Improved Regularity for the hyperbolic IBVP

We have established the existence of a weak solution to the IBVP:

$$\begin{aligned} u_{tt} + Lu &= f && \text{in } U_T \\ u &= 0 && \text{on } \partial^* U_T \\ u = \psi,\ u_t &= \psi' && \text{on } \Sigma_0 \end{aligned} \qquad (1)$$

Provided $\psi \in H^1_0(U)$, $\psi' \in L^2(U)$, $f \in L^2(U_T)$ a unique solution $u \in H^1(U_T)$ exists solving the equation in a weak sense. As in the elliptic BVP we'd like to improve the regularity of the solution so that we have a strong solution in the sense that (1) holds pointwise almost everywhere. To motivate our arguments, let's suppose that $u \in C^\infty(U_T)$ solves:

$$\begin{aligned} u_{tt} - \Delta u &= f && \text{in } U_T \\ u &= 0 && \text{on } \partial^* U_T \\ u = \psi,\ u_t &= \psi' && \text{on } \Sigma_0 \end{aligned}$$

Since $u \in C^\infty(U_T)$, we can differentiate the eqn w.r.t. t to find

$$u_{ttt} - \Delta u_t = f_t$$

Noting also that

$$u_{tt}\big|_{t=0} = (\Delta u + f)\big|_{t=0} = \Delta \psi + f\big|_{t=0}$$

we deduce that $w = u_t$ solves

$$\begin{aligned} w_{tt} - \Delta w &= f_t && \text{in } U_T \\ w &= 0 && \text{on } \partial^* U_T \\ w = \psi',\ w_t &= \Delta \psi + f\big|_{t=0} && \text{on } \Sigma_0 \end{aligned}$$

Multiplying the eqn. by $e^{-\lambda t}w_t$ and integrating we find:

$$\int_0^\tau dt \int_U dx \, \left(w_{tt} w_t e^{-\lambda t} - \Delta w \, w_t e^{-\lambda t} \right) = \int_0^\tau dt \int_U f_t w_t e^{-\lambda t}$$

$$\Rightarrow \int_0^\tau dt \int_U dx \left\{ \frac{d}{dt}\left[\left(\tfrac{1}{2} w_t^2 + \tfrac{1}{2} |Dw|^2 \right) e^{-\lambda t} \right] + \frac{\lambda}{2}\left(w_t^2 + |Dw|^2 \right) \right\}$$

$$\leq \frac{1}{2} \int_0^\tau dt \int_U dx \left(f_t^2 + w_t^2 \right) e^{-\lambda t}$$

$$\Rightarrow \sup_{\tau \in [0,T]} \int_{\Sigma_\tau} dx \left(\tfrac{1}{2} w_t^2 + \tfrac{1}{2} |Dw|^2 \right) + \int_0^T dt \int_U dx \left(\tfrac{1}{2} w_t^2 + |Dw|^2 \right)$$

$$\leq C_T \left[\int_0^T dt \int_U dx \, f_t^2 + \int_{\Sigma_0} dx \left(\tfrac{1}{2} w_t^2 + |Dw|^2 \right) \right]$$

$$\Rightarrow \quad \|w\|_{L^\infty((0,T);H^1(U))} + \|w_t\|_{L^\infty((0,T);L^2(U))} + \|w\|_{H^1(U_T)}$$

$$\leq C \left(\|f\|_{H^1(U_T)} + \|\psi\|_{H^2(U_T)} + \|\psi'\|_{H^1(U_T)} \right)$$

Where we introduce the spaces $L^\infty((0,T);H^k(U))$ with norm:

$$\|g\|_{L^\infty((0,T);H^k(U))} = \operatorname*{ess\,sup}_{t \in [0,T]} \|g(t,\cdot)\|_{H^k(U)}$$

This estimate gives us control of u_{tt}, u_{tx_i} in $L^\infty((0,T);L^2(U))$. To recover the other second derivatives of u, the $u_{x_i x_j}$ derivatives, we note that for each time t:

$$-\Delta u = f - u_{tt} \in L^2(U), \quad u = 0 \text{ on } \partial U$$

So an elliptic estimate implies:

$$\|u(t,\cdot)\|_{H^2(U)} \leq C\left(\|f(t,\cdot)\|_{L^2(U)} + \|u_{tt}(t,\cdot)\|_{L^2(U)} \right)$$

but we already control the RHS, so that

$$\|u\|_{L^\infty((0,T);H^2(U))} \leq C\left(\|f\|_{H^1(U_T)} + \|\psi\|_{H^2(U_T)} + \|\psi'\|_{H^1(U_T)}\right).$$

Thus, by differentiating the wave equation for u and performing an energy estimate we gain $L^\infty_t L^2_x$ - control of u_{tt}, u_{tx_i}. Using the wave equation as an elliptic eqn. for u in terms of f, u_{tt} then gives $L^\infty_t L^2_x$ control of $u_{x_i x_j}$.

