Chapter 4

# The Fourier Transform and Sobolev Spaces

## 4.1 The Fourier transform on $L^1(\mathbb{R}^n)$

The Fourier transform is an extremely powerful tool across the full range of mathematics. Loosely speaking, the idea is to consider a function on $\mathbb{R}^n$ as a superposition of plane waves with different frequencies. For $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by:

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

Sine $\left| f(x) e^{-ix\cdot\xi} \right| \leqslant |f(x)|$, the integral is absolutely convergent, and $\hat{f}(\xi)$ makes sense for each $\xi \in \mathbb{R}^n$.

**Example 16.** *i) Suppose $f \in L^1(\mathbb{R})$ is the "top hat" function, defined by:*

$$f(x) = \begin{cases} 1 & -1 < x < 1, \\ 0 & |x| \geqslant 1. \end{cases}$$

*We calculate:*

$$\hat{f}(\xi) = \int_{-1}^{1} e^{-ix\xi} dx = \left[ \frac{e^{-ix\xi}}{-i\xi} \right]_{-1}^{1} = 2\frac{\sin \xi}{\xi}$$

*Notice that $\hat{f}(\xi)$ is continuous (in fact smooth) on $\mathbb{R}$. We also have $\hat{f}(\xi) \to 0$ as $\xi \to \infty$.*

*ii) Suppose $f \in L^1(\mathbb{R})$ is defined by:*

$$f(x) = \begin{cases} e^{x} & x < 0, \\ e^{-x} & x \geqslant 0. \end{cases}$$

*Then:*

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{0} e^{x(1-i\xi)}dx + \int_{0}^{\infty} e^{x(-1-i\xi)}dx \\ &= \left[\frac{e^{x(1-i\xi)}}{1-i\xi}\right]_{-\infty}^{0} + \left[\frac{e^{x(-1-i\xi)}}{-1-i\xi}\right]_{0}^{\infty} \\ &= \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2}\end{aligned}$$

*Again, notice that $\hat{f}$ is smooth and decays for large $\xi$.*

*iii) Consider $g \in L^1(\mathbb{R})$ given by*

$$g(x) = \frac{1}{1+x^2}.$$

*We have:*

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{1+x^2}dx$$

*We can consider this as a limit of contour integrals:*

$$\hat{g}(\xi) = \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-iz\xi}}{1+z^2}dx.$$

*Where $\gamma_R = \{\Im(z) = 0, \ |\Re(z)| < R\}$. For $\xi \geqslant 0$, we can close the contour with a semi-circle in the lower half-plane, and we pick up a contribution from the pole at $z = -i$. The contribution from the curved part of the contour tends to zero as $R \to \infty$ by Jordan's lemma, and we find:*

$$\hat{g}(\xi) = \pi e^{-\xi}, \quad \xi \geqslant 0.$$

*For $\xi < 0$, we close the contour in the upper half-plane, picking up a contribution from the pole at $z = i$ and again discard the contribution from the curved part of the contour in the limit. We find:*

$$\hat{g}(\xi) = \pi e^{\xi}, \quad \xi < 0.$$

*In conclusion, we have:*

$$\hat{g}(\xi) = \begin{cases} \pi e^{\xi} & \xi < 0, \\ \pi e^{-\xi} & \xi \geqslant 0. \end{cases}$$

*iv) Consider now for $x \in \mathbb{R}^n$ the Gaussian $f(x) = e^{-\frac{1}{2}|x|^2}$. We calculate:*

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2 - i\xi\cdot x}dx \\ &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x-i\xi)\cdot(x-i\xi) - \frac{1}{2}|\xi|^2}dx \\ &= e^{-\frac{1}{2}|\xi|^2}\left(\int_{\mathbb{R}} e^{-\frac{1}{2}(x_1-i\xi_1)^2}dx_1\right)\cdots\left(\int_{\mathbb{R}} e^{-\frac{1}{2}(x_n-i\xi_n)^2}dx_n\right)\end{aligned}$$

*By shifting a contour in the complex plane, which is justified since $e^{-z^2}$ is entire and rapidly decaying as $z$ approaches infinity along any line parallel to the real axis, we can show that:*

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(x_1 - i\xi_1)^2} dx_1 = \int_{\mathbb{R}} e^{-\frac{1}{2}x_1^2} dx_1 = \sqrt{2\pi}.$$

*We deduce that:*

$$\hat{f}(\xi) = (2\pi)^{\frac{n}{2}} e^{-\frac{1}{2}|\xi|^2}$$

*Notice that this is (as a function) equal to $f$ up to a factor.*

We will make some casual observations at this stage. In all of our examples, we saw that the Fourier transformed function decays towards infinity. For examples iii), iv) we see very rapid (exponential) decay of the Fourier transform, while in examples i), ii) the decay is only polynomial. In all examples the transformed function is continuous. In examples i), ii), iv) it is in fact smooth, while for iii) the Fourier transform has a discontinuous first derivative. Reflecting on this, one sees that these two features appear to be dual to one another: if $f$ is smooth, then $\hat{f}$ has rapid decay towards infinity. If $f$ decays rapidly near infinity, then $\hat{f}$ is smooth. This is in fact a general feature of the Fourier transform smoothness and decay are dual to one another under the transform.

We shall now make some of these observations more precise.

**Lemma 4.1** (Riemann-Lebesgue Lemma). *Suppose $f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^0(\mathbb{R}^n)$ with the estimate:*

$$\sup_{\xi \in \mathbb{R}^n} \left| \hat{f}(\xi) \right| \leqslant \|f\|_{L^1} \tag{4.1}$$

*and moreover $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

*Proof.* To establish the continuity of $\hat{f}$, we use the continuity of the exponential, together with the dominated convergence theorem. Let $\{\xi_j\}_{j=1}^{\infty}$ be any sequence with $\xi_j \to \xi$ as $j \to \infty$. Recalling the definition of the integral, we have:

$$\hat{f}(\xi_j) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi_j} dx.$$

Now, clearly for $x \in \mathbb{R}^n$ we have:

$$f(x)e^{-ix\cdot\xi_j} \to f(x)e^{-ix\cdot\xi}, \qquad \text{as } j \to \infty$$

so we have pointwise convergence of the integrand. We can also estimate:

$$\left| f(x)e^{-ix\cdot\xi_j} \right| \leqslant |f(x)|$$

so the integrand is dominated by an integrable function, since $f \in L^1(\mathbb{R}^n)$. Applying the Dominated Convergence Theorem, we conclude:

$$\hat{f}(\xi_j) \to \hat{f}(\xi), \qquad \text{as } j \to \infty.$$

This implies that $\hat{f}(\xi)$ is continuous. We can readily estimate:

$$\sup_{\xi \in \mathbb{R}^n} \left| \hat{f}(\xi) \right| = \sup_{\xi \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx \right| \leqslant \sup_{\xi \in \mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{L^1}.$$

This establishes the first part of the Lemma. To establish the second part, we make use of an approximation argument. Chapter 1 that if $f \in L^1(\mathbb{R}^n)$, we can approximate $f$ by an element of $C_0^\infty(\mathbb{R}^n)$. Given $\epsilon > 0$, there exists $f_\epsilon \in C_0^\infty(\mathbb{R}^n)$ with

$$\|f - f_\epsilon\|_{L^1} < \frac{\epsilon}{2}.$$

Now, in the integral for $\hat{f}_\epsilon$ we can integrate by parts::

$$\begin{aligned} \hat{f}_\epsilon(\xi) &= \int_{\mathbb{R}^n} f_\epsilon(x) e^{-ix\cdot\xi} dx \\ &= \int_{\mathbb{R}^n} f_\epsilon(x) \operatorname{div}\left( \frac{\xi}{-i\,|\xi|^2} e^{-ix\cdot\xi} \right) dx \\ &= -\int_{\mathbb{R}^n} \frac{\xi}{-i\,|\xi|^2} \cdot Df_\epsilon(x) e^{-ix\cdot\xi} dx \end{aligned}$$

so that for each $i = 1, \ldots, n$ we have, by the Cauchy-Schwarz inequality:

$$\begin{aligned} \left| \hat{f}_\epsilon(\xi) \right| &= \left| \int_{\mathbb{R}^n} \frac{\xi}{i\,|\xi|^2} \cdot Df_\epsilon(x) e^{-ix\cdot\xi} dx \right| \\ &\leqslant \int_{\mathbb{R}^n} \left| \frac{\xi}{i\,|\xi|^2} \cdot Df_\epsilon(x) e^{-ix\cdot\xi} \right| dx \\ &\leqslant \int_{\mathbb{R}^n} \frac{1}{|\xi|} |Df_\epsilon(x)|\, dx \\ &= \frac{1}{|\xi|} \Big\| \, |Df_\epsilon(x)| \, \Big\|_{L^1} \end{aligned} \tag{4.2}$$

From this, we conclude that there exists $R > 0$ such that if $|\xi| > R$, we have $\left| \hat{f}_\epsilon(\xi) \right| < \frac{\epsilon}{2}$. For $|\xi| > R$ we calculate:

$$\begin{aligned} \left| \hat{f}(\xi) \right| &= \left| \hat{f}(\xi) - \hat{f}_\epsilon(\xi) + \hat{f}_\epsilon(\xi) \right| \\ &\leqslant \left| \hat{f}_\epsilon(\xi) \right| + \left| \hat{f}(\xi) - \hat{f}_\epsilon(\xi) \right| \\ &\leqslant \left| \hat{f}_\epsilon(\xi) \right| + \|f - f_\epsilon\|_{L^1} < \epsilon. \end{aligned}$$

In the last line, we have used (4.1), together with the linearity of the Fourier transform. Since $\epsilon > 0$ was arbitrary, we have shown that $\left| \hat{f}(\xi) \right| \to 0$. □

**Remark.** *The argument above is another example of an approximation argument where one first proves the result on a suitably nice dense subset then extends to the full space by* *continuity. In this case, we are using the fact that the Fourier transform is a bounded (hence continuous) linear operator from $L^1(\mathbb{R}^n)$ to the Banach space of continuous functions decaying at infinity equipped with the uniform norm. The dense set is $C_0^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.*

*Another tactic which we used here was to use integration by parts to exploit the rapid oscillations in the $e^{-ix\cdot\xi}$ factor when $|\xi|$ is large.*

*One might be tempted to infer from (4.2) that $\left|\hat{f}(\xi)\right| \leqslant C(1+|\xi|)^{-1}$. While this is true for each $f_\epsilon$ approximating $f$, in general the constant $C$ will grow larger and larger as $\epsilon \to 0$, so we cannot quite come to this conclusion.*

**Exercise(*).** For $\xi \in \mathbb{R}^n$, define $e_\xi(x) = e^{i\xi\cdot x}$. Show that $T_{e_\xi} \in \mathscr{S}'$, and that:

$$T_{e_\xi} \to 0, \qquad \text{as } |\xi| \to \infty$$

in the weak-$*$ topology of $\mathscr{S}'$.

We shall prove some important properties of the Fourier transform. Recall that $\tau_y f(x) = f(x-y)$, and introduce the character $e_y(x) = e^{iy\cdot x}$.

**Lemma 4.2** (Properties of the Fourier transform). *i) Suppose $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\lambda > 0$ and $f_\lambda(y) = \lambda^{-n} f(\lambda^{-1}y)$. Then*

$$\hat{f}_\lambda(\xi) = \hat{f}(\lambda\xi) \qquad \widehat{(e_x f)}(\xi) = \tau_x \hat{f}(\xi) \qquad \widehat{\tau_x f}(\xi) = e_{-x}(\xi)\hat{f}(\xi)$$

*ii) Suppose $f, g \in L^1(\mathbb{R}^n)$. Then $f \star g \in L^1(\mathbb{R}^n)$ and:*

$$\widehat{f \star g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

*Proof.* i) Writing out the expression for $\hat{f}_\lambda(\xi)$, and changing the integration variable to $z = \lambda^{-1}x$, we see

$$\hat{f}_\lambda(\xi) = \int_{\mathbb{R}^n} f_\lambda(x) e^{-i\xi\cdot x} dx = \int_{\mathbb{R}^n} f(\lambda^{-1}x) e^{-i\xi\cdot x} \lambda^{-n} dx = \int_{\mathbb{R}^n} f(y) e^{-i\lambda\xi\cdot z} dz = \hat{f}(\lambda\xi).$$

Next, we calculate:

$$\widehat{(e_x f)}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot y} f(y) e^{-i\xi\cdot y} dy = \int_{\mathbb{R}^n} f(y) e^{-i(\xi-x)\cdot y} dy = \tau_x \hat{f}(\xi).$$

Finally, we have:

$$\widehat{\tau_x f}(\xi) = \int_{\mathbb{R}^n} f(y-x) e^{-i\xi\cdot y} dy = \int_{\mathbb{R}^n} f(z) e^{-i\xi\cdot(z+x)} dz = e^{-i\xi\cdot x} \int_{\mathbb{R}^n} f(z) e^{-i\xi\cdot z} dz = e_{-x}(\xi)\hat{f}(\xi),$$

where we have used the substitution $z = y - x$.

ii) First we show that $f \star g \in L^1(\mathbb{R}^n)$. To see this, we first estimate:

$$|f \star g(x)| = \left| \int_{\mathbb{R}^n} f(y) g(x-y) dy \right| \leqslant \int_{\mathbb{R}^n} |f(y) g(x-y)| \, dy$$

Integrating and applying Fubini's theorem, we have:

$$\begin{aligned}\|f \star g\|_{L^1} &\leqslant \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(y)g(x-y)| \, dy \right) dx \\ &= \int_{\mathbb{R}^n} |f(y)| \left( \int_{\mathbb{R}^n} |g(x-y)| \, dx \right) dy \\ &= \int_{\mathbb{R}^n} |f(y)| \, \|g\|_{L^1} \, dy = \|f\|_{L^1} \, \|g\|_{L^1}\end{aligned}$$

Now, we can calculate the Fourier transform:

$$\begin{aligned}\widehat{f \star g}(\xi) &= \int_{\mathbb{R}^n} f \star g(x) e^{-i\xi\cdot x} dx \\ &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y)g(x-y)dy \right) e^{-i\xi\cdot x} dx \\ &= \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} g(x-y)e^{-i\xi\cdot x} dx \right) dy \\ &= \int_{\mathbb{R}^n} f(y) \widehat{\tau_y g}(\xi) dy \\ &= \int_{\mathbb{R}^n} f(y) \widehat{g}(\xi) e^{-i\xi\cdot y} dy = \hat{f}(\xi)\hat{g}(\xi)\end{aligned}$$

□

**Exercise(∗).** Calculate the Fourier transform of the following functions $f \in L^1(\mathbb{R})$:

a) $f(x) = \dfrac{\sin x}{1+x^2}$.

b) $f(x) = \dfrac{1}{\epsilon^2 + x^2}$, for $\epsilon > 0$ a constant.

c) $f(x) = \sqrt{\dfrac{\sigma}{t}} e^{-\sigma \frac{(x-y)^2}{t}}$, where $\sigma > 0$, $t > 0$ and $y$ are constants.

*d) $f(x) = \dfrac{1}{\cosh x}$.

We saw with the examples that there is a duality between the decay of a function and the regularity of its Fourier transform and vice versa. To make this more precise we prove the following result, which tells us, roughly speaking, that the Fourier transform swaps coordinate functions $x_j$ multiplying $f$ for derivatives $iD_j$ acting on $\hat{f}$.

