $\gamma \in \mathbb{N}^m$, $\delta \in \mathbb{N}^n$ are multi-indices. We have that:

$$\underline{\underline{B}}_{j,\gamma,\delta} = \frac{D_z^\gamma D_x^\delta \underline{\underline{B}}_j(0,0)}{\gamma!\delta!}, \qquad \underline{c}_{\gamma,\delta} = \frac{D_z^\gamma D_x^\delta \underline{c}(0,0)}{\gamma!\delta!}, \tag{5}$$

for $j = 1, \dots, n-1$ and all multi-indices γ, δ. In components $\underline{\underline{B}}_{j,\gamma,\delta} = ((b^{kl}_{j,\gamma,\delta}))$ and $\underline{c}_{\gamma,\delta} = (c^k_{\gamma,\delta})$.

3. Since $\underline{u}(x',0) = 0$ by assumption (2), by differentiating in the x' directions we deduce that
$$\underline{u}_\alpha = \frac{D^\alpha \underline{u}(x',0)}{\alpha!} = 0, \text{ for all multi-indices } \alpha \text{ with } \alpha_n = 0. \tag{6}$$
Setting $t = 0$ in (1), and using that $\underline{u}(x',0) = u_{x_i}(x',0) = 0$, we deduce that:
$$\begin{aligned} \underline{u}_t(x',0) &= \sum_{j=1}^{n-1} \underline{\underline{B}}_j(0,x') \cdot \underline{u}_{x_j}(x',0) + \underline{c}(0,x') \\ &= \underline{c}(0,x') \end{aligned}$$
Now, differentiating this equation in the x' directions we deduce that if α is a multi-index of the form $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 1) = (\alpha', 1)$ then:
$$D^\alpha u^k(0) = D_x^{\alpha'} c^k(0,0).$$
We have thus computed $\underline{u}_\alpha$ for all α with $\alpha_n \leq 1$. We will proceed inductively to compute all $\underline{u}_\alpha$ by making use of the equation.

4. Suppose $\alpha = (\alpha', \alpha_n)$ for $\alpha_n \geq 1$ and let $\tilde{\alpha} = (\alpha', \alpha_n - 1)$. We can act on (3) with $D^{\tilde{\alpha}}$ to deduce
$$D^\alpha u^k = D^{\tilde{\alpha}} \left(\sum_{j=1}^{n-1} \sum_{l=1}^{m} b_j^{kl}(\underline{u}, x') u^l_{x_j} + c^k(\underline{u}, x') \right)$$
Now, by the chain rule and Leibniz rule we can expand the right hand side to produce a polynomial expression in $D_z^\gamma D_x^\delta b_j^{kl}$, $D_z^\gamma D_x^\delta c^k$ and $D^\beta u^l$ where $\beta_n \leq \alpha_n - 1$. Moreover the coefficients of this polynomial are non-negative (indeed even integers) as they arise as sums of multinomial coefficients. We deduce that:
$$u_\alpha^k = q_\alpha^k(b^{kl}_{j,\gamma,\delta}, c^k_{\gamma,\delta}, u^l_\beta)$$
where q_α^k is a polynomial with non-negative coefficients and $\beta_n \leq \alpha_n - 1$ for each multi-index on the right-hand side. Since we know u^l_β for $\beta_n = 0, 1$ we can construct $\underline{u}_\alpha$ inductively for all α.

5. So far, we have shown that *if* a smooth solution $\underline{u}$ of (1) subject to (2) exists, then we can find all of the derivatives of $\underline{u}$ at $x = 0$ in terms of known quantities. This tells us immediately that there is at most one analytic solution, since analytic function is determined by its Taylor expansion about a point. Now we make use of the method of majorants to show that the formal power series for $\underline{u}$ about the origin indeed converges.

   We suppose that
$$\underline{\underline{B}}^*_j \gg \underline{\underline{B}}_j, \quad \text{for } j = 1, \ldots, n-1, \qquad \underline{c}^* \gg \underline{c}$$
where
$$\underline{\underline{B}}^*_j(z,x) = \sum_{\gamma,\delta} \underline{\underline{B}}^*_{j,\gamma,\delta} z^\gamma x^\delta, \quad \text{for } j = 1, \ldots, n-1, \qquad \underline{c}^*(z,x) = \sum_{\gamma,\delta} \underline{c}^*_{\gamma,\delta} z^\gamma x^\delta,$$

with these power series converging for $|x| + |x'| < s$ for some (possibly smaller) s. In components we write $\underline{\underline{B}}^*_{j,\gamma,\delta} = ((b^{*kl}_{j,\gamma,\delta}))$ and $\underline{c}^*_{\gamma,\delta} = (c^{*k}_{\gamma,\delta})$. Then for all j, γ, δ we have:

$$0 \leq \left| b^{kl}_{j,\gamma,\delta} \right| \leq b^{*kl}_{j,\gamma,\delta}, \qquad 0 \leq \left| c^{k}_{\gamma,\delta} \right| \leq c^{*k}_{\gamma,\delta}.$$