To establish our improved regularity result for (1), we essentially follow the argument above, however since we do not know our weak solution admits second derivatives, we instead have to work with the approximating sequence for u we constructed by Galerkin's method, and then pass to a limit.

<u>Theorem</u> (Improved regularity for the hyperbolic IBVP)

Suppose $a^{ij}, b^i, b, c \in C^2(\overline{U}_T)$ and ∂U is C^2. Then for $\psi \in H^2(U) \cap H^1_0(U)$, $\psi' \in H^1_0(U)$, $f, f_t \in L^2(U)$ the unique weak solution to (1) in fact satisfies:

$$u \in H^2(U_T) \cap L^\infty((0,T);H^2(U))$$

$$u_t \in L^\infty((0,T);H^1_0(U))$$

$$u_{tt} \in L^\infty((0,T);L^2(U))$$

With corresponding estimates in terms of $\|\psi\|_{H^2(U)}, \|\psi'\|_{H^1(U)}, \|f\|_{L^2(U_T)}$ and $\|f_t\|_{L^2(U_T)}$.

<u>Proof</u> We may assume $f \in C^\infty(\bar{\mathcal{U}}_T)$, $\psi, \psi' \in C^\infty_c(\mathcal{U})$ by approximation. We return to our Galerkin approximations. Recall

$$u^N(x,t) = \sum_{k=1}^{N} u^N_k(t)\,\varphi_k(x)$$

where we take φ_k to be the eigenfunctions of the Dirichlet Laplacian and $u^N_k(t)$ solve the ODE:

$$\ddot{u}^N_k(t) + \int_{\Sigma_t} \sum_{i,j} a^{ij} u^N_{x_i} (\varphi_k)_{x_j} + \sum_i b^i u^N_{x_i} \varphi_k + b\dot{u}^N \varphi_k + c u^N \varphi_k \, dx \tag{2}$$

$$= (f, \varphi_k)_{L^2(\mathcal{U})} \qquad k = 1, \dots, N$$

with initial conditions

$$u_k(0) = (\psi, \varphi_k)_{L^2(\mathcal{U})} \qquad \dot{u}_k(0) = (\psi', \varphi_k)_{L^2(\mathcal{U})}.$$

Under our hypothesis, (2) is a linear system of equations with coefficients in $C^2(\overline{(0,T)})$ and RHS also in $C^1(\overline{(0,T)})$. We can thus differentiate the system wrt t to find:

$$\dddot{u}^N_k(t) + \int_{\Sigma_t} \sum_{i,j} a^{ij} \dot{u}^N_{x_i} (\varphi_k)_{x_j} + \sum_i b^i \dot{u}^N_{x_i} \varphi_k + b\ddot{u}^N \varphi_k + c\dot{u}^N \varphi_k \, dx \tag{3}$$

$$= (\dot{f}, \varphi_k)_{L^2(\mathcal{U})} - \int_{\Sigma_t} \sum_{i,j} \dot{a}^{ij} u^N_{x_i} (\varphi_k)_{x_j} + \dot{b}\dot{u}^N \varphi_k + \sum_i \dot{b}^i u^N_{x_i} \varphi_k + \dot{c} u^N \varphi_k \, dx$$

We multiply this expression by $\ddot{u}^N_k e^{-\lambda t}$, sum over $k = 1, \dots, N$ and integrate $\int_0^\tau dt$. We then deal with the LHS as for the proof of existence of weak solutions.

We find:

$$\int_0^\tau dt \int_U dx \Big\{ \dddot{u}^N \ddot{u}^N e^{-\lambda t} + \sum_{i,j} a^{ij} \dot{u}^N_{x_i} \ddot{u}^N_{x_j} e^{-\lambda t} + \sum_i b^i \dot{u}^N_{x_i} \ddot{u}^N e^{-\lambda t}$$
$$+ b(\ddot{u}^N)^2 e^{-\lambda t} + c\dot{u}^N \ddot{u}^N e^{-\lambda t} + \sum_{i,j} \dot{a}^{ij} u^N_{x_i} \ddot{u}^N_{x_j} e^{-\lambda t} \Big\}$$

$$= \int_0^\tau \int_U dx \Big\{ \dot{f} \ddot{u}^N e^{-\lambda t} - \dot{b} \dot{u}^N \ddot{u}^N e^{-\lambda t} - \sum_i \dot{b}^i u^N_{x_i} \ddot{u}^N e^{-\lambda t}$$
$$- \dot{c} u^N \ddot{u}^N e^{-\lambda t} \Big\}$$