**Theorem 4.3.** *i) Suppose $f \in C^1(\mathbb{R}^n)$ and that $f, D_j f \in L^1(\mathbb{R}^n)$ for all $j = 1, \ldots n$. Then*

$$\widehat{D_j f}(\xi) = i\xi_j \hat{f}(\xi)$$

*ii) Suppose* $(1+|x|)f \in L^1(\mathbb{R}^n)$*. Then* $\hat{f} \in C^1(\mathbb{R}^n)$*, and:*

$$D_j \hat{f}(\xi) = -i\,\widehat{x_j f}(\xi)$$

*Proof.* i) We again appeal to an approximation result. For $f \in C^1(\mathbb{R}^n)$ with $f, D_i f \in L^1(\mathbb{R}^n)$, then for any $\epsilon > 0$ there exists $f_\epsilon \in C^1_0(\mathbb{R}^n)$ such that $\|f - f_\epsilon\|_{L^1} < \epsilon$ and $\|D_j f - D_j f_\epsilon\|_{L^1} < \epsilon$. Integrating by parts, we readily calculate:

$$\begin{aligned}\widehat{D_j f_\epsilon}(\xi) &= \int_{\mathbb{R}^n} D_j f_\epsilon(x) e^{-i\xi\cdot x} dx \\ &= -\int_{\mathbb{R}^n} f_\epsilon(x) D_j(e^{-i\xi\cdot x}) dx \\ &= i\xi_j \int_{\mathbb{R}^n} f_\epsilon(x) e^{-i\xi\cdot x} dx\end{aligned}$$

so that $\widehat{D_j f_\epsilon}(\xi) = i\xi_j \hat{f}_\epsilon(\xi)$. Now, we calculate:

$$\begin{aligned}\left|\widehat{D_j f}(\xi) - i\xi_j \hat{f}(\xi)\right| &= \left|\widehat{D_j f}(\xi) - \widehat{D_j f_\epsilon}(\xi) + i\xi_j \hat{f}_\epsilon(\xi) - i\xi_j \hat{f}(\xi)\right| \\ &\leqslant \|D_j f - D_j f_\epsilon\|_{L^1} + |\xi|\,\|f - f_\epsilon\|_{L^1} \\ &\leqslant \epsilon(1+|\xi|)\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we must have that $\left|\widehat{D_j f}(\xi) - i\xi_j \hat{f}(\xi)\right| = 0$, and the result follows.

ii) From the condition on $f$ it is clear that $x_j f \in L^1(\mathbb{R}^n)$, so $-i\,\widehat{x_j f}$ is continuous. It suffices to prove then that:

$$\Delta_j^{h_k} \hat{f}(\xi) \to -i\,\widehat{x_j f}(\xi), \qquad \text{as } k \to \infty$$

for any sequence $\{h_k\}_{k=1}^\infty \subset \mathbb{R}$ with $h_k \to 0$. We calculate:

$$\Delta_j^{h_k} \hat{f}(\xi) = \frac{1}{h_k}\left(\hat{f}(\xi + h_k e_j) - \hat{f}(\xi)\right) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\left(\frac{e^{-ix_j h_k} - 1}{h_k}\right) dx.$$

Now for $x \in \mathbb{R}^n$ we have:

$$f(x) e^{-ix\cdot\xi}\left(\frac{e^{-ix_j h_k} - 1}{h_k}\right) \to -ix_j f(x) e^{-ix\cdot\xi}$$

as $k \to \infty$. Noting that $\left|e^{i\theta} - 1\right| = 2\left|\sin\frac{\theta}{2}\right| \leqslant \theta$ for any $\theta \in \mathbb{R}$, we have that:

$$\left|f(x) e^{-ix\cdot\xi}\left(\frac{e^{-ix_j h_k} - 1}{h_k}\right)\right| \leqslant |x_j f(x)|$$

where the right hand side is integrable. By the Dominated Convergence Theorem, we have:

$$\lim_{k\to\infty} \Delta_j^{h_k} \hat{f}(\xi) = \int_{\mathbb{R}^n} -ix_j f(x) e^{-ix\cdot\xi} dx = -i\,\widehat{x_j f}(\xi).$$

We deduce that $\hat{f} \in C^1(\mathbb{R}^n)$.

□

**Exercise($*$).** Suppose $f \in C^1(\mathbb{R}^n)$ and that $f, D_j f \in L^1(\mathbb{R}^n)$. Fix $\epsilon > 0$. Show that there exists $f_\epsilon \in C^1_0(\mathbb{R}^n)$ such that

$$\|f - f_\epsilon\|_{L^1} + \|D_j f - D_j f_\epsilon\|_{L^1} < \frac{\epsilon}{2}.$$

**Corollary 4.4.** *i) Suppose $f \in C^k(\mathbb{R}^n)$ and $D^\alpha f \in L^1(\mathbb{R}^n)$ for $|\alpha| \leqslant k$. Then there is some constant $C_k > 0$ depending only on $k$ such that:*

$$\sup_{\xi \in \mathbb{R}^n} \left| (1 + |\xi|)^k \hat{f}(\xi) \right| \leqslant C_k \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^1}$$

*ii) Suppose $(1 + |x|)^k f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^k(\mathbb{R}^n)$ and for any $|\alpha| \leqslant k$ we have:*

$$\sup_{\xi \in \mathbb{R}^n} \left| D^\alpha \hat{f}(\xi) \right| \leqslant \left\| (1 + |x|)^k f \right\|_{L^1}$$

*iii) The Fourier transform is a continuous linear map from $\mathscr{S}$ into $\mathscr{S}$:*

$$\mathcal{F} : \mathscr{S} \to \mathscr{S}.$$

*Proof.* i) First we note the algebraic fact that for any $k$ there is some constant $C_k$ such that[1]:

$$(1 + |\xi|)^k \leqslant C_k \sum_{|\alpha| \leqslant k} |\xi^\alpha|$$

holds for any $\xi \in \mathbb{R}^n$. Repeatedly applying the part $i)$ of Theorem 4.3 we know that:

$$i^{|\alpha|} \xi^\alpha \hat{f}(\xi) = \widehat{D^\alpha f}(\xi).$$

We therefore have:

$$(1 + |\xi|)^k \left| \hat{f}(\xi) \right| \leqslant C_k \sum_{|\alpha| \leqslant k} \left| i^{|\alpha|} \xi^\alpha \hat{f}(\xi) \right| = C_k \sum_{|\alpha| \leqslant k} \left| \widehat{D^\alpha f}(\xi) \right|$$

taking the supremum over $\xi \in \mathbb{R}^n$ and applying the estimate (4.1) we conclude

$$\sup_{\xi \in \mathbb{R}^n} \left| (1 + |\xi|)^k \hat{f}(\xi) \right| \leqslant C_k \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^1}.$$

ii) By iterating part $ii)$ of Theorem 4.3 we have that for $|\alpha| \leqslant k$:

$$D^\alpha \hat{f}(\xi) = (-i)^{|\alpha|} \widehat{x^\alpha f}(\xi).$$

Taking the supremum of the absolute value over $\xi \in \mathbb{R}^n$ and applying the estimate (4.1) we have:

$$\sup_{\xi \in \mathbb{R}^n} \left| D^\alpha \hat{f}(\xi) \right| \leqslant \|x^\alpha f\|_{L^1} \leqslant \left\| (1 + |x|)^k f \right\|_{L^1}$$

[1] recall that $\xi^\alpha := \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$

iii) Note that if:

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N|f(x)|<K,$$

we have:

$$\|f\|_{L^1}=\int_{\mathbb{R}^n}|f(x)|\,dx\leqslant K\int_{\mathbb{R}^n}\frac{1}{(1+|x|)^N}dx<\infty$$

provided $N>n$. Thus in particular if $f\in\mathscr{S}$ then there exists some constant $C_n$ such that:

$$\left\|(1+|x|)^M D^\alpha f\right\|_{L^1}\leqslant C_n\sup_{x\in\mathbb{R}^n}(1+|x|)^{M+n+1}|D^\alpha f(x)|$$

for all $M\in\mathbb{N}$ and all multi-indices $\alpha$. Applying the previous two parts we conclude that $\hat{f}\in C^\infty(\mathbb{R}^n)$ and:

$$\sup_{\xi\in\mathbb{R}^n,|\beta|\leqslant M}(1+|\xi|)^N\left|D^\beta\hat{f}(\xi)\right|\leqslant C_{N,M,n}\sup_{x\in\mathbb{R}^n,|\alpha|\leqslant N}(1+|x|)^{M+n+1}|D^\alpha f(x)|$$

For some constant $C_{N,M,n}$ depending only on $N,M,n$. Thus $\hat{f}\in\mathscr{S}$. Moreover, if $\{f_j\}_{j=1}^\infty\subset\mathscr{S}$ is a sequence with $f_j\to 0$ in $\mathscr{S}$, then $\hat{f}_j\to 0$ in $\mathscr{S}$, so that $\mathcal{F}$ is continuous.

□

Notice that while the Fourier transform maps $\mathscr{S}$ to itself, the same is not true of $\mathscr{D}(\mathbb{R}^n)$. Suppose $f\in C_0^\infty(\mathbb{R}^n)$, then provided $\operatorname{supp} f\subset K$ for $K$ a compact set we have:

$$\hat{f}(\xi)=\int_K f(x)e^{-ix\cdot\xi}dx$$

By repeatedly differentiating, it is possible to show that $\hat{f}$ is in fact *real analytic*, and hence $\hat{f}$ cannot vanish on any open set without vanishing everywhere. In particular, $\hat{f}$ cannot vanish outside a compact set.

**Exercise 3.8.** Suppose $f\in L^1(\mathbb{R}^n)$, with $\operatorname{supp} f\subset B_R(0)$ for some $R>0$.

a) Show that $\hat{f}\in C^\infty(\mathbb{R}^n)$ and for any multi-index:

$$\sup_{\xi\in\mathbb{R}^n}\left|D^\alpha\hat{f}(\xi)\right|\leqslant R^{|\alpha|}\|f\|_{L^1}$$

b) (*)Show that $\hat{f}$ is real analytic, with an infinite radius of convergence, i.e.:

$$\hat{f}(\xi)=\sum_\alpha D^\alpha\hat{f}(0)\frac{\xi^\alpha}{\alpha!}$$

holds for all $\xi\in\mathbb{R}^n$. Deduce that if $\hat{f}(\xi)$ vanishes on an open set, it must vanish everywhere.

*You may assume the following form of Taylor's theorem. Suppose $g \in C^{k+1}(\overline{B_r(0)})$. Then for $x \in B_r(0)$:*

$$g(x) = \sum_{|\alpha|\leqslant k} D^\alpha g(0) \frac{x^\alpha}{\alpha!} + \sum_{|\beta|=k+1} R_\beta(x) x^\beta$$

*where the remainder $R_\beta(x)$ satisfies the following estimate in $B_r(0)$:*

$$|R_\beta(x)| \leqslant \frac{1}{\beta!} \max_{|\alpha|=|\beta|} \max_{y \in \overline{B_r(0)}} |D^\alpha g(y)| \,.$$

**Exercise 3.9.** Recall that $L^\infty(\mathbb{R}) = L^1(\mathbb{R})'$. Consider the sequence $(f_n)_{n=1}^\infty$, where $f_n \in L^\infty(\mathbb{R})$ is given by $f_n(x) = \sin(nx)$. Show that $f_n \overset{*}{\rightharpoonup} 0$. Show that $f_n^2 \overset{*}{\rightharpoonup} g$ for some $g \in L^\infty(\mathbb{R})$ which you should find.

To complete this section, we are going to establish the invertibility of the Fourier transform, under some reasonable assumptions on $f$ and $\hat{f}$. In particular, this will permit us to show that $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is in fact a bijection.

**Theorem 4.5** (Fourier inversion theorem)**.** *Suppose $f \in L^1(\mathbb{R}^n)$, and assume $\hat{f} \in L^1(\mathbb{R}^n)$, then for almost every $x$:*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi. \tag{4.3}$$

*Proof.* We shall establish the result by looking at the limit $\epsilon \to 0$ of

$$I_\epsilon(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\frac{1}{2}\epsilon^2|\xi|^2} e^{ix\cdot\xi} d\xi.$$

in two different ways. Firstly note that for $\xi \in \mathbb{R}^n$ we have:

$$\hat{f}(\xi) e^{-\frac{1}{2}\epsilon^2|\xi|^2} e^{ix\cdot\xi} \to \hat{f}(\xi) e^{ix\cdot\xi}.$$

Moreover, we can estimate

$$\left| \hat{f}(\xi) e^{-\frac{1}{2}\epsilon^2|\xi|^2} e^{ix\cdot\xi} \right| \leqslant \left| \hat{f}(\xi) \right|$$

so that the integrand is dominated by an integrable function. Thus by the Dominated Convergence Theorem we have:

$$I_\epsilon(x) \to \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi, \qquad \text{as } \epsilon \to 0.$$

On the other hand, we have, using Fubini's thorem:

$$\begin{aligned}
I_\epsilon(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y) e^{-i\xi\cdot y} dy \right) e^{-\frac{1}{2}\epsilon^2|\xi|^2} e^{ix\cdot\xi} d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} e^{-\frac{1}{2}\epsilon^2|\xi|^2} e^{-i\xi\cdot(y-x)} d\xi \right) dy \\
&= \int_{\mathbb{R}^n} f(y) \frac{1}{\epsilon^n (2\pi)^{\frac{n}{2}}} e^{-\frac{|y-x|^2}{2\epsilon^2}} dy \\
&= f \star \psi_\epsilon(x)
\end{aligned}$$

where $\psi_\epsilon(x) = \epsilon^{-n}\psi(\epsilon^{-1}x)$ for

$$\psi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2}.$$

Note that $\psi \in C^\infty(\mathbb{R}^n)$, $\psi(x) \geqslant 0$ and

$$\int_{\mathbb{R}^n} \psi(x)dx = 1$$

so by Theorem 1.13, $b$) we have that:

$$f \star \psi_\epsilon \to f,$$

in $L^1(\mathbb{R}^n)$, thus we must have that

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi$$

for almost every $x$. □

Note that by the Riemann Lebesgue Lemma the map

$$x \mapsto \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi$$

is continuous. Thus under the conditions of the theorem, if $f$ is additionally assumed to be continuous, then we can upgrade the almost everywhere convergence to convergence everywhere. Alternatively, our result shows that if both $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then $f$ must be almost everywhere equal to a continuous function.

We can summarise the inversion formula quite neatly by noting that (on a suitable $f$):

$$\mathcal{F}^2 f = (2\pi)^n \check{f}.$$

An immediate corollary of the above result is that $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is a bijection, and that $\mathcal{F}^{-1} : \mathscr{S} \to \mathscr{S}$ is continuous.

**Exercise($*$).** Consider the following ODE problem. Given $f : \mathbb{R} \to \mathbb{C}$, find $\phi$ such that:

$$-\phi'' + \phi = f. \tag{4.4}$$

a) Show that if $f \in \mathscr{S}$, there is a unique $\phi \in \mathscr{S}$ solving (4.4), and give an expression for $\hat{\phi}$.

b) Show that

$$\phi(x) = \int_{\mathbb{R}} f(y)G(x-y)dy$$

where

$$G(x) = \begin{cases} \frac{1}{2}e^{x} & x < 0, \\ \frac{1}{2}e^{-x} & x \geqslant 0. \end{cases}$$

**Exercise($*$).** Suppose $f \in L^1(\mathbb{R}^3)$ is a radial function, i.e. $f(Rx) = f(x)$, whenever $R \in SO(3)$ is a rotation.

a) Show that $\hat{f}$ is radial.

b) Suppose that $\xi = (0, 0, \zeta)$. By writing the Fourier integral in polar coordinates, show that

$$\hat{f}(\xi) = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(r) e^{-i\zeta r \cos\theta} r^2 \sin\theta d\theta dr d\phi.$$

c) Making the substitution $s = \cos\theta$, and using the fact that $\hat{f}$ is radial, deduce:

$$\hat{f}(\xi) = 4\pi \int_0^\infty f(r) \frac{\sin r\,|\xi|}{r\,|\xi|} r^2 dr$$

for any $\xi \in \mathbb{R}^n$.

## 4.2 The Fourier transform on $L^2(\mathbb{R}^n)$

Having defined the Fourier transform acting on functions in $L^1(\mathbb{R}^n)$, we are going to extend it to act on more general functions (and eventually distributions). Firstly, we shall see how the Fourier transform extends very nicely to act on functions in $L^2(\mathbb{R}^n)$. As we have already seen, this is a particularly nice function space because is is a *Hilbert space*. We recall the inner product:

$$(f, g) = \int_{\mathbb{R}^n} \overline{f}(x) g(x) dx,$$

which induces the norm via:

$$\|f\|_{L^2} = (f, f)^{\frac{1}{2}}$$

and moreover it is complete, which means that all Cauchy sequences converge in $L^2(\mathbb{R}^n)$.

We shall first establish that the Fourier transform maps $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, and moreover show that the $L^2$ inner product is preserved by the Fourier transform (up to multiplication by a constant).