We consider the modified Cauchy problem:

$$\underline{u}^*_t = \sum_{j=1}^{n-1} \underline{\underline{B}}^*_j(\underline{u}^*, x') \cdot \underline{u}^*_{x_j} + \underline{c}^*(\underline{u}^*, x'), \quad \text{ on } B^n_r, \tag{7}$$

$$\underline{u}^* = 0, \quad \text{ on } B^{n-1}_r. \tag{8}$$

and as above seek a real analytic solution

$$\underline{u}^*(x) = \sum_\alpha \underline{u}^*_\alpha x^\alpha, \quad \text{where} \quad \underline{u}^*_\alpha = \frac{D^\alpha \underline{u}^*(0)}{\alpha!}.$$

6. We claim that for each multi-index α:

$$0 \leq \left| u^k_\alpha \right| \leq u^{*k}_\alpha.$$

The proof is by induction on α_n. For $\alpha_n = 0$ we have $u^k_\alpha = u^{*k}_\alpha$. Suppose the inequality holds for $\alpha_n < N$. Let α be a multi-index with $\alpha_n = N$. We estimate:

$$\begin{aligned} \left| u^k_\alpha \right| &= \left| q^k_\alpha(b^{kl}_{j,\gamma,\delta}, c^k_{\gamma,\delta}, u^l_\beta) \right| \\ &\leq q^k_\alpha(\left| b^{kl}_{j,\gamma,\delta} \right|, \left| c^k_{\gamma,\delta} \right|, \left| u^l_\beta \right|) \\ &\leq q^k_\alpha(b^{*kl}_{j,\gamma,\delta}, c^{*k}_{\gamma,\delta}, u^{*l}_\beta) \\ &= u^{*k}_\alpha \end{aligned}$$

This uses crucially that q^k_α is a polynomial with positive coefficients, and moreover that $\beta_n < N$ for all terms on the right-hand side. We deduce that $\underline{u}^* \gg \underline{u}$, so to establish that the series for $\underline{u}$ converges, it suffices to find $\underline{\underline{B}}^*_j$, $\underline{c}^*$ such that the corresponding series for $\underline{u}^*$ converges near $x = 0$.

7. We make a particular choice for $\underline{\underline{B}}^*_j$, $\underline{c}^*$ which enables us to solve the equation explicitly. We recall from lectures that

$$\begin{aligned} \underline{\underline{B}}^*_j &= \frac{C\rho}{\rho - (x_1 + \ldots + x_{n-1}) - (z_1 + \ldots + z_m)} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \\ \underline{c}^* &= \frac{C\rho}{\rho - (x_1 + \ldots + x_{n-1}) - (z_1 + \ldots + z_m)}(1, \ldots, 1) \end{aligned}$$

will majorise $\underline{\underline{B}}_j$, $\underline{c}$ respectively provided that we fix C to be large enough and ρ small enough, and their power series about 0 will converge for $|x'| + |z| < s$, for some sufficiently small s.

With these choices of majorants the modified equation (7) reads

$$\underline{u}_t^* = \frac{C\rho}{\rho - (x_1 + \ldots + x_{n-1}) - (u^{*1} + \ldots + u^{*m})} \left(\sum_{l=1}^{m} \sum_{j=1}^{n-1} u_{x_j}^{*l} + 1 \right), \qquad \text{on } B_r^n, \quad (9)$$

$$\underline{u}^* = 0, \qquad \text{on } B_r^{n-1}. \quad (10)$$

This problem has an explicit solution, namely

$$\underline{u}^* = v^*(1, \ldots, 1)$$

where

$$v^*(x) = \frac{1}{nm} \left(\rho - (x_1 + \ldots + x_{n-1}) - \sqrt{(\rho - (x_1 + \ldots + x_{n-1}))^2 - 2mnCrx_n} \right)$$

V^* is analytic for $|x| < r$ provided $r > 0$ is sufficiently small, thus $\underline{u}^*$ is given by a convergent series and as $\underline{u}^* \gg \underline{u}$, the power series (4) converges for $|x| < r$.

8. Since the Taylor expansion of the analytic function

$$F(x) := \underline{u}_t - \sum_{j=1}^{n-1} \underline{\underline{B}}_j(\underline{u}, x') \cdot \underline{u}_{x_j} + \underline{c}(\underline{u}, x')$$

vanishes to all orders at $x = 0$, we deduce that $F = 0$ in B_r^n and so $\underline{u}$ given by (4) is indeed a solution.

$\square$