We rearrange this to write:

$$\int_0^\tau dt \int_U dx \Bigg(\frac{d}{dt} \Big[e^{-\lambda t} \Big\{ \tfrac{1}{2}(\ddot{u}^N)^2 + \tfrac{1}{2} \sum_{i,j} a^{ij} \dot{u}^N_{x_i} \dot{u}^N_{x_j} + \sum_{i,j} \dot{a}^{ij} u^N_{x_i} \dot{u}^N_{x_j} \Big\} \Big]$$
$$+ \lambda e^{-\lambda t} \Big\{ \tfrac{1}{2}(\ddot{u}^N)^2 + \tfrac{1}{2} \sum_{i,j} a^{ij} \dot{u}^N_{x_i} \dot{u}^N_{x_j} + \sum_{i,j} \dot{a}^{ij} u^N_{x_i} \dot{u}^N_{x_j} \Big\} \Bigg)$$

$$= \int_0^\tau dt \int_U dx \Big(\dot{f} \ddot{u}^N e^{-\lambda t} + \sum_{i,j} \big(\ddot{a}^{ij} u^N_{x_i} \dot{u}^N_{x_j} + \tfrac{3}{2} \dot{a}^{ij} \dot{u}^N_{x_i} \dot{u}^N_{x_j} \big) e^{-\lambda t}$$
$$- \sum_i \big(b^i \dot{u}^N_{x_i} \ddot{u}^N + \dot{b}^i u^N_{x_i} \ddot{u}^N \big) e^{-\lambda t} - b(\ddot{u}^N)^2 e^{-\lambda t}$$
$$- \dot{b} \dot{u}^N \ddot{u}^N e^{-\lambda t} - c \dot{u}^N \ddot{u}^N e^{-\lambda t} - \dot{c} u^N \ddot{u}^N e^{-\lambda t} \Big)$$

Which we write as $A = B$. We deal first with the expression A, where we can perform the t integration in the first term to find:

$$A = e^{-\lambda\tau}\int_{\Sigma_\tau} dx\left\{\frac{1}{2}(\ddot{u}^N)^2 + \frac{1}{2}\sum_{i,j} a^{ij}\dot{u}^N_{x_i}\dot{u}^N_{x_j} + \sum_{i,j}\dot{a}^{ij}u^N_{x_i}\dot{u}^N_{x_j}\right\}$$

$$- \int_{\Sigma_0} dx\left\{\frac{1}{2}(\ddot{u}^N)^2 + \frac{1}{2}\sum_{i,j} a^{ij}\dot{u}^N_{x_i}\dot{u}^N_{x_j} + \sum_{i,j}\dot{a}^{ij}u^N_{x_i}\dot{u}^N_{x_j}\right\}$$

$$+ \int_0^\tau dt\int_U \lambda e^{-\lambda t}\left\{\frac{1}{2}(\ddot{u}^N)^2 + \frac{1}{2}\sum_{i,j} a^{ij}\dot{u}^N_{x_i}\dot{u}^N_{x_j} + \sum_{i,j}\dot{a}^{ij}u^N_{x_i}\dot{u}^N_{x_j}\right\}$$

Now, recall that by the uniform ellipticity

$$\sum_{i,j} a^{ij}\dot{u}^N_{x_i}\dot{u}^N_{x_j} \geq \theta\,|D\dot{u}^N|^2$$

Moreover, by Young's inequality we can estimate:

$$\left|\sum_{i,j}\dot{a}^{ij}u^N_{x_i}\dot{u}^N_{x_j}\right| \leq \frac{\theta}{4}|D\dot{u}^N|^2 + C_{\theta,a}|Du^N|^2$$

We deduce

$$A \geq e^{-\lambda\tau}\left(\frac{1}{2}\|\ddot{u}^N(\tau)\|^2_{L^2(U)} + \frac{\theta}{4}\|D\dot{u}^N(\tau)\|^2_{L^2(U)}\right)$$

$$+ \lambda e^{-\lambda\tau}\left(\frac{1}{2}\|\ddot{u}^N\|^2_{L^2(U_\tau)} + \frac{\theta}{4}\|D\dot{u}^N\|^2_{L^2(U_\tau)}\right)$$

$$- C_1\left(\|Du^N(\tau)\|^2_{L^2(U)} + \lambda\|Du^N\|^2_{L^2(U_\tau)} + \|\ddot{u}^N(0)\|^2_{L^2(U)} + \|D\dot{u}(0)\|^2_{L^2(U)}\right)$$

where C_1 is some constant depending on θ, a^{ij} but not λ.