**Theorem 4.6** (Parseval's Formula)**.** *Suppose $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$ and moreover:*

$$(f, g) = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g}).$$

*Proof.* We will again use a density argument to prove this result. First suppose that $f, g \in \mathscr{S}$. Then using the Fourier Inversion Theorem (Theorem 4.5) and Fubini's theorem we can calculate:

$$\begin{aligned}(f, g) &= \int_{\mathbb{R}^n} \overline{f}(x)g(x)dx \\ &= \int_{\mathbb{R}^n} \overline{f}(x) \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi)e^{ix\cdot\xi}d\xi \right) dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \overline{f}(x)e^{ix\cdot\xi}dx \right) \hat{g}(\xi)d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\left( \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx \right)} \hat{g}(\xi)d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{g}(\xi)d\xi = \frac{1}{(2\pi)^n}(\hat{f}, \hat{g})\end{aligned}$$

Now suppose that $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By Theorem 1.13 part *b*), there exists a sequence $\{f_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^n) \subset \mathscr{S}$ such that:

$$\|f_j - f\|_{L^1} + \|f_j - f\|_{L^2} < \frac{1}{j}$$

and similarly for $g$. We know that:

$$\sup_{\xi\in\mathbb{R}^n} \left| \hat{f}_j(\xi) - \hat{f}(\xi) \right| \leqslant \|f_j - f\|_{L^1} < \frac{1}{j}$$

so that $\hat{f}_j \to f$ uniformly on $\mathbb{R}^n$. We also have by the calculation above:

$$\left\| \hat{f}_j - \hat{f}_k \right\|_{L^2} = (2\pi)^{\frac{n}{2}} \|f_j - f_k\|_{L^2} .$$

Now since $f_j \to f$ in $L^2(\mathbb{R}^n)$, we have that $\{f_j\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. Thus $\hat{f}_j$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. By the completeness of $L^2(\mathbb{R}^n)$, we have that $\hat{f}_j$ converges in $L^2(\mathbb{R}^n)$ and hence $\hat{f} \in L^2(\mathbb{R}^n)$. Furthermore, we know that

$$(f_j, g_j) = \frac{1}{(2\pi)^n}(\hat{f}_j, \hat{g}_j)$$

since each of the sequences $\{f_j\}, \{g_j\}, \{\hat{f}_j\}, \{\hat{g}_j\}$ converge in $L^2(\mathbb{R}^n)$, we can take the limit[2] $j \to \infty$ to conclude:

$$(f, g) = \frac{1}{(2\pi)^n}(\hat{f}, \hat{g})$$ □

Thus we have shown that the Fourier transform $\mathcal{F}$ maps $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Moreover, we have that it is a *bounded* as an operator from $L^2(\mathbb{R}^n)$ to itself, since

$$\left\| \hat{f} \right\|_{L^2} \leqslant (2\pi)^{\frac{n}{2}} \|f\|_{L^2} .$$

This means that $\mathcal{F}$ is a bounded linear map defined on a dense subset of a Banach space. A general result tells us that the map extends uniquely to a bounded linear map on the entire space. Rather than invoke an abstract result, we can show this directly.

[2]You should check that you understand why this is valid.

**Corollary 4.7.** *There is a unique continuous linear operator* $\overline{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *such that:*

$$\overline{\mathcal{F}}[f] = \mathcal{F}[f], \qquad \text{for all } f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{4.5}$$

*We say that* $\overline{\mathcal{F}}$ *is the extension of the Fourier transform to* $L^2(\mathbb{R}^n)$*. It is sometimes known as the* Fourier-Plancherel *transform.*

*Proof.* For any $f \in L^2(\mathbb{R}^n)$, we can take a sequence $\{f_j\}_{j=1}^\infty \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $f_j \to f$ in $L^2(\mathbb{R}^n)$ (for example by approximating $f$ with smooth functions of compact support). By Theorem 4.6 we have that:

$$\left\| \hat{f}_j - \hat{f}_k \right\|_{L^2} = (2\pi)^{\frac{n}{2}} \left\| f_j - f_k \right\|_{L^2}. \tag{4.6}$$

Now, since $f_j$ converges in $L^2(\mathbb{R}^n)$, it is in particular a Cauchy sequence in $L^2(\mathbb{R}^n)$. Equation (4.6) shows that $\hat{f}_j$ is also a Cauchy sequence in $L^2(\mathbb{R}^n)$, hence has a limit, say $F \in L^2(\mathbb{R}^n)$ by the completeness of $L^2(\mathbb{R}^n)$. Suppose $\{f'_j\}_{j=1}^\infty \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is another sequence with $f'_j \to f$, and suppose $\hat{f}_j \to F'$. Then we have:

$$\left\| F - F' \right\|_{L^2} = \lim_{j\to\infty} \left\| \hat{f}_j - \hat{f}'_j \right\|_{L^2} = \lim_{j\to\infty} (2\pi)^{\frac{n}{2}} \left\| f_j - f'_j \right\|_{L^2} = 0$$

since both $f_j$ and $f'_j$ tend to $f$. Thus $F$ depends only $f$, and not on the sequence $f_j$ which we chose to approximate $f$.

We define $\overline{\mathcal{F}}[f] = F$, i.e.:

$$\overline{\mathcal{F}}[f] = \lim_{j\to\infty} \mathcal{F}[f_j], \quad \text{where } \{f_j\}_{j=1}^\infty \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad f_j \to f \text{ in } L^2(\mathbb{R}^n),$$

and the limit is to be understood to be in $L^2(\mathbb{R}^n)$. This certainly satisfies (4.5), since we can take our approximating sequence to be the constant sequence $f_j = f$ for all $j$ when $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. $\overline{\mathcal{F}}$ is clearly linear and moreover, we have that

$$\begin{aligned}
\left\| \overline{\mathcal{F}}[f] \right\|_{L^2} &= \left\| \lim_{j\to\infty} \mathcal{F}[f_j] \right\|_{L^2} \\
&= \lim_{j\to\infty} \left\| \mathcal{F}[f_j] \right\|_{L^2} \\
&= \lim_{j\to\infty} (2\pi)^{\frac{n}{2}} \left\| f_j \right\|_{L^2} = (2\pi)^{\frac{n}{2}} \left\| f \right\|_{L^2},
\end{aligned}$$

so $\overline{\mathcal{F}}$ is bounded and hence continuous[3]. It remains to show that $\overline{\mathcal{F}}$ is unique. Suppose that $\overline{\mathcal{F}}'$ is another continuous linear operator satisfying (4.5). For any $f \in L^2(\mathbb{R}^n)$, take a sequence $\{f_j\}_{j=1}^\infty \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $f_j \to f$ in $L^2(\mathbb{R}^n)$. We have:

$$\overline{\mathcal{F}}'[f] = \lim_{j\to\infty} \overline{\mathcal{F}}'[f_j] = \lim_{j\to\infty} \overline{\mathcal{F}}[f_j] = \overline{\mathcal{F}}[f]$$

so that $\overline{\mathcal{F}}' = \overline{\mathcal{F}}$. □

[3] If $\{f_j\}_{j=1}^\infty \subset L^2(\mathbb{R}^n)$ is a sequence with $f_j \to f$ in $L^2(\mathbb{R}^n)$, then

$$\left\| \overline{\mathcal{F}}[f_j] - \overline{\mathcal{F}}[f] \right\|_{L^2} = \left\| \overline{\mathcal{F}}[f_j - f] \right\|_{L^2} = (2\pi)^{\frac{n}{2}} \left\| f_j - f \right\|_{L^2} \to 0$$

so $\overline{\mathcal{F}}[f_j] \to \overline{\mathcal{F}}[f]$ in $L^2(\mathbb{R}^n)$.

**Exercise(∗).** (*) Suppose that $f, g \in L^2(\mathbb{R}^n)$, and denote the Fourier-Plancherel transform by $\overline{\mathcal{F}}$. You may assume any results already established for the Fourier transform.

a) Show that

$$(f, g) = \frac{1}{(2\pi)^n}\left(\overline{\mathcal{F}}[f], \overline{\mathcal{F}}[g]\right).$$

b) Recall that $\check{f}(y) = f(-y)$. Show that:

$$\overline{\mathcal{F}}\left[\overline{\mathcal{F}}[f]\right] = (2\pi)^n \check{f}.$$

Hence, or otherwise, deduce that $\overline{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bijection, and that $\overline{\mathcal{F}}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear map.

c) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \lim_{R\to\infty} \int_{B_R(0)} f(x) e^{-ix\cdot\xi} dx$$

with convergence in the sense of $L^2(\mathbb{R}^n)$.

d) Suppose that $f \in C^1(\mathbb{R}^n)$ and $f, D_j f \in L^2(\mathbb{R}^n)$. Show that $\xi_j \overline{\mathcal{F}}[f](\xi) \in L^2(\mathbb{R}^n)$ and:

$$\overline{\mathcal{F}}[D_j f](\xi) = i\xi_j \overline{\mathcal{F}}[f](\xi)$$

e) For $x \in \mathbb{R}$ let:

$$f(x) = \frac{\sin x}{x}$$

   i) Show that $f \in L^2(\mathbb{R})$.

   ii) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \begin{cases} \pi & -1 < \xi < 1, \\ 0 & |\xi| \geqslant 1. \end{cases}$$

f) i) Show that for all $x \in \mathbb{R}^n$:

$$|f \star g(x)| \leqslant \|f\|_{L^2} \|g\|_{L^2}.$$

   ii) Show that $f \star g \in C^0(\mathbb{R}^n)$ and:

$$f \star g = \mathcal{F}^{-1}\left[\overline{\mathcal{F}}[f] \cdot \overline{\mathcal{F}}[g]\right]$$

   where:

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi\cdot x} d\xi.$$

*[Hint for parts a), b), d), f): approximate by Schwartz functions]*

**Exercise(∗).** Work in $\mathbb{R}^3$. For $k > 0$, define the function:

$$G(x) = \frac{e^{-k|x|}}{4\pi |x|}$$

a) Show that $G \in L^1(\mathbb{R}^3)$.

b) Show that:

$$\hat{G}(\xi) = \frac{1}{|\xi|^2 + k^2}$$

*[Hint: use Exercise 4.1, part c)]*

**Exercise 3.10.** Suppose $f \in \mathscr{S}(\mathbb{R}^n)$. By observing that

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} \frac{1}{n} (\text{div } x) |f(x)|^2 \, dx,$$

or otherwise, show that:

$$(2\pi)^{\frac{n}{2}} \|f\|_{L^2}^2 \leqslant \frac{2}{n} \big\| |x| \, f(x) \big\|_{L^2} \big\| \, |\xi| \, \hat{f}(\xi) \big\|_{L^2}$$

with equality if and only if $f(x) = ae^{-\lambda |x|^2}$ for some $a \in \mathbb{C}, \lambda > 0$. Deduce that if $x_0, \xi_0 \in \mathbb{R}^n$:

$$(2\pi)^{\frac{n}{2}} \|f\|_{L^2}^2 \leqslant \frac{2}{n} \big\| |x - x_0| \, f(x) \big\|_{L^2} \big\| \, |\xi - \xi_0| \, \hat{f}(\xi) \big\|_{L^2}.$$

Explain how this shows that a function $f \in L^2(\mathbb{R}^n)$ cannot be sharply localised in both physical and Fourier space simultaneously. This is the *uncertainty principle.*

Usually one does not labour the distinction between the Fourier transform acting on $L^1(\mathbb{R}^n)$ and the Fourier-Plancherel transform acting on $L^2(\mathbb{R}^n)$. From now on we shall use the same notation for both, so that for $f \in L^2(\mathbb{R}^n)$ we write $\overline{\mathcal{F}}[f] = \mathcal{F}[f] = \hat{f}$. Since the two transforms agree wherever both are defined, there is no ambiguity in this. The majority of the results that we have already established for the Fourier transform extend to the Fourier-Plancherel transform in a straightforward way, see Exercise 4.2.

## 4.3 The Fourier transform on $\mathscr{S}'$

We are now going to extend the Fourier transform in a slightly different way, such that it acts on distributions. Suppose that $f \in L^1(\mathbb{R}^n)$, and $\phi \in \mathscr{S}$. Then since $\hat{f} \in C^0(\mathbb{R}^n)$ and $\hat{f}$ decays towards infinity, we have that $T_{\hat{f}} \in \mathscr{S}'$. By Fubini we have:

$$\begin{aligned} T_{\hat{f}}[\phi] = \int_{\mathbb{R}^n} \hat{f}(x)\phi(x)dx &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(y) e^{-ix\cdot y} dy \right) \phi(x) dx \\ &= \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot y} dx \right) dy \\ &= \int_{\mathbb{R}^n} f(x) \hat{\phi}(x) dx. \end{aligned}$$

Thus for $f \in L^1(\mathbb{R}^n)$ we have that:

$$T_{\hat{f}}[\phi] = T_f[\hat{\phi}], \qquad \text{for all } \phi \in \mathscr{S}.$$

Motivated by this, we define:

**Definition 4.1.** *For a distribution $u \in \mathscr{S}'$, we define the Fourier transform of $u$, written $\hat{u} \in \mathscr{S}'$ to be the distribution satisfying:*

$$\hat{u}[\phi] = u[\hat{\phi}], \qquad \textit{for all } \phi \in \mathscr{S}.$$

Notice that the definition makes sense because the Fourier transform maps $\mathscr{S}$ to $\mathscr{S}$ continuously. If we tried to use the above definition but with $\phi \in \mathscr{D}(\mathbb{R}^n)$ and $u \in \mathscr{D}'(\mathbb{R}^n)$, we would run into difficulties because $\hat{\phi} \notin \mathscr{D}(\mathbb{R}^n)$.

**Example 17.** *a) For $\xi \in \mathbb{R}^n$ we have:*

$$\widehat{\delta_\xi} = T_{e_{-\xi}}$$

*To see this, we use the definition. For $\phi \in \mathscr{S}$:*

$$\widehat{\delta_\xi}[\phi] = \delta_\xi[\hat{\phi}] = \hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(x)dx = T_{e_{-\xi}}[\phi]$$

*Since $\phi$ was arbitrary, the distributions are equal.*

*b) For $x \in \mathbb{R}^n$ we have:*

$$\widehat{T_{e_x}} = (2\pi)^n \delta_x.$$

*To see this, we note for $\phi \in \mathscr{S}$:*

$$\widehat{T_{e_x}}[\phi] = T_{e_x}[\hat{\phi}] = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\phi}(\xi)d\xi = (2\pi)^n\phi(x) = (2\pi)^n\delta_x[\phi].$$

*Again, as $\phi$ is arbitrary the distributions are equal. Note that a particular case is $\widehat{T_1} = (2\pi)^n\delta_0$.*

*c) For $\alpha$ a multi-index, denote by $X^\alpha$ the map*

$$X^\alpha : x \mapsto x^\alpha.$$

*Then we have:*

$$\widehat{T_{X^\alpha}} = (2\pi)^n i^{|\alpha|} D^\alpha \delta_0$$

*For $\phi \in \mathscr{S}$:*

$$\begin{aligned}
\widehat{T_{X^\alpha}}[\phi] = T_{X^\alpha}[\hat{\phi}] &= \int_{\mathbb{R}^n} \xi^\alpha \hat{\phi}(\xi)d\xi \\
&= (-i)^{|\alpha|}\int_{\mathbb{R}^n} \widehat{D^\alpha\phi}(\xi)d\xi \\
&= (2\pi)^n(-i)^{|\alpha|}D^\alpha\phi(0) = (2\pi)^n i^{|\alpha|} \times (-1)^{|\alpha|}\delta_0\,[D^\alpha\phi] \\
&= (2\pi)^n i^{|\alpha|} D^\alpha \delta_0[\phi]
\end{aligned}$$

Most of the properties of the Fourier transform defined on $\mathscr{S}$ are inherited by the transform defined on $\mathscr{S}'$. We first need to define a couple of operations on $\mathscr{S}'$. Recall that if $\phi \in \mathscr{S}$, then $\tau_x\phi \in \mathscr{S}$ is the translate of $\phi$, given by $\tau_x\phi(y) = \phi(y-x)$, and $\check{\phi} \in \mathscr{S}$ is given by $\check{\phi}(y) = \phi(-y)$. For $u \in \mathscr{S}$, we define:

$$\tau_x u[\phi] = u[\tau_{-x}\phi], \qquad \check{u}[\phi] = u[\check{\phi}]$$

Notice also that if $f \in C^\infty(\mathbb{R}^n)$ is a function of tempered growth, i.e., if for each $\alpha$ and there exists a constant $C_\alpha$ and integer $N_\alpha$ such that:

$$|D^\alpha f(x)| \leqslant C_\alpha(1+|x|)^{N_\alpha}, \qquad \forall x \in \mathbb{R}^n.$$

then $\phi f \in \mathscr{S}$ when $\phi \in \mathscr{S}$ and we can define $fu \in \mathscr{S}'$ by

$$fu[\phi] = u[f\phi]$$

**Exercise(∗).** Verify that if $f \in L^1_{loc.}$ is such that $T_f \in \mathscr{S}'$, then:

$$\tau_x T_f = T_{\tau_x f}, \quad \text{and} \quad \check{T}_f = T_{\check{f}}$$

**Lemma 4.8.** *Suppose $u \in \mathscr{S}'$ is a tempered distribution. Then:*

$$\widehat{e_x u} = \tau_x \hat{u}, \qquad \widehat{\tau_x u} = e_{-x}\hat{u}, \qquad \widehat{D^\alpha u} = i^{|\alpha|}X^\alpha \hat{u} \qquad D^\alpha \hat{u} = (-i)^{|\alpha|}\widehat{X^\alpha u}$$

*Moreover:*

$$\hat{\hat{u}} = (2\pi)^n \check{u},$$

*so that the Fourier transform on $\mathscr{S}'$ is invertible.*

*Proof.* These are all calculations using the corresponding results for $\mathscr{S}$. Take $\phi \in \mathscr{S}$. We have:

$$\widehat{e_x u}[\phi] = e_x u[\hat{\phi}] = u[e_x\hat{\phi}] = u\left[\widehat{\tau_{-x}\phi}\right] = \hat{u}[\tau_{-x}\phi] = \tau_x u[\phi].$$

Since $\phi$ was arbitrary, we have $\widehat{e_x u} = \tau_x \hat{u}$. Similarly, we calculate:

$$\widehat{\tau_x u}[\phi] = \tau_x u[\hat{\phi}] = u[\tau_{-x}\hat{\phi}] = u\left[\widehat{e_{-x}\phi}\right] = \hat{u}[e_{-x}\phi] = e_{-x}u[\phi].$$

Next we have

$$\begin{aligned}\widehat{D^\alpha u}[\phi] &= D^\alpha u\left[\hat{\phi}\right] = (-1)^{|\alpha|}u\left[D^\alpha\hat{\phi}\right] \\ &= (-1)^{|\alpha|}u\left[(-i)^{|\alpha|}\widehat{X^\alpha\phi}\right] = i^{|\alpha|}u\left[\widehat{X^\alpha\phi}\right] \\ &= i^{|\alpha|}\hat{u}\left[X^\alpha\phi\right] = \left(i^{|\alpha|}X^\alpha\hat{u}\right)[\phi]\end{aligned}$$

similarly:

$$\begin{aligned}D^\alpha\hat{u}[\phi] &= (-1)^{|\alpha|}\hat{u}[D^\alpha\phi] = (-1)^{|\alpha|}u[\widehat{D^\alpha\phi}] \\ &= (-1)^{|\alpha|}u[i^{|\alpha|}X^\alpha\hat{\phi}] = (-i)^{|\alpha|}X^\alpha u[\hat{\phi}] \\ &= \left((-i)^{|\alpha|}\widehat{X^\alpha u}\right)[\phi].\end{aligned}$$

Finally, we have

$$\hat{u}[\phi] = \hat{u}[\hat{\phi}] = u[\hat{\hat{\phi}}] = u[(2\pi)^n \check{\phi}] = (2\pi)^n \check{u}[\phi]$$

Since $\check{\check{u}} = u$, we have that the Fourier transform is invertible. □

Importantly, the Fourier transform is also a continuous linear map $\mathscr{S}' \to \mathscr{S}'$.

**Lemma 4.9.** *The map:*

$$\begin{array}{cccc} \mathcal{F} & : & \mathscr{S}' & \to & \mathscr{S}' \\ & & u & \mapsto & \hat{u} \end{array}$$

*is a linear homeomorphism.*

*Proof.* We already have that $\mathcal{F}$ is linear. From the definition of the weak-$\star$ topology, a sequence $\{u_j\}_{j=1}^{\infty} \subset \mathscr{S}'$ converges to $u$ if

$$u_j[\phi] \to u[\phi]$$

for all $\phi \in \mathscr{S}$. Suppose that we have such a convergent sequence in $\mathscr{S}'$. We calculate:

$$\hat{u}_j[\phi] = u_j[\hat{\phi}] \to u[\hat{\phi}] = \hat{u}[\phi].$$

Thus if $u_j \to u$ we have $\mathcal{F}(u_j) \to \mathcal{F}(u)$. Thus $\mathcal{F}$ is continuous. Since $\mathcal{F}^4 = (2\pi)^{2n}\iota$, we have that $\mathcal{F}$ is invertible and the inverse is also continuous. □

**Remark.** *Strictly, we have only established that $\mathcal{F}$ is sequentially continuous with respect to the weak-$\star$ topology induced on $\mathscr{S}'$ by $\mathscr{S}$. Establishing genuine continuity is not difficult, but requires the full description of the weak-$\star$ topology, and we leave this as an exercise.*

**Exercise 3.11.** Let $f : \mathbb{R} \to \mathbb{R}$ be the sign function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$$

and define $f_R(x) = f(x)\mathbb{1}_{[-R,R]}(x)$.

a) Sketch $f_R(x)$, and show that $T_{f_R} \to T_f$ in $\mathscr{S}'(\mathbb{R})$ as $R \to \infty$.

b) Compute $\hat{f}_R(\xi)$, and show that for $\phi \in \mathscr{S}(\mathbb{R})$:

$$T_{\hat{f}_R}[\phi] = -2i \int_0^\infty \frac{\phi(x) - \phi(-x)}{x} dx + 2i \int_0^\infty \left( \frac{\phi(x) - \phi(-x)}{x} \right) \cos Rx dx.$$

Deduce $\widehat{T_f} = -2iP.V. \left(\frac{1}{x}\right)$, where we define the distribution $P.V. \left(\frac{1}{x}\right)$ by:

$$P.V. \left( \frac{1}{x} \right) [\phi] = \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} \frac{\phi(x)}{x} dx, \qquad \phi \in \mathscr{S}(\mathbb{R}).$$

c) Write down $\widehat{T_H}$, where $H$ is the Heaviside function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$$

By considering $e^{-\epsilon x}H(x)$, or otherwise, find an expression for the distribution $u$ which acts on $\phi \in \mathscr{S}(\mathbb{R})$ by:

$$u[\phi] := \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{\phi(x)}{x + i\epsilon} dx.$$

**Exercise 3.12.** Suppose $\phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^m)$. For each $y \in \mathbb{R}^m$ let $\phi_y : \mathbb{R}^n \to \mathbb{C}$ be given by $\phi_y(x) = \phi(x, y)$. Let $u \in \mathscr{D}'(\mathbb{R}^n)$.

a) Show that $\psi : y \mapsto u[\phi_y]$ is smooth and find an expression for $D^\alpha \psi$. Deduce that

$$\int_{\mathbb{R}^m} \psi(y)dy = u[\Psi], \quad \text{where} \quad \Psi(x) = \int_{\mathbb{R}^m} \phi(x, y)dy.$$

b) Show that there exists a sequence of smooth functions $f_n \in C_c^\infty(\mathbb{R}^n)$ such that $T_{f_n} \to u$ in the weak-$*$ topology of $\mathscr{D}'(\mathbb{R}^n)$.

### 4.3.1 Convolutions

We have generalised almost all of the properties of the Fourier transform to distributions. The final result that we shall establish concerns convolutions. Recall that if $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$ then $u \star \phi \in C^\infty(\mathbb{R}^n)$ is given by:

$$u \star \phi(x) = u\left[\tau_x \check{\phi}\right].$$

Notice that this definition continues to make sense for each $x$, provided $u \in \mathscr{S}'$ and $\phi \in \mathscr{S}$, although it is no longer clear that the resulting function is smooth. We have the following results concerning this convolution.

**Theorem 4.10.** *Suppose $u \in \mathscr{S}'$ and $\phi \in \mathscr{S}$ are given. Then the function:*

$$u \star \phi : \mathbb{R}^n \to \mathbb{C}$$

*has the following properties*

*a) $u \star \phi \in C^\infty(\mathbb{R}^n)$ with*

$$D^\alpha(u \star \phi) = D^\alpha u \star \phi = u \star D^\alpha \phi.$$

*b) There exist constants $N \in \mathbb{N}$, $K > 0$ depending on $u$ and $\phi$ such that:*

$$|u \star \phi(x)| \leqslant K(1 + |x|)^N.$$

*c) $T_{u \star \phi} \in \mathscr{S}'$ and moreover:*

$$\widehat{T_{u \star \phi}} = \hat{\phi}\hat{u}.$$

*d) For any $\psi \in \mathscr{S}$, we have:*

$$(u \star \phi) \star \psi = u \star (\phi \star \psi)$$

*e) We have:*

$$T_{\hat{u} \star \hat{\phi}} = (2\pi)^n \widehat{\phi u}$$

*Proof.* a) The smoothness of $u \star \phi$ is proven exactly as in Lemma 3.10, *ii*). The only modification to the argument required is to note that for $\phi \in \mathscr{S}$, we have

$$\Delta_i^h \phi \to D_i \phi \quad \text{in} \quad \mathscr{S}, \quad \text{as} \quad h \to 0.$$

b) First, we note the following simple inequality which holds for all $x, y \in \mathbb{R}^n$:

$$1 + |x + y| \leqslant 1 + |x| + |y| \leqslant (1 + |x|)(1 + |y|).$$

Next, recall from Lemma 3.15 that there exist $N, k \in \mathbb{N}$ and $C > 0$ such that:

$$|u[\psi]| \leqslant C \sup_{y \in \mathbb{R}^n; |\alpha| \leqslant k} \left| (1 + |y|)^N D^\alpha \psi(y) \right|, \qquad \text{for all } \psi \in \mathscr{S}.$$

Applying this inequality with $\psi = \tau_x \check{\phi}$, we calculate:

$$\begin{aligned} |u \star \phi(x)| = \left| u \left[ \tau_x \check{\phi} \right] \right| &\leqslant C \sup_{y \in \mathbb{R}^n; |\alpha| \leqslant k} \left| (1 + |y|)^N D^\alpha \phi(y - x) \right| \\ &= C \sup_{z \in \mathbb{R}^n; |\alpha| \leqslant k} \left| (1 + |z + x|)^N D^\alpha \phi(z) \right| \\ &\leqslant \left[ C \sup_{z \in \mathbb{R}^n; |\alpha| \leqslant k} \left| (1 + |z|)^N D^\alpha \phi(z) \right| \right] (1 + |x|)^N \end{aligned}$$

which gives the result on setting:

$$K = C \sup_{y \in \mathbb{R}^n; |\alpha| \leqslant k} \left| (1 + |z|)^N D^\alpha \phi(z) \right|.$$

c) Combining the above two results, we have that $T_{u \star \phi} \in \mathscr{S}'$, since $u \star \phi \in L^1_{loc.}(\mathbb{R}^n)$ and $u \star \phi$ grows at most polynomially. It therefore makes sense to consider the Fourier transform. Suppose that $\psi \in \mathscr{D}(\mathbb{R}^n)$. We calculate:

$$\begin{aligned}
\widehat{T_{u\star\phi}}[\hat{\psi}] &= T_{u\star\phi}\left[\hat{\hat{\psi}}\right] = (2\pi)^n T_{u\star\phi}\left[\check{\psi}\right] && \text{Fourier Inversion Thm} \\
&= (2\pi)^n \int_{\mathbb{R}^n} u\star\phi(x)\psi(-x)dx && \text{Defn. of } T_f \\
&= (2\pi)^n \int_{\mathbb{R}^n} u\left[\tau_x\check{\phi}\right]\psi(-x)dx && \text{Defn. of } u\star\phi \\
&= (2\pi)^n \int_{\mathbb{R}^n} u\left[\psi(-x)\tau_x\check{\phi}\right]dx && \text{Linearity of } u \\
&= (2\pi)^n u\left[\int_{\mathbb{R}^n} \psi(-x)\tau_x\check{\phi}dx\right] && \text{(!!)} \\
&= (2\pi)^n u\left[(\phi\check{\star}\psi)\right] && \text{Defn. of } \phi\star\psi \\
&= u\left[\widehat{\widehat{\phi\star\psi}}\right] = \hat{u}\left[\widehat{\phi\star\psi}\right] && \text{Fourier Inversion Thm} \\
&= \hat{u}\left[\hat{\phi}\hat{\psi}\right] = \left(\hat{\phi}\hat{u}\right)[\hat{\psi}] && \text{F.T. of convolution}
\end{aligned}$$

Most of the manipulations here are relatively straightforward. We have used Theorems 4.2, 4.5 in addition to various definitions. The step marked (!!), in which we interchanged an integration and an application of $u$ requires some justification. Crudely this is true because we can replace the integration with an appropriately convergent Riemann sum and use the linearity and continuity of $u$. We shall justify this step in Lemma 4.11. The conclusion of this calculation is that:

$$\widehat{T_{u\star\phi}}[\hat{\psi}] = \left(\hat{\phi}\hat{u}\right)[\hat{\psi}]$$

This holds for all $\psi \in \mathscr{D}(\mathbb{R}^n)$. Now, since $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}$ and $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is a homeomorphism, we have that:

$$\mathcal{F}\left[\mathscr{D}(\mathbb{R}^n)\right] = \left\{\hat{\psi} : \psi \in \mathscr{D}(\mathbb{R}^n)\right\}$$

is dense in $\mathscr{S}$. Thus, by approximation,

$$\widehat{T_{u\star\phi}}[\chi] = \left(\hat{\phi}\hat{u}\right)[\chi]$$

holds for any $\chi \in \mathscr{S}$ and we're done.

d) Note that in the process of proving the previous part, we established that for any $\psi \in \mathscr{D}(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} u\star\phi(x)\psi(-x)dx = u\left[(\phi\check{\star}\psi)\right]$$

which is equivalent to:

$$(u\star\phi)\star\psi(0) = u\star(\phi\star\psi)(0). \tag{4.7}$$

Now, note that:

$$u \star \tau_y \phi = \tau_y (u \star \phi), \qquad \phi \star \tau_y \psi = \tau_y (\phi \star \psi)$$

as can be easily seen from the definitions. Applying (4.7) with $\psi$ replaced by $\tau_y \psi$, we conclude that:

$$(u \star \phi) \star \psi(y) = u \star (\phi \star \psi)(y).$$

Since this holds for any $\psi \in \mathscr{D}(\mathbb{R}^n)$ and $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}$, we're done.

e) This result follows by applying part *c)* to $\hat{u} \star \hat{\phi}$ and repeatedly making use of the Fourier inversion theorem. We calculate:

$$\begin{aligned} \widehat{T_{\hat{u}\star\hat{\phi}}} &= \hat{\hat{\phi}}\hat{\hat{u}} = (2\pi)^{2n} \check{\phi}\check{u} \\ &= (2\pi)^{2n} (\check{\phi u}) = (2\pi)^n \widehat{\widehat{(\phi u)}} \end{aligned}$$

Since the Fourier transform is a bijection on $\mathscr{S}'$, the result follows.

□

In order to complete the proof of the above result, we need to justify the step marked (!!) in which a tempered distribution and an integration were interchanged. We will first prove a result concerning the convergence of Riemann sums, which will enable us to establish that the (!!) step was justified. Let us suppose that $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ are open and that $f \in C^0(\Omega \times \Omega')$ is uniformly continuous. We will also assume that there is some $R > 0$ such that:

$$\operatorname{supp} f(\cdot, y) \subset [-R, R)^n \subset \Omega$$

for each $y \in \Omega'$.

Next, we define a dyadic family of partitions of $[-R, R)^n$ into cubes as follows:

$$\Pi_k = \left\{ \left[ -\frac{R}{2^k} i_1, \frac{R}{2^k}(i_1 + 1) \right) \times \cdots \times \left[ -\frac{R}{2^k} i_n, \frac{R}{2^k}(i_n + 1) \right) : i_l \in [-2^k, 2^k - 1] \cap \mathbb{Z} \right\}$$

where $k = 0, 1, \ldots$. The $(k+1)^{st}$ partition is obtained by chopping each cube in the $k^{th}$ partition into cubes with half the side length. Clearly for each fixed $k$:

$$\bigcup \Pi_k = [-R, R)^n$$

For $\pi \in \Pi_k$, we define $x_\pi$ to be the point at the centre of the cube $\pi$. We define the $k^{th}$ Riemann sum with respect to this partition by:

$$S_k(y) = \sum_{\pi \in \Pi_k} f(x_\pi, y) \, |\pi| \, .$$

**Lemma 4.11.** *With the definitions as above,*

$$S_k(y) \to \int_\Omega f(x, y) dx$$

*uniformly in* $y \in \Omega'$.