Next we consider B. By applying Young's inequality and using $e^{-\lambda t} \geq 1$ we find:

$$B \leq C_2 \Big(\|f\|^2_{L^2(U_\tau)} + \|\ddot{u}^N\|^2_{L^2(U_\tau)} + \|D\dot{u}^N\|^2_{L^2(U_\tau)} + \|Du^N\|^2_{L^2(U_\tau)} + \|\dot{u}^N\|^2_{L^2(U_\tau)} + \|u^N\|^2_{L^2(U_\tau)} \Big)$$

where C_2 is some constant depending on a^{ij}, b^i, b, c but not λ. Taking λ sufficiently large we can absorb the $\|\ddot{u}^N\|^2_{L^2(U_\tau)}$ and $\|D\dot{u}^N\|^2_{L^2(U_\tau)}$ terms on the LHS when we set $A = B$. We recall from the proof of existence of weak solns that

$$\|Du^N\|^2_{L^2(U_T)} + \|\dot{u}^N\|^2_{L^2(U_T)} + \|u^N\|^2_{L^2(U_T)} \leq C \big(\|f\|^2_{L^2(U_T)} + \|\psi\|^2_{H^1(U)} + \|\psi'\|^2_{L^2(U)}$$

We also note that

$$\|D\dot{u}^N(0)\|^2_{L^2(U)} \leq C \|D\psi'\|^2_{L^2(U)}$$

by Bessel's inequality, and using the eqn. (2) we can estimate

$$\|\ddot{u}^N(0)\|_{L^2(U)} \leq C \big(\|\psi\|^2_{H^2(U)} + \|\psi'\|^2_{H^1(U)} + \|f(0)\|_{L^2(U)} \big)$$

Putting this all together, we find

$$\sup_{\tau \in [0,T]} \Big\{ \|\ddot{u}^N(\tau)\|^2_{L^2(U)} + \|D\dot{u}^N(\tau)\|^2_{L^2(U)} \Big\} \leq C \Big(\|\psi\|^2_{H^2(U)} + \|\psi'\|^2_{H^1(U)} + \|f\|^2_{L^2(U_T)} + \|\dot{f}\|^2_{L^2(U_T)} \Big)$$

Passing, through a subsequence, to the unique weak solution, we conclude

$$\ddot{u} \in L^\infty((0,T); L^2(U)), \quad \dot{u} \in L^\infty((0,T); H^1_0(U)).$$

We can finally recover spatial regularity by noting that for almost every time τ, $u(\tau)$ is a weak solution of:

$$-\sum_{i,j}(a^{ij}u_{x_i})_{x_j} = -f - \ddot{u} - \sum_i b^i u_{x_i} - b\dot{u} - cu \qquad \text{in } U$$
$$= \tilde{f}$$

with $u=0$ on ∂U. We've already established $\tilde{f} \in L^2(U)$, so $u(\tau) \in H^2(U)$ with

$$\|u(\tau)\|^2_{H^2(U)} \leq C\|\tilde{f}\|^2_{L^2(U)}$$

but
$$\|\tilde{f}\|^2_{L^2(U)} \leq C\left(\|\varphi\|^2_{H^2(U)} + \|\varphi'\|^2_{H^1(U)} + \|f\|^2_{L^2(U_T)} + \|\dot{f}\|^2_{L^2(U_T)}\right)$$

Hence $u \in L^\infty((0,T); H^2(U))$. $\square$

We may iterate our argument to establish

<u>Thm</u> If $a^{ij}, b^i, b, c \in C^{k+1}(\overline{U}_T)$, ∂U is C^{k+1} and

$$\partial_t^i u\big|_{\Sigma_0} \in H_0^1(U) \qquad i=0,\dots,k$$
$$\partial_t^{k+1} u\big|_{\Sigma_0} \in L^2(U)$$
$$\partial_t^i f \in L^2((0,T); H^{k-i}) \qquad i=0,\dots,k$$

Then $u \in H^{k+1}(U_T)$ and $\quad \partial_t^i u \in L^\infty((0,T); H^{k+1-i}(U)) \quad i=0,\dots,k+1.$

Note that since $u_{tt}|_{\Sigma_0} = (f - Lu)|_{\Sigma_0}$ etc., the conditions at Σ_0 can be reduced to the requirement that $\varphi \in H^{k+1}(U)$, $\varphi' \in H^k(U)$ and certain compatibility conditions hold at $\partial\Sigma_0$.