*Proof.* First note that $x \mapsto f(x,y)$ is continuous and of compact support, hence Riemann integrable on $\Omega$. Thus for each fixed $y$ we have:

$$S_k(y) \to \int_\Omega f(x,y)dx$$

Next consider $k' \geqslant k$. We have that $\Pi_{k'}$ is a refinement of $\Pi_k$, i.e. if $\pi' \in \Pi_{k'}$, then there is a unique $\pi \in \Pi_k$ with $\pi' \subset \pi$. We calculate:

$$\begin{aligned} S_k(y) - S_{k'}(y) &= \sum_{\pi \in \Pi_k} f(x_\pi, y)\, |\pi| - \sum_{\pi \in \Pi_k} \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} f(x_{\pi'}, y)\, |\pi'| \\ &= \sum_{\pi \in \Pi_k} \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} \left( f(x_\pi, y) - f(x_{\pi'}, y) \right) |\pi'| \end{aligned}$$

here we have used that:

$$|\pi| = \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} |\pi'|$$

Now, since $f$ is uniformly continuous, we know that for any $\epsilon > 0$ there exists a $\delta$, independent of $y$, such that

$$\left| f(x,y) - f(x',y) \right| < \epsilon$$

for all $|x - x'| < \delta$. Notice that for $\pi' \subset \pi$ we have:

$$\left| x'_\pi - x_\pi \right| \leqslant \frac{R}{2^{k+1}} \sqrt{n}.$$

Thus given $\epsilon > 0$, there exists $K$ such that for all $k \geqslant K$:

$$|f(x_\pi, y) - f(x_{\pi'}, y)| < \frac{\epsilon}{(2R)^n}.$$

Now suppose $k' \geqslant k \geqslant K$. We estimate:

$$\begin{aligned} |S_k(y) - S_{k'}(y)| &\leqslant \sum_{\pi \in \Pi_k} \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} |f(x_\pi, y) - f(x_{\pi'}, y)|\, |\pi'| \\ &\leqslant \frac{\epsilon}{(2R)^n} \sum_{\pi \in \Pi_k} \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} |\pi'| = \epsilon, \end{aligned}$$

since the sum over the partition simply gives us back the volume of the large cube. Sending $k'$ to infinity, we have the result we require. □

This result allows us to establish the result we require:

**Corollary 4.12.** *Suppose $u \in \mathscr{S}'$, $\phi \in \mathscr{S}$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$. Then:*

$$u \left[ \int_{\mathbb{R}^n} \psi(-x) \tau_x \check{\phi} dx \right] = \int_{\mathbb{R}^n} u \left[ \psi(-x) \tau_x \check{\phi} \right] dx$$

*Proof.* Fix $\Omega$, $R > 0$ such that $\operatorname{supp} \check{\psi} \subset [-R, R)^n \subset \Omega$. Define the map:

$$\begin{array}{rccl} f & : & \Omega \times \mathbb{R}^n & \to & \mathbb{C} \\ & & (x, y) & \mapsto & \psi(-x)\phi(y - x) \end{array}$$

Notice that $(1+|y|)^N D_y^\alpha f$ is uniformly continuous on $\Omega \times \mathbb{R}^n$ for any $\alpha$, $N$. Thus applying Lemma 4.11 we deduce that:

$$S_k \to \int_{\mathbb{R}^n} \psi(-x)\tau_x \check{\phi} dx, \qquad \text{in } \mathscr{S}.$$

By the continuity of $u$, we deduce that:

$$u\left[\int_{\mathbb{R}^n} \psi(-x)\tau_x \check{\phi} dx\right] = u\left[\lim_{k\to\infty} S_k\right] = \lim_{k\to\infty} u\,[S_k]$$

By the linearity of $u$, we calculate:

$$u\,[S_k] = u\left[\sum_{\pi\in\Pi_k} f(x_\pi, \cdot)\,|\pi|\right] = \sum_{\pi\in\Pi_k} u\,[f(x_\pi, \cdot)]\,|\pi| = \sum_{\pi\in\Pi_k} u\left[\psi(-x_\pi)\tau_{x_\pi}\check{\phi}\right]|\pi|$$

But $x \mapsto u\left[\psi(-x)\tau_x\check{\phi}\right]$ is smooth, hence Riemann integrable, and we have that

$$\lim_{k\to\infty} \sum_{\pi\in\Pi_k} u\left[\psi(-x_\pi)\tau_{x_\pi}\check{\phi}\right]|\pi| = \int_{\mathbb{R}^n} u\left[\psi(-x)\tau_x\check{\phi}\right] dx.$$

□

## 4.4 The Fourier–Laplace transform on $\mathscr{E}'(\mathbb{R}^n)$

Recall that $\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ is a continuously embedded subspace, consisting of the distributions of *compact support*. These distributions are precisely those which extend to continuous linear maps from $\mathscr{E}(\mathbb{R}^n)$ to $\mathbb{C}$ (see Theorem 3.14). For these distributions, we can express the Fourier transform in a very clean fashion.

**Theorem 4.13.** *Suppose that $u \in \mathscr{E}'(\mathbb{R}^n)$. Then $\hat{u} = T_{\hat{v}}$ for some $\hat{v} \in C^\infty(\mathbb{R}^n)$ with:*

$$\hat{v}(\xi) = u[e_{-\xi}].$$

*Proof.* Suppose that $\operatorname{supp} u \subset B_R(0)$. Pick $\psi \in \mathscr{D}(\mathbb{R}^n)$ with $\psi = 1$ on $B_{R+1}(0)$, so that $\psi u = u$. We calculate:

$$\hat{u} = \widehat{\psi u} = \frac{1}{(2\pi)^n} T_{\hat{u}\star\hat{\psi}}.$$

By Theorem 4.10 $e$). Thus we have $\hat{u} = T_{\hat{v}}$ with $\hat{v} = (2\pi)^{-n}\hat{u}\star\hat{\psi} \in C^\infty(\mathbb{R}^n)$, by Theorem 4.10 $a$).

Now let $\phi \in \mathscr{S}$ be such that $\hat{\phi} = \psi$. We calculate:

$$\begin{aligned} \hat{v}(\xi) &= \frac{1}{(2\pi)^n}\hat{u}\star\hat{\psi}(\xi) = \hat{u}\star\check{\phi}(\xi) \\ &= \hat{u}\,[\tau_\xi\phi] = u\left[\widehat{\tau_\xi\phi}\right] = u[e_{-\xi}\psi] = (\psi u)[e_{-\xi}] \\ &= u[e_{-\xi}]. \end{aligned}$$

□

In practice, one does not distinguish between the distribution $\hat{u}$ and the function $\hat{v}$ and one uses the same letter to denote both. Notice that for $u \in \mathscr{E}'(\mathbb{R}^n)$, the expression $u[e_{-z}]$ makes sense for $z \in \mathbb{C}^n$. Moreover, this function is in fact *holomorphic* on $\mathbb{C}^n$. The analytic extension of a Fourier transform from $\mathbb{R}^n$ to $\mathbb{C}^n$ (or a subset thereof) is sometimes called the *Fourier-Laplace* transform.

## 4.5 Periodic distributions and Poisson's summation formula

Recall that the translate of a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ is defined by:

$$\tau_z u[\phi] = u\left[\tau_{-z}\phi\right], \qquad \text{for all } \phi \in \mathscr{D}(\mathbb{R}^n),$$

Bearing this in mind, we can make a very natural definition of what it means to be periodic:

**Definition 4.2.** *We say that a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ is* periodic *if for each $g \in \mathbb{Z}^n$ we have:*

$$\tau_g u = u.$$

**Example 18.** *a) The distribution $u = T_{e_{2\pi g}}$ is periodic for any $g \in \mathbb{Z}^n$. Suppose $g' \in \mathbb{Z}^n$. Then:*

$$\begin{aligned}\tau_{g'} T_{e_{2\pi g}}[\phi] &= T_{e_{2\pi g}}\left[\tau_{-g'}\phi\right] = \int_{\mathbb{R}^n} e^{2\pi i g\cdot y}\phi(y+g')dy \\ &= \int_{\mathbb{R}^n} e^{2\pi i g\cdot(z-g')}\phi(z)dz = e^{-2\pi i g\cdot g'}\int_{\mathbb{R}^n} e^{2\pi i g\cdot z}\phi(z)dz \\ &= T_{e_{2\pi g}}[\phi]\end{aligned}$$

*b) Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$. Then*

$$u = \sum_{g\in\mathbb{Z}^n} \tau_g v$$

*is periodic. It is straightforward to show that $u$ defines a distribution (see Exercise below). To check periodicity, we have, for $g \in \mathbb{Z}^n$:*

$$\tau_{g'}u[\phi] = u[\tau_{-g'}\phi] = \sum_{g\in\mathbb{Z}^n}\tau_g v\left[\tau_{-g'}\phi\right] = \sum_{g\in\mathbb{Z}^n} v\left[\tau_{-g-g'}\phi\right] = \sum_{g\in\mathbb{Z}^n}\tau_{g+g'}v\left[\phi\right] = u[\phi],$$

*where we shift the dummy variable in the sum for the last step.*

**Exercise($*$).** Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$ and let:

$$u = \sum_{g\in\mathbb{Z}^n}\tau_g v.$$

Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp}\phi \subset K$ for some compact $K \subset \mathbb{R}^n$ then

$$u[\phi] = \sum_{g\in A}\tau_g v[\phi],$$

for some finite set $A \subset \mathbb{Z}^n$ which depends only on $K$. Deduce that $u$ defines a distribution.

When dealing with a periodic function $f \in C^\infty(\mathbb{R}^n)$, many quantities of interest are averages over a fundamental cell of the periodic lattice:

$$q = \left\{ x \in \mathbb{R}^n : -\frac{1}{2} \leqslant x_i < \frac{1}{2}, i = 1, \ldots, n \right\}$$

For example:

$$M(f) = \int_q f(x)dx$$

is the mean value that $f$ attains. We'd like to extend this notion to makes sense for periodic functions, but we're presented with a difficulty. The obvious definition would be to set:

$$M(u) \stackrel{!}{=} u[\mathbb{1}_q]$$

but of course $\mathbb{1}_q \notin \mathscr{D}(\mathbb{R}^n)$ so we're not able to do this. Instead we will 'smear out' the function $\mathbb{1}_q$. To do this, notice that a crucial property of $\mathbb{1}_q$ is the following identity:

$$\sum_{g \in \mathbb{Z}^n} \tau_g \mathbb{1}_q = 1,$$

which tells us that $\mathbb{1}_q$ generates a partition of unity.

We shall construct a smooth 'partition of unity', which will allow us to localise various objects, and thus render them easier to deal with, and will enable us to define the mean of a periodic distribution. This is a slightly technical result, but the basic idea is important and crops up in many areas of analysis.

**Lemma 4.14.** *Let*

$$Q = \{x \in \mathbb{R}^n : |x_i| < 1, i = 1, \ldots, n\}$$

*be the cube of side length* 2 *centred at the origin. There exists a function* $\psi \in C^\infty(\mathbb{R}^n)$ *with* $\psi \geqslant 0$ *and* $\operatorname{supp} \psi \subset Q$ *such that:*

$$\sum_{g \in \mathbb{Z}^n} \tau_g \psi = 1.$$

*Suppose that* $u \in \mathscr{D}'(\mathbb{R}^n)$ *is periodic, and* $\psi, \psi'$ *are both as above. Then:*

$$u[\psi] = u[\psi']$$

*We then define:*

$$M(u) := u[\psi]$$

*Proof.* Note

$$\overline{q} = \left\{ x \in \mathbb{R}^n : |x_i| \leqslant \frac{1}{2}, i = 1, \ldots, n \right\}.$$

By Lemma 1.14, there exists a function $\psi_0 \in C_0^\infty(Q)$, with $\psi_0(x) = 1$ for $x \in \overline{q}$ and $\psi_0 \geqslant 0$. Consider:

$$S(x) := \sum_{g \in \mathbb{Z}^n} \psi_0(x - g).$$

For any bounded open set $\Omega$, we have that

$$A = \{g \in \mathbb{Z}^n : (\Omega - g) \cap Q \neq \emptyset\}$$

is finite. For $x \in \Omega$, we have:

$$S(x) = \sum_{g \in A} \psi_0(x - g),$$

so $S(x)$ is smooth. Moreover, for each $x \in \mathbb{R}^n$, there is at least one $g \in \mathbb{Z}^n$ with $x - g \in \overline{q}$. Thus $S(x) \geqslant 1$. We can thus take:

$$\psi(x) = \frac{\psi_0(x)}{S(x)}.$$

This is smooth, positive, supported in $Q$ and moreover:

$$\sum_{g \in \mathbb{Z}^n} \tau_g \psi(x) = \frac{1}{S(x)} \sum_{g \in \mathbb{Z}^n} \psi_0(x - g) = 1.$$

Now suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic and $\psi, \psi'$ are both partitions of unity as above. We calculate:

$$\begin{aligned} u[\psi] &= u\left[\psi \sum_{g \in \mathbb{Z}^n} \tau_g \psi'\right] = \sum_{g \in \mathbb{Z}^n} u\left[\psi \tau_g \psi'\right] \\ &= \sum_{g \in \mathbb{Z}^n} \tau_{-g} u\left[\tau_{-g} \psi \psi'\right] = u\left[\psi' \sum_{g \in \mathbb{Z}^n} \tau_{-g} \psi\right] = u[\psi'] \end{aligned}$$

□

We thus have an acceptable definition of the mean of a periodic distribution. Notice that if $u = T_f$ for some locally integrable periodic function $f$, then by choosing a bounded sequence of $\psi_j$'s such that $\psi_j \to \mathbb{1}_{\overline{q}}$ pointwise, we can show that:

$$M(T_f) = \int_{\overline{q}} f(x) dx,$$

justifying calling $M$ the mean of the distribution.

To see why this technical lemma is useful, let us apply it to show that a periodic distribution is necessarily tempered, and in fact the periodic distributions we found by translating a compact distribution are indeed all of the periodic distributions.

**Lemma 4.15.** *Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$ is a compact distribution. Then:*

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v \tag{4.8}$$

*converges in $\mathscr{S}'$. Conversely, suppose that $u \in \mathscr{D}'(\mathbb{R}^n)$ is a periodic distribution. Then there exists $v \in \mathscr{E}'(\mathbb{R}^n)$ such that (4.8) holds and thus $u$ extends uniquely to a tempered distribution $u \in \mathscr{S}'$.*

*Proof.* Let $K = \operatorname{supp} v$. Since $v \in \mathscr{E}'(\mathbb{R}^n)$, by Lemma 3.13 there exists $C > 0, N$ such that:

$$|v[\phi]| \leqslant C \sup_{x\in K;|\alpha|\leqslant N} |D^\alpha \phi(x)|, \qquad \text{for all } \phi \in \mathscr{E}(\mathbb{R}^n).$$

Now suppose $\phi \in \mathscr{S} \subset \mathscr{E}(\mathbb{R}^n)$. We have:

$$|\tau_g v\,[\phi]| = |v\,[\tau_{-g}\phi]| \leqslant C \sup_{x\in K;|\alpha|\leqslant N} |D^\alpha \phi(x+g)|.$$

Since $K$ is bounded, we have that $K \subset B_R(0)$ for some $R > 0$. We calculate:

$$1 + |g| = 1 + |x + g - x| \leqslant 1 + R + |x+g| \leqslant (1+R)(1+|x+g|)$$

for all $x \in K$, so that:

$$1 \leqslant (1+R)\frac{1+|x+g|}{1+|g|}.$$

We conclude that for any $M \geqslant 1$:

$$\begin{aligned}
|\tau_g v\,[\phi]| &\leqslant \frac{C(1+R)^M}{(1+|g|)^M} \sup_{x\in K;|\alpha|\leqslant N} (1+|x+g|)^M |D^\alpha\phi(x+g)| \\
&\leqslant \frac{C(1+R)^M}{(1+|g|)^M} \sup_{y\in \mathbb{R}^n;|\alpha|\leqslant N} (1+|y|)^M |D^\alpha\phi(y)|.
\end{aligned}$$

Since $\phi \in \mathscr{S}$, in particular we have:

$$|\tau_g v\,[\phi]| \leqslant \frac{C'}{(1+|g|)^{n+1}}$$

where $C'$ depends on $v$, $\phi$. Now, since:

$$\sum_{g\in\mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty,$$

(see Exercise below) we deduce that for each $\phi \in \mathscr{S}$ the sum:

$$\sum_{g\in\mathbb{Z}^n} \tau_g v\,[\phi]$$

converges. This is precisely the statement that the sum in (4.8) converges in $\mathscr{S}'$.

Now suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic, and take $\psi$ as in Lemma 4.14. Suppose $\phi \in \mathscr{D}(\mathbb{R}^n)$ is arbitrary. We have:

$$u[\phi] = \left(\sum_{g\in\mathbb{Z}^n} \tau_g\psi\right) u[\phi] = \sum_{g\in\mathbb{Z}^n} u\,[\tau_g\psi\phi]. \tag{4.9}$$

Now, since $u$ is periodic,:

$$u\,[\tau_g\psi\phi] = \tau_g u\,[\tau_g\psi\phi] = u\,[\psi\tau_{-g}\phi] = (\psi u)\,[\tau_{-g}\phi] = \tau_g(\psi u)[\phi]$$

Now $\psi u$ has compact support, so by Theorem 3.14 extends uniquely to $v \in \mathscr{E}'(\mathbb{R}^n)$. Thus we have:

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v$$

which by the first part of the proof converges in $\mathscr{S}'$, thus $u \in \mathscr{S}'$. $\square$

**Exercise($*$).** Recall that for $x \in \mathbb{R}^n$:

$$\|x\|_1 := \sum_{i=1}^{n} |x_i| .$$

For $k \in \mathbb{N}$ set:

$$Q_k = \left\{ g \in \mathbb{Z}^n : k - \frac{1}{2} \leqslant \|g\|_1 < k + \frac{1}{2} \right\}$$

a) Show that:

$$\#Q_k = (2k+1)^n - (2k-1)^n$$

so that $\#Q_k \leqslant c(1+k)^{n-1}$ for some $c > 0$.

b) By applying the Cauchy-Schwarz identity to estimate $a \cdot b$ for $a = (1, \ldots, 1)$ and $b = (|g_1|, \ldots, |g_n|)$, deduce that:

$$\|g\|_1 \leqslant \sqrt{n}\, |g|$$

c) Show that there exists a constant $C > 0$, depending only on $n$ such that:

$$\sum_{g \in \mathbb{Z}^n; \|g\|_1 \leqslant K} \frac{1}{(1+|g|)^{n+1}} \leqslant 1 + C \sum_{k=1}^{K} \frac{1}{k^2}$$

holds for all $K \in \mathbb{N}$. Deduce that:

$$\sum_{g \in \mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty.$$

We have shown that a periodic distribution is necessarily tempered. It is therefore reasonable to ask what one can say about the Fourier transform of a periodic distribution. In fact, it will turn out that the Fourier transform of a periodic distribution has a very simple form: it consists of a sum over $\delta$-distributions with support on the points of an integer lattice.

Before we establish this, we will first need a technical result regarding distributions.

**Lemma 4.16.** *Suppose that $u \in \mathscr{S}$ satisfies:*

$$\left(e_{-g'} - 1\right) u = 0 \tag{4.10}$$

*for all $g' \in \mathbb{Z}^n$. Then:*

$$u = \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g},$$

*where* $c_g \in \mathbb{C}$ *satisfy the bound*

$$|c_g| \leqslant K(1+|g|)^N$$

*for some* $K > 0$, $N \in \mathbb{Z}$, *and the sum converges in* $\mathscr{S}'$.

*Proof.* First, we claim that $\operatorname{supp} u \subset \Lambda$, with

$$\Lambda = \{2\pi g : g \in \mathbb{Z}^n\}.$$

Suppose $\phi \in \mathscr{D}(R^n)$ with $\operatorname{supp}\phi \subset \mathbb{R}^n\backslash\Lambda$. Then for each $g' \in \mathbb{Z}^n$, we have $\left(e_{-g'}-1\right)^{-1}\phi \in \mathscr{S}$, since $\phi$ vanishes near any zeros of $e_{-g'}-1$. Applying the condition (4.10), we deduce:

$$0 = \left(e_{-g'}-1\right) u \left[\left(e_{-g'}-1\right)^{-1}\phi\right] = u[\phi]$$

so $u$ vanishes. Thus $\operatorname{supp} u \subset \Lambda$.

Now, let us take $\psi$ as in Lemma 4.14, and define $\tilde{\psi}(x) = \psi\left(\frac{x}{2\pi}\right)$. It's straightforward to check that:

$$\sum_{g\in\mathbb{Z}^n} \tau_{2\pi g}\tilde{\psi} = 1, \qquad \operatorname{supp}\tilde{\psi} \subset \{x \in \mathbb{R}^n : |x_i| < 2\pi\}.$$

For $g \in \mathbb{Z}^n$, let us consider $v_g = (\tau_{2\pi g}\tilde{\psi})u$. This distribution is supported at $2\pi g$, and by multiplying (4.10) by $\tau_{2\pi g}\tilde{\psi}$ we have:

$$\left(e_{-g'}-1\right) v_g = 0$$

In particular, we have, taking $g' = l_j$ for $j = 1, \ldots n$, where $\{l_j\}$ is the canonical basis for $\mathbb{R}^n$:

$$\left(e^{-i(x_j-2\pi g_j)}-1\right) v_g = 0.$$

Now,

$$\left(e^{-i(x_j-2\pi g_j)}-1\right) = (x_j-2\pi g_j)\kappa(x_j)$$

where $\kappa(x_j)$ is non-zero on a neighbourhood of $g_j$. Thus we conclude that:

$$(x_j - 2\pi g_j)v_g = 0, \qquad j = 1, \ldots n.$$

Now suppose $\phi \in \mathscr{S}$. We can write:

$$\phi(x) = \phi(2\pi g) + \sum_{j=1}^{n}(x_j - 2\pi g_j)\phi_j(x)$$

where $\phi_j(x) \in C^\infty(\mathbb{R}^n)$. Since $v_g$ has compact support, it extends to smoothly to act on $\mathscr{E}(\mathbb{R}^n)$ and we calculate:

$$v_g[\phi] = v_g[\phi(2\pi g)] + \sum_{j=1}^{n}(x_j - 2\pi g_j)v_g[\phi_j] = v_g[\phi(2\pi g)]$$

Returning to the definition of $v_g$, we have:

$$(\tau_{2\pi g}\tilde{\psi})u[\phi] = (\tau_{2\pi g}\tilde{\psi})u[\phi(2\pi g)] = u[\tau_{2\pi g}\tilde{\psi}]\delta_{2\pi g}[\phi]$$

so that

$$(\tau_{2\pi g}\tilde{\psi})u = u[\tau_{2\pi g}\tilde{\psi}]\delta_{2\pi g}$$

Summing over $g \in \mathbb{Z}^n$, we recover:

$$\sum_{g\in\mathbb{Z}^n}(\tau_{2\pi g}\tilde{\psi})u = \left(\sum_{g\in\mathbb{Z}^n}(\tau_{2\pi g}\tilde{\psi})\right)u = u = \sum_{g\in\mathbb{Z}^n} c_g\delta_{2\pi g}$$

Where

$$c_g = u[\tau_{2\pi g}\tilde{\psi}].$$

To establish the estimate for $c_g$, we recall from Lemma 3.15 that there exist $N, k \in \mathbb{N}$ and $C > 0$ such that:

$$|u[\phi]| \leqslant C \sup_{x\in\mathbb{R}^n;|\alpha|\leqslant k}\left|(1+|x|)^N D^\alpha\phi(x)\right|, \qquad \text{for all } \phi \in \mathscr{S}.$$

Applying this to $\tau_{2\pi g}\tilde{\psi}$, we have:

$$\begin{aligned}
|c_g| &\leqslant C \sup_{x\in\mathbb{R}^n;|\alpha|\leqslant k}\left|(1+|x|)^N D^\alpha\tilde{\psi}(x-2\pi g)\right| \\
&\leqslant C \sup_{x\in\mathbb{R}^n;|\alpha|\leqslant k}\left|(1+|x+2\pi g|)^N D^\alpha\tilde{\psi}(x)\right| \\
&\leqslant C' \sup_{x\in\mathbb{R}^n;|\alpha|\leqslant k}\left|(1+|x|)^N D^\alpha\tilde{\psi}(x)\right| \times (1+|g|)^N \\
&\leqslant K(1+|g|)^N
\end{aligned}$$

With this bound, it is a straightforward exercise to verify that the sum converges in $\mathscr{S}'$. □

**Exercise($*$).** Show that if $c_g$ satisfy:

$$|c_g| \leqslant K(1+|g|)^N$$

for some $K > 0$ and $N \in \mathbb{N}$, then:

$$\sum_{g\in\mathbb{Z}^n} c_g\delta_{2\pi g}$$

converges in $\mathscr{S}'$.

This now enables us to establish a result regarding the Fourier series of a periodic distribution.

**Theorem 4.17.** *Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is a periodic distribution. Then there exist constants $c_g \in \mathbb{C}$ such that:*

$$u = \sum_{g \in \mathbb{Z}^n} c_g T_{e_{2\pi g}}.$$

*with $c_g$ are given by:*

$$c_g = M(e_{-2\pi g} u).$$

*and satisfy the bound:*

$$|c_g| \leqslant K(1 + |g|)^N \tag{4.11}$$

*for some $K > 0$, $N \in \mathbb{Z}$.*

*Proof.* Since $u$ is periodic, it is tempered by Lemma 4.15. Thus we may take the Fourier transform. Noting that:

$$\tau_{g'} u = u$$

for all $g' \in \mathbb{Z}^n$, we have that

$$e_{-g'} \hat{u} = \hat{u} \qquad \Longrightarrow \qquad (e_{-g'} - 1)\hat{u} = 0.$$

By Lemma 4.16, we deduce that:

$$\hat{u} = (2\pi)^n \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g},$$

for some $c_g$ satisfying (4.11), where the sum converges in $\mathscr{S}'$. We can apply the inverse Fourier transform, making use of the fact that it is continuous on $\mathscr{S}'$ to deduce:

$$u = \sum_{g \in \mathbb{Z}^n} c_g T_{e_{2\pi g}},$$

with convergence again in $\mathscr{S}'$. To establish the formula for $c_g$, we make use of the comments after Lemma 4.14 to note that:

$$M(e_{-2\pi g} T_{e_{2\pi g'}}) = \int_{\overline{q}} e^{2\pi i (g - g')\cdot x} dx = \delta_{gg'}$$

Since $u \mapsto M(e_{-2\pi g} u)$ is a continuous map from $\mathscr{S}'$ to $\mathbb{C}$, we deduce that:

$$M(e_{-2\pi g} u) = \sum_{g' \in \mathbb{Z}^n} c_g M(e_{-2\pi g} T_{e_{2\pi g'}}) = c_{g'}.$$

□

**Remark.** *Usually one writes the Fourier series for $u$ as:*

$$u = \sum_{g \in \mathbb{Z}^n} c_g e_{2\pi g},$$

*ignoring the distinction between the function $e_{2\pi g}$ and the distribution it defines.*

As a simple example, let us consider the distribution:

$$u = \sum_{g \in \mathbb{Z}^n} \delta_g.$$

By Lemma 4.15, this defines a periodic distribution, since $\delta_g = \tau_g \delta_0$ and $\delta_0 \in \mathscr{E}'(\mathbb{R}^n)$. Notice also that if $\psi$ satisfies the conditions of Lemma 4.14, then since $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^n : |x_j| < 1\}$, we have that $\tau_g \psi(0) = 0$ for $g \in \mathbb{Z}^n$ with $g \neq 0$. Thus, since $\sum_{g \in \mathbb{Z}^n} \tau_g \psi = 1$, we must have $\psi(0) = 1$. We can then calculate:

$$c_g = M(e_{-2\pi g} u) = u[\psi e_{-2\pi g}] = \psi(0) e^{-2\pi i g \cdot 0} = 1.$$

Thus we have established *Poisson's formula*:

$$\sum_{g \in \mathbb{Z}^n} \delta_g = \sum_{g \in \mathbb{Z}^n} T_{e_{2\pi g}},$$

where we understand both sums to converge in $\mathscr{S}'$. This is sometimes written, with an abuse of notation:

$$\sum_{g \in \mathbb{Z}^n} \delta(x-g) = \sum_{g \in \mathbb{Z}^n} e^{2\pi g \cdot x i}$$

We can specialise various results concerning Fourier transforms to the case of Fourier series.

**Corollary 4.18.** *i) Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic and may be written as:*

$$u = \sum_{g \in \mathbb{Z}^n} c_g T_{e_{2\pi g}}.$$

*Then $D_j u \in \mathscr{D}'(R^n)$ is periodic and has Fourier series:*

$$D_j u = \sum_{g \in \mathbb{Z}^n} (2\pi i g_j c_g) T_{e_{2\pi g}}.$$

*ii) Suppose $f \in L^1_{loc.}(\mathbb{R}^n)$, then:*

$$|c_g| \leqslant \|f\|_{L^1(q)},$$

*and moreover, $c_g \to 0$ as $|g| \to \infty$.*

*iii) Suppose $f \in C^{n+1}(\mathbb{R}^n)$ is periodic. Then:*

$$f(x) = \sum_{g \in \mathbb{Z}^n} c_g e^{2\pi i g \cdot x}$$

*with the sum converging uniformly.*

*iv) Suppose $f, h \in L^2_{loc.}(\mathbb{R}^n)$ are periodic with Fourier coefficients $f_g, h_g$ respectively. Then:*

$$\int_q \overline{f}(x)h(x)dx = \sum_{g\in\mathbb{Z}^n} \overline{f}_g h_g.$$

*This is the Fourier series version of Parseval's formula. Moreover,*

$$f(x) = \sum_{g\in\mathbb{Z}^n} c_g e^{2\pi i g\cdot x}$$

*holds, with the sum converging in $L^2(q)$.*

*Proof.* i) Since the Fourier series for $u$ converges in $\mathscr{S}'$, we may differentiate term by term (as differentiation is a continuous operation from $\mathscr{S}'$ to itself). Since

$$D_j T_{e_{2\pi g}} = (2\pi i g_j) T_{e_{2\pi g}},$$

the result follows.

ii) Note that if $f \in L^1_{loc.}(\mathbb{R}^n)$, then:

$$|c_g| = \left| \int_q e^{-2\pi i g\cdot x} f(x) dx \right| \leqslant \int_q |f(x)|\, dx = \|f\|_{L^1(q)}\,.$$

Now, given $\epsilon > 0$, we can approximate[4] $f$ by a smooth periodic function $f_\epsilon$, with Fourier coefficients $c'_g$, such that

$$\|f - f_\epsilon\|_{L^1(q)} < \frac{\epsilon}{2}.$$

Since $D_j D_j f_\epsilon \in L^1_{loc.}(\mathbb{R}^n)$, we have that $|g|^2 \left|c'_g\right| < C$, for each $j = 1, \dots, n$ so there exists $R > 0$ such that $\left|c'_g\right| < \frac{\epsilon}{2}$ for $|g| > R$. We have:

$$\left|c_g - c'_g\right| \leqslant \|f - f_\epsilon\|_{L^1(q)} < \frac{\epsilon}{2},$$

so we conclude that for $|g| > R$:

$$|c_g| = \left|c_g - c'_g + c'_g\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $c_g \to 0$ as $|g| \to \infty$.

iii) Since $f \in C^{k+1}(\mathbb{R}^n)$, we have that $D^\alpha f \in L^1_{loc.}(\mathbb{R}^n)$ for $|\alpha| < n+1$. Applying the previous two results we conclude that $|c_g| \leqslant K(1 + |g|)^{-n+1}$ for some $K > 0$. Thus the partial sums:

$$F_n(x) = \sum_{g\in\mathbb{Z}^n, |g|\leqslant n} c_g e^{2\pi i g\cdot x}$$

[4] See Exercise 4.5

converge uniformly to some continuous function $F$ by the Weierstrass $M-$test. We have:

$$T_f = \lim_{n\to\infty} \sum_{g\in\mathbb{Z}^n, |g|\leqslant n} c_g T_{e_{2\pi g}} = \lim_{n\to\infty} T_{F_n} = T_F$$

since uniform convergence implies convergence in $\mathscr{S}'$. By the injectivity of the mapping between continuous functions and distributions we conclude $f = F$.

iv) Suppose $f, h \in C^\infty(\mathbb{R}^n)$ are periodic. Then:

$$f(x) = \sum_{g\in\mathbb{Z}^n} f_g e^{2\pi i g\cdot x}, \qquad h(x) = \sum_{g\in\mathbb{Z}^n} h_g e^{2\pi i g\cdot x}$$

with $\sup_{g\in\mathbb{Z}^n}(1+|g|)^N |f_g| < \infty$ for all $N\in\mathbb{N}$, and similarly for $h_g$. We calculate:

$$\begin{aligned}
\int_q \overline{f}(x)g(x)dx &= \int_q \left(\sum_{g\in\mathbb{Z}^n} \overline{f}_g e^{-2\pi i g\cdot x}\right)\left(\sum_{g'\in\mathbb{Z}^n} h_{g'} e^{2\pi i g'\cdot x}\right) dx \\
&= \sum_{g\in\mathbb{Z}^n}\sum_{g'\in\mathbb{Z}^n} \overline{f}_g h_{g'} \int_q e^{2\pi i (g'-g)\cdot x} dx \\
&= \sum_{g\in\mathbb{Z}^n}\sum_{g'\in\mathbb{Z}^n} \overline{f}_g h_{g'}\delta_{gg'} = \sum_{g\in\mathbb{Z}^n} \overline{f}_g h_g .
\end{aligned}$$

In particular, we have that:

$$\|f\|_{L^2(q)} = \|f_g\|_{\ell^2(\mathbb{Z}^n)},$$

where for a sequence $\{a_g\}_{g\in\mathbb{Z}^n}$, we define:

$$\|a_g\|_{\ell^2(\mathbb{Z}^n)} = \left(\sum_{g\in\mathbb{Z}^n} |a_g|^2\right)^{\frac{1}{2}}.$$

Now suppose $f \in L^2_{loc.}(\mathbb{R}^n)$. Given $k>0$, we can find $f^{(k)} \in C^\infty(\mathbb{R}^n)$ with Fourier coefficients $f_g^{(k)}$ such that:

$$\left\| f - f^{(k)} \right\|_{L^2(q)} < \frac{1}{k}.$$

Since by Cauchy-Schwarz we have:

$$\|f\|_{L^1(q)} = \int_q |f(x)|\, dx \leqslant \left(\int_q |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_q dx\right)^{\frac{1}{2}} = \|f\|_{L^2(q)}$$

we have that:

$$\left\| f_g - f_g^{(k)} \right\|_{\ell^\infty(\mathbb{Z}^n)} := \sup_{g\in\mathbb{Z}^n} \left| f_g - f_g^{(k)} \right| < \frac{1}{k},$$

Now, $f^{(k)}$ is a Cauchy sequence in $L^2(q)$, so $\{f_g^{(k)}\}$ is a Cauchy sequence in $\ell^2(\mathbb{Z}^n)$. We conclude that $f_g^{(k)}$ converges in $\ell^2(\mathbb{Z}^n)$, however we also know that $f^{(k)} \to f$

in $\ell^\infty(\mathbb{Z}^n)$, thus we must have $f^{(k)} \to f$ in $\ell^2(\mathbb{Z}^n)$. Taking a similar sequence of $h^{(k)} \in C^\infty(\mathbb{R}^n)$ approximating $h$ with Fourier coefficients $h_g^{(k)}$, and recalling that:

$$\left(f^{(k)}, h^{(k)}\right)_{L^2(q)} = \left(f^{(k)}, h^{(k)}\right)_{\ell^2(\mathbb{Z}^n)},$$

the result follows on sending $k \to \infty$. The convergence of the Fourier series in $L^2(q)$ follows by showing that the partial sums form a Cauchy sequence in $L^2(q)$.

□

**Exercise(∗).** Suppose $f \in L^p_{loc.}(\mathbb{R}^n)$ is a periodic function. Fix $\epsilon > 0$, and let:

$$Q = \{x \in \mathbb{R}^n : |x_j| < 1, j = 1, \ldots, n\}, \qquad q = \left\{x \in \mathbb{R}^n : |x_j| < \frac{1}{2}, j = 1, \ldots, n\right\}$$

a) Show that there exists $h_\epsilon \in C^\infty(\mathbb{R}^n)$ with:

$$\operatorname{supp} h_\epsilon \subset Q$$

such that:

$$\|f\mathbb{1}_q - h_\epsilon\|_{L^p(\mathbb{R}^n)} < \epsilon.$$

Define

$$f_\epsilon = \sum_{g \in \mathbb{Z}^n} \tau_g h_\epsilon$$

b) Show that $f_\epsilon$ is smooth and periodic.

c) Show that there exists a constant $c_n$ depending only on $n$ such that:

$$\|f - f_\epsilon\|_{L^p(q)} < c_n \epsilon.$$

**Exercise(∗).** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = x \text{ for } |x| < \frac{1}{2}, \qquad f(x+1) = f(x).$$

Show that:

$$f(x) = \sum_{n \in \mathbb{Z}, n \neq 0} \frac{i(-1)^n}{2\pi n} e^{2\pi i n x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n x),$$

with convergence in $L^2_{loc.}(\mathbb{R})$.

**Exercise(∗).** Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = \begin{cases} -1 & -\frac{1}{2} < x \leqslant 0 \\ 1 & 0 < x \leqslant \frac{1}{2} \end{cases}, \qquad f(x+1) = f(x).$$

a) Show that:

$$f(x) = \frac{1}{\pi i}\sum_{n=-\infty}^{\infty}\frac{2}{2n+1}e^{2\pi i(2n+1)x} = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left[2\pi(2n+1)x\right]$$

With convergence in $L^2_{loc.}(\mathbb{R}^n)$.

Define the partial sum:

$$S_N(x) = 8\sum_{n=0}^{N-1}\frac{1}{2\pi(2n+1)}\sin\left[2\pi(2n+1)x\right].$$

b) Show that:

$$S_N(x) = 8\int_0^x\sum_{n=0}^{N-1}\cos\left[2\pi(2n+1)t\right]dt.$$

c) Show that:

$$\cos\left[2\pi(2n+1)t\right]\sin 2\pi t = \frac{1}{2}\left(\sin\left[2\pi(2n+2)t\right] - \sin\left[4\pi nt\right]\right)$$

And deduce:

$$S_N(x) = 8\int_0^x\frac{\sin 4\pi Nt}{2\sin 2\pi t}dt.$$

d) Show that the first local maximum of $S_N$ occurs at $x = \frac{1}{4N}$, and:

$$S_N\left(\frac{1}{4N}\right) \geqslant 8\int_0^{\frac{1}{4N}}\frac{\sin 4\pi Nt}{4\pi t}dt = \frac{2}{\pi}\int_0^{\pi}\frac{\sin s}{s}ds \simeq 1.179\ldots$$

e) Conclude that the sum in part $a)$ does not converge uniformly.

This lack of uniform convergence of a Fourier series at a point of discontinuity is known as Gibbs Phenomenon.

**Exercise($*$).** (*) Suppose that $\lambda = \{\lambda_1, \ldots \lambda_n\}$ is a basis for $\mathbb{R}^n$. We define the lattice generated by $\lambda$ to be:

$$\Lambda = \left\{\sum_{j=1}^{n} z_j\lambda_j : z_j \in \mathbb{Z}\right\}.$$

Define the fundamental cell:

$$q_\Lambda = \left\{\sum_{j=1}^{n} x_j\lambda_j : |x_j| < \frac{1}{2}\right\}.$$

We say that $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic if:

$$\tau_g u = u \qquad \text{for all } g \in \Lambda.$$

a) Show that there exists $\psi \in C_0^\infty(2q_\Lambda)$ such that $\psi \geqslant 0$ and

$$\sum_{g\in\Lambda} \tau_g \psi = 1.$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic and $\psi$, $\psi'$ are both as in part $a$), then

$$\frac{1}{|q_\Lambda|} u[\psi] = \frac{1}{|q_\Lambda|} u[\psi'] =: M(u)$$

c) Define the *dual lattice* by:

$$\Lambda^* := \{x \in \mathbb{R}^n : g \cdot x \in 2\pi\mathbb{Z}, \ \forall g \in \Lambda\}$$

Show that there exists a basis $\lambda^* = \{\lambda_1^*, \dots \lambda_n^*\}$ such that $\lambda_j^* \cdot \lambda_k = \delta_{jk}$, and $\Lambda^*$ is the lattice induced by $\lambda^*$.

d) Show that if $g \in \Lambda^*$ then $e_g$ is $\Lambda-$periodic.

e) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$\hat{u} = \sum_{g\in\Lambda^*} c_g \delta_g$$

for some $c_g \in \mathbb{C}$ satisfying $|c_g| \leqslant K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$.

f) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$u = \sum_{g\in\Lambda^*} d_g T_{e_g}$$

where $|d_g| \leqslant K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$ are given by:

$$d_g = M(e_{-g}u)$$

## 4.6 Sobolev spaces

### 4.6.1 The spaces $W^{k,p}(\Omega)$

Suppose $\Omega \subset \mathbb{R}^n$ is an open set. For $k \in \mathbb{Z}_{\geqslant 0}$ and $1 \leqslant p \leqslant \infty$, we say that $f \in L^p(\Omega)$ belongs to the Sobolev space $W^{k,p}(\Omega)$ if for any $|\alpha| \leqslant k$ there exists $f^\alpha \in L^p(\Omega)$ with:

$$D^\alpha T_f = T_{f^\alpha}.$$

We call $f^\alpha$ the weak, or distributional derivative of $f$ and write $D^\alpha f := f^\alpha$. We can equip $W^{k,p}(\Omega)$ with the norm:

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{|\alpha|\leqslant k} \|D^\alpha f\|_{L^p(\Omega)}.$$

With this norm, $W^{k,p}(\Omega)$ is complete, and hence a Banach space. The Sobolev spaces are particularly well suited to the study of PDE, and form the starting point for many modern PDE investigations.

We can think of $k$ as telling us how *differentiable* our function is, while $p$ tells us how *integrable* our function is. Roughly speaking spaces with larger $k$ contain smoother functions, while spaces with larger $p$ contain less 'spiky' functions. We shall see that (roughly speaking) one can trade smoothness for integrability: a function that belongs to $W^{k,p}(\mathbb{R}^n)$ belongs to certain $W^{l,q}(\mathbb{R}^n)$ where $l < k$ and $p > q$. If $k$ and $p$ are large enough we can even conclude that the function must be classically differentiable.

We will frame the result as concerning the embedding of $W^{k,p}(\mathbb{R}^n)$ spaces. Recall that a Banach space $(X, \|\cdot\|_X)$ is said to embed continuously into the Banach space $(Y, \|\cdot\|_Y)$ if $X \subset Y$ and there exists a constant $C$ such that:

$$\|x\|_Y \leqslant C \|x\|_X , \qquad \text{for all } x \in X.$$

**Theorem 4.19** (Sobolev embedding theorem)**.** *Suppose $k > l$ and $1 \leqslant p < q < \infty$ satisfy $(k - l)p < n$ and:*

$$\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n}.$$

*Then $W^{k,p}(\mathbb{R}^n)$ embeds continuously into $W^{l,q}(\mathbb{R}^n)$.*

*If $kp > n$, then $W^{k,p}(\mathbb{R}^n)$ embeds continuously into the Hölder space $C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\mathbb{R}^n)$, where $[x]$ is the largest integer less than or equal to $x$, and*

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p} & \frac{n}{p} \notin \mathbb{Z}, \\ \textit{any element of } (0,1) & \frac{n}{p} \in \mathbb{Z}. \end{cases}$$

Here we have introduced the Hölder space $C^{m,\kappa}(\mathbb{R}^n)$ which consists of $f \in C^m(\mathbb{R}^n)$ such that:

$$\|f\|_{C^{m,\kappa}(\mathbb{R}^n)} := \sum_{\alpha \leqslant m} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| + \sum_{\alpha = m} \sup_{x,y \in \mathbb{R}^n} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\kappa} < \infty.$$

We shan't attempt to prove the general Sobolev embedding theorems, but will establish a special case later on.

### 4.6.2 The space $H^s(\mathbb{R}^n)$

We shall immediately specialise to the case $p = 2$ and $\Omega = \mathbb{R}^n$. This is an important special case for two reasons. Firstly, $W^{k,2}(\Omega)$ is a Hilbert space (in addition to being a Banach space), and so carries additional structure. Secondly, $W^{k,2}(\mathbb{R}^n)$ is very well adapted to the Fourier transform. To see this, we recall that if $f \in L^2(\mathbb{R}^n)$, then:

$$\widehat{T_f} = T_{\hat{f}}$$

where $\hat{f} \in L^2(\mathbb{R}^n)$ is the Fourier-Plancherel transform of $f$. We immediately obtain an alternative characterisation of the space $W^{k,2}(\mathbb{R}^n)$. A function $f \in L^2(\mathbb{R}^n)$ belongs to

$W^{k,2}(\mathbb{R}^n)$ if and only if:

$$\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^k \left|\hat{f}(\xi)\right|^2 d\xi < \infty.$$

Notice that in this characterisation there is no need to restrict $k$ to be an integer, nor in fact for $f$ to belong to $L^2(\mathbb{R}^n)$. This motivates the following definition. For $s \in \mathbb{R}$ we say that $f \in \mathcal{S}'$ belongs to the space $H^s(\mathbb{R}^n)$ provided $\hat{f} \in L^2_{loc.}(\mathbb{R}^n)$ and:

$$\|f\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} < \infty.$$

$H^s(\mathbb{R}^n)$ is complete, and moreover is a Hilbert space. We see that if $k \in \mathbb{Z}_{\geqslant 0}$ then $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$, where we make the canonical identification between a functions $f \in L^2(\mathbb{R}^n)$ and the distribution $T_f \in \mathscr{S}'(\mathbb{R}^n)$. From now on, we shall use $f$ to mean both the function and the distribution.

**Exercise 4.1.** Let $s \in \mathbb{R}$.

a) Show that $\mathscr{S}$ is a dense subset of $H^s(\mathbb{R}^n)$.

b) Find a condition on $s$ such that $\delta_x \in H^s(\mathbb{R}^n)$.

c) Show that $H^t(\mathbb{R}^n)$ is continuously embedded in $H^s(\mathbb{R}^n)$ for $s < t$.

d) Show that the derivative $D^\alpha$ is a bounded linear map from $H^{s+k}(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$, where $k = |\alpha|$.

e) (*) Show that the pairing $\langle , \rangle : H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{C}$, which acts on $f \in H^{-s}(\mathbb{R}^n), g \in H^s(\mathbb{R}^n)$ by

$$\langle f, g\rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi) d\xi$$

is well defined, and show that the map $g \mapsto \langle f, g\rangle$ is a bounded linear operator on $H^s(\mathbb{R}^n)$. Deduce that $H^s(\mathbb{R}^n)'$ may be identified with $H^{-s}(\mathbb{R}^n)$. How does this relate to your answer to part b)?

### 4.6.3 Sobolev Embedding

An important feature of the Sobolev spaces $H^s(\mathbb{R}^n)$ is that for $s$ sufficiently large, they embed into $C^k(\mathbb{R}^n)$. More previsely:

**Theorem 4.20.** *Fix $k \in \mathbb{Z}_{\geqslant 0}$. Suppose that $f \in H^s(\mathbb{R}^n)$ for some $s > k + \frac{n}{2}$, then (possibly after redefinition on a set of measure zero) $f \in C^k(\mathbb{R}^n)$. That is, we have:*

$$H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n).$$

*Proof.* First suppose $f \in \mathscr{S}(\mathbb{R}^n)$. Then by the Fourier inversion theorem we have for $|\alpha| \leqslant k$:

$$D^\alpha f(x) = \frac{i^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{f}(\xi) d\xi.$$

We estimate with the Cauchy-Schwarz inequality:

$$\begin{aligned} |D^\alpha f(x)| &= \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{f}(\xi) d\xi \right| \\ &\leqslant \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| \xi^\alpha \hat{f}(\xi) \right| d\xi \\ &\leqslant \frac{1}{(2\pi)^n} \left( \int_{\mathbb{R}^n} (1+|\xi|^2)^s \left| \hat{f}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \frac{|\xi^\alpha|^2}{(1+|\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \end{aligned}$$

Now, since $|\xi^\alpha|^2 \leqslant c_k(1+|\xi|^2)^k$ for some $c_k > 0$, we have that:

$$\frac{1}{(2\pi)^n} \left( \int_{\mathbb{R}^n} \frac{|\xi^\alpha|^2}{(1+|\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \leqslant \frac{c_k}{(2\pi)^n} \left( \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^{s-k}} d\xi \right)^{\frac{1}{2}} =: C_{n,k,s} < \infty$$

where we have used $s > k + \frac{n}{2}$ in order to ensure that the integral converges. We thus have that:

$$\sup_{|\alpha|\leqslant k, x\in\mathbb{R}^n} |D^\alpha f(x)| \leqslant C_{n,k,s} \|f\|_{H^s(\mathbb{R}^n)}. \tag{4.12}$$

Now suppose $f \in H^s(\mathbb{R}^n)$. We can approximate $f$ by a sequence $(f_m)_{m=1}^\infty$ with $f_m \in \mathscr{S}(\mathbb{R}^n)$ and $f_m \to f$ in $H^s(\mathbb{R}^n)$ and pointwise almost everywhere. In particular, $(f_m)$ is Cauchy in $H^s(\mathbb{R}^n)$, so by the estimate (4.12) applied to $f_m - f_l$ we have that $(f_m)$ is Cauchy in $C^k(\mathbb{R}^n)$, thus there exists $f^* \in C^k(\mathbb{R}^n)$ such that $D^\alpha f_m \to D^\alpha f^*$ uniformly for all $|\alpha| \leqslant k$. Since $f_m \to f$ pointwise almost everywhere, we deduce that $f = f^*$ almost everywhere. □

**Exercise 4.2.** a) Suppose $s = \frac{n}{2} + \gamma$ for some $0 < \gamma < 1$. Show that there exists a constant $C_{n,\gamma} > 0$ such that for all $x, y \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \frac{\left|e^{ix\cdot\xi} - e^{iy\cdot\xi}\right|^2}{|\xi|^{2s}} d\xi \leqslant C_{n,\gamma} |x-y|^{2\gamma}$$

b) Show that if $s = \frac{n}{2} + k + \gamma$ for some $k \in \mathbb{Z}_{\geqslant 0}$, $0 < \gamma < 1$, then

$$H^s(\mathbb{R}^n) \subset C^{k,\gamma}(\mathbb{R}^n).$$

**Exercise 4.3.** Fix $s \in \mathbb{R}$, and suppose that $f \in H^s(\mathbb{R}^n)$.

a) Show that there exists a unique $u \in H^{s+4}(\mathbb{R}^n)$ which solves:

$$\Delta^2 u + u = f.$$

b) Show further that there exists $C > 0$ such that $\|u\|_{H^{s+4}} \leqslant C \|f\|_{H^s}$.

c) For what values of $s$ does the equation hold in the sense of classical derivatives (possibly after redefining $u, f$ on a set of measure zero)?

### 4.6.4 The trace theorem

We are often interested in the restriction of a function defined on $\mathbb{R}^n$, or some open subset, to some hypersurface $\Sigma \subset \mathbb{R}^n$. For example, when studying a PDE problem posed in some nice domain $\Omega$ we might wish to impose a boundary condition on $\partial\Omega$. If we work with functions in $H^s(\mathbb{R}^n)$ for $s > 0$, which are defined only almost everywhere, then this is a problem, since for nice domains $\partial\Omega$ will have Lebesgue measure zero. The trace theorem allows us to make sense of the restriction of a function in $H^s$ to a hypersurface $\Sigma$, even when we don't have $f \in C^0$ by Sobolev embedding. We restrict to the problem of defining $f|_{\{x^n=0\}}$ when $f \in H^s(\mathbb{R}^n)$ is given, however by combining this result with coordinate transformations it is fairly easy to see how to generalise to the case of smoothly embedded submanifolds.

**Theorem 4.21.** *Let $s > \frac{1}{2}$. Then there is a bounded linear map $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^n)$ such that*

$$Tf = f|_{\{x^n=0\}}$$

*for all $f \in H^s(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$.*

*Proof.* See Exercise 4.4. □

**Exercise 4.4.** Assume $s > \frac{1}{2}$ and suppose $u \in \mathscr{S}(\mathbb{R}^n)$. Define $Tu \in \mathscr{S}(\mathbb{R}^{n-1})$ by:

$$Tu(x') = u(x', 0), \qquad x' \in \mathbb{R}^{n-1}.$$

a) Show that if $\xi' \in \mathbb{R}^{n-1}$:

$$\widehat{Tu}(\xi') = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(\xi', \xi_n) d\xi_n.$$

b) Deduce that:

$$\left|\widehat{Tu}(\xi')\right|^2 \leqslant \frac{1}{2\pi} \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s \left|\hat{u}(\xi', \xi_n)\right|^2 d\xi_n \right) \left( \int_{\mathbb{R}} \frac{d\xi_n}{\left(1 + |\xi|^2\right)^s} \right),$$

where $\xi = (\xi', \xi_n)$.

c) By changing variables in the second integral above to $\xi_n = t\sqrt{1 + |\xi'|^2}$, conclude that there exists a constant $C(s)$ such that:

$$\|Tu\|_{\hat{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leqslant C(s) \|u\|_{\hat{H}^s(\mathbb{R}^n)}.$$

d) Conclude that $T$ extends to a bounded linear operator $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.

e) (*) Suppose $v \in \mathscr{S}(\mathbb{R}^{n-1})$ and let $\phi \in C_c^\infty(\mathbb{R})$ satisfy $\int_{\mathbb{R}} \phi(t)dt = \sqrt{2\pi}$. Define $u$ through its Fourier transform by:

$$\hat{u}(\xi', \xi_n) = \frac{\hat{v}(\xi')}{\sqrt{1+|\xi'|^2}} \phi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right).$$

Show that there exists a constant $C > 0$ such that:

$$\|u\|_{H^s(\mathbb{R}^n)} \leqslant C \|v\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})}$$

and that $Tu = v$. Conclude that $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is surjective.

### 4.6.5 The space $H_0^1(\Omega)$

Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$. For any function $f \in C_c^\infty(\Omega)$, we can trivially extend to an element of $C_c^\infty(\mathbb{R}^n)$ by $f(x) = 0$ for $x \in \Omega^c$, so can abuse notation slightly to denote by $C_c^\infty(\Omega)$ the space of smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ with support in some compact $K \subset \Omega$. We define $H_0^1(\Omega)$ to be the completion of $C_c^\infty(\Omega)$ with respect to the $H^1(\mathbb{R}^n)$-norm. $H_0^1(\Omega)$ is a Hilbert space, equipped with the inner product:

$$(u, v)_{H^1} = \int_\Omega \left(\overline{Du(x)} \cdot Dv(x) + \overline{u(x)}v(x)\right) dx.$$

Let $u \in H_0^1(\Omega)$. Then by definition there exists a sequence $(\phi_n)_{n=1}^\infty$, with $\phi_n \in C_c^\infty(\Omega)$ and $\phi_n \to \phi$ in $H^1(\mathbb{R}^n)$. Since for any open $U \subset \mathbb{R}^n$ we have

$$\|f_n - f\|_{L^2(U)} \leqslant \|f_n - f\|_{L^2(\mathbb{R}^n)} \leqslant \|f_n - f\|_{H^1(\mathbb{R}^n)},$$

we deduce that $f_n|_U \to f|_U$ in $L^2$. If we choose $U = \Omega^c$, we conclude that if $f \in H_0^1(\Omega)$ then $f|_{\Omega^c} = 0$ almost everywhere.

If we assume the boundary of $\Omega$ is smooth, i.e. is an embedded smooth $(n-1)$-manifold, then we can make sense of the restriction of $f$ to $\partial\Omega$ in the trace sense, and since the trace operator is a continuous map from $H^1(\mathbb{R}^n)$, we find that $f$ vanishes on $\partial\Omega$ in the trace sense.

For many PDE problems, one wishes to solve some equation in an open set $\Omega$, subject to the condition that the solution vanishes on the boundary of $\Omega$. Seeking a solution in $H_0^1(\Omega)$ is often a convenient way to encode this boundary condition.

### 4.6.6 Rellich–Kondrachov

The Rellich–Kondrachov theorem is an important result concerning Sobolev spaces, with applications in PDE, calculus of variations and beyond. It concerns compact embedding for Sobolev spaces defined on a bounded domain. We shall prove a version of the result for the space $H_0^1(\Omega)$, where $\Omega$ is a bounded open set.

**Theorem 4.22** (Rellich–Kondrachov)**.** *Suppose that $\Omega$ is a bounded open set and that $(u_i)_{i=1}^\infty$ is a bounded sequence in $H_0^1(\Omega)$. Then there exists $u \in H_0^1(\Omega)$ and a subsequence $(u_{i_j})_{j=1}^\infty$ such that:*

*i)* $u_{i_j} \rightharpoonup u$ *in* $H_0^1(\Omega)$*, and*

*ii)* $u_{i_j} \to u$ *in* $L^2(\Omega)$.

*Proof.* By assumption, we have that

$$\|u_i\|_{L^2(\Omega)} \leqslant \|u_i\|_{H_0^1(\Omega)} \leqslant K$$

so $(u_i)_{i=1}^\infty$ is bounded in both $H_0^1(\Omega)$, and $L^2(\Omega)$, and we immediately deduce from the Banach–Alaoglu theorem that there exists $u \in H_0^1(\Omega)$ and a subsequence $(u_{i_j})_{j=1}^\infty$ such that $u_{i_j} \rightharpoonup u$ in $H_0^1(\Omega)$, and $u_{i_j} \rightharpoonup u$ in $L^2(\Omega)$. For convenience, let us set $w_j = u_{i_j}$ so that $w_j \rightharpoonup u$ in $H_0^1(\Omega)$, and $w_j \rightharpoonup u$ in $L^2(\Omega)$. Thus our goal is to improve the weak-$L^2$ convergence of $(w_j)$ to strong-$L^2$ convergence.

Fix $\epsilon > 0$. We make use of Parseval's Formula (Theorem 4.6) to give:

$$\begin{aligned}\|w_j - u\|_{L^2}^2 &= \frac{1}{(2\pi)^n}\|\hat{w}_j - \hat{u}\|_{L^2}^2 \\ &= \frac{1}{(2\pi)^n}\int_{|\xi|<R} |\hat{w}_j(\xi) - \hat{u}(\xi)|^2\,d\xi + \frac{1}{(2\pi)^n}\int_{|\xi|\geqslant R} |\hat{w}_j(\xi) - \hat{u}(\xi)|^2\,d\xi\end{aligned}$$

We deal with the two integrals on the final line separately. First we estimate:

$$\begin{aligned}\frac{1}{(2\pi)^n}\int_{|\xi|\geqslant R} |\hat{w}_j(\xi) - \hat{u}(\xi)|^2\,d\xi &\leqslant \frac{2}{(2\pi)^n R^2}\int_{|\xi|\geqslant R} |\xi|^2\,(|\hat{w}_j(\xi)|^2 + |\hat{u}(\xi)|^2)d\xi \\ &\leqslant \frac{2K^2}{(2\pi)^n R^2} < \epsilon,\end{aligned}$$

provided $R > 0$ is chosen sufficiently large.

Now consider the remaining integral that we need to bound. First, we note that

$$\hat{w}_j(\xi) = \int_\Omega w_j(x)e^{-ix\cdot\xi}dx = (w_i, e_{-\xi})_{L^2(\Omega)},$$

where we recall $e_y(x) = e^{ix\cdot y}$. Noting that $e_{-\xi} \in L^2(\Omega)$ since $|\Omega| < \infty$, and that $w_j \rightharpoonup u$ in $L^2(\Omega)$, we deduce that for each $\xi \in \mathbb{R}^n$:

$$\hat{w}_j(\xi) \to \hat{u}(\xi).$$

We can also estimate, for $|\xi| < R$:

$$\begin{aligned}|\hat{w}_j(\xi) - \hat{u}(\xi)|^2 &\leqslant 2\,|\hat{w}_j(\xi)|^2 + 2\,|\hat{u}(\xi)|^2 \leqslant 2\left(\|\hat{w}_j\|_{L^\infty}^2 + \|\hat{u}\|_{L^\infty}^2\right) \\ &\leqslant 2\left(\|w_j\|_{L^1(\Omega)}^2 + \|u\|_{L^1(\Omega)}^2\right) \leqslant 2|\Omega|\left(\|w_j\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right) \\ &\leqslant 4K^2|\Omega| \in L^1(B_R(0))\end{aligned}$$

So by the dominated convergence theorem we deduce that

$$\frac{1}{(2\pi)^n}\int_{|\xi|<R} |\hat{w}_j(\xi) - \hat{u}(\xi)|^2\,d\xi \to 0$$

as $j \to \infty$, so that for $j$ sufficiently large we have established:

$$\|w_j - u\|_{L^2}^2 < 2\epsilon$$

□

**Corollary 4.23.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose that $L : L^2(\Omega) \to H_0^1(\Omega)$ is a bounded linear operator, then $L : L^2(\Omega) \to L^2(\Omega)$ is compact.*

**Exercise 4.5.** Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded. For $u \in H_0^1(\Omega)$, define the Dirichlet energy:

$$E[u] = \int_\Omega |Du|^2 \, dx.$$

a) Suppose that $(u_i)_{i=1}^\infty$ is a sequence with $u_i \in H_0^1(\Omega)$ such that $u_i \rightharpoonup u$. Show that $E[u] \leqslant \liminf_i E[u_i]$.

b) Consider the set

$$\mathcal{E}_1 = \{E[u] : u \in H_0^1(\Omega), \|u\|_{L^2} = 1\}$$

Let $\lambda_1 := \inf \mathcal{E}$. Show that there exists $w_1 \in H_0^1(\Omega)$ with $\|w_1\|_{L^2} = 1$ and $E[w_1] = \lambda_1$, and deduce $\lambda_1 > 0$.

c) Deduce that:

$$\lambda_1 \|u\|_{L^2}^2 \leqslant \int_\Omega |Du|^2 \, dx$$

holds for all $u \in H_0^1(\Omega)$, with equality for $u = w_1$. This is *Poincaré's inequality.*

d) By considering $u = w_1 + t\phi$ for $t \in \mathbb{R}$, $\phi \in \mathscr{D}(\Omega)$, or otherwise, show that $w_1$ satisfies

$$-\Delta w_1 = \lambda_1 w_1,$$

where we understand this equation as holding in $\mathscr{D}'(\Omega)$.

e) (*) Suppose $\chi \in C_c^\infty(\Omega)$, and let $v = \chi w_1$. Show that $v$ satisfies $-\Delta v + v = f$, where we understand the equation as holding in $\mathscr{S}'(\mathbb{R}^n)$, where $f \in L^2(\mathbb{R}^n)$. Deduce that $v \in H^2(\mathbb{R}^n)$. By iterating this argument, deduce that $w_1 \in H_0^1(\Omega) \cap C^\infty(\Omega)$.

f) (*) By considering

$$\mathcal{E}_2 = \{E[u] : u \in H_0^1(\Omega), \|u\|_{L^2} = 1, (u, w_1)_{L^2} = 0\},$$

or otherwise, show that there exists $\lambda_2 \geqslant \lambda_1$ and $w_2 \in H_0^1(\Omega) \cap C^\infty(\Omega)$ with $w_2 \neq w_1$, $\|w_2\|_{L^2} = 1$ solving

$$-\Delta w_2 = \lambda_2 w_2.$$

## 4.7 PDE Examples

### 4.7.1 Elliptic equations on $\mathbb{R}^n$

Consider the following equation on $\mathbb{R}^n$, with $k > 0$:

$$-\Delta u + k^2 u = f,$$

where $f$ is given and we wish to find $u$. Suppose that $f \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. We claim that there is a unique solution $u \in H^{s+2}(\mathbb{R}^n)$. Our assumptions on $u, f$ permit us to take the Fourier transform of the equation so that:

$$(|\xi|^2 + k^2)\hat{u}(\xi) = \hat{f}(\xi)$$

holds pointwise almost everywhere. Since $|\xi|^2 + k^2 \geqslant C(1 + |\xi|^2) > 0$ for some $C$, we can divide through to find

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 + k^2},$$

again using $|\xi|^2 + k^2 \geqslant C(1 + |\xi|^2) > 0$ we deduce:

$$\|u\|_{H^{s+2}(\mathbb{R}^n)} \leqslant C \|f\|_{H^s(\mathbb{R}^n)}.$$

Thus we indeed have that $u \in H^{s+2}(\mathbb{R}^n)$. Uniqueness follows from the injectivity of the Fourier transform. Note that if $s > \frac{n}{2}$ then $f \in C^0(\mathbb{R}^n)$ and $u \in C^2(\mathbb{R}^n)$, so that we in fact have a classical solution to the PDE. Note also that the solution is *more regular* than the data. This is an example of a phenomenon known as *elliptic regularity.*

### 4.7.2 Elliptic boundary value problems

Suppose that $\Omega \subset \mathbb{R}^n$ is open, assume $f : \Omega \to \mathbb{R}$ is given, and consider the equation:

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

We wish to reformulate this so that we can solve it. In order to incorporate the boundary condition, we shall seek a solution $u \in H^1_0(\Omega)$. Since an element of $H^1_0(\Omega)$ only has weak derivatives in $L^2$ up to first order, we need to recast the equation in a form that makes sense. To do this, suppose we have a sufficiently regular solution, conjugate the equation and multiply it by $v \in C^\infty_c(\Omega)$ to deduce, after integrating by parts:

$$\int_\Omega \left(\overline{Du} \cdot Dv + \overline{u}v\right) dx = \int_\Omega \overline{f} v dx \tag{4.14}$$

holds for all $v \in C^\infty_c(\Omega)$. We realise that, if $f \in L^2(\Omega)$, we are seeking $u \in H^1_0(\Omega)$ such that:

$$(u, v)_{H^1} = (\overline{f}, v)_{L^2}$$

for all $v \in C^\infty_c(\Omega)$. We also notice that since $C^\infty_c(\Omega)$ is dense in $H^1_0(\Omega)$, this is equivalent to requiring the condition holds for $v \in H^1_0(\Omega)$. We say that $u \in H^1_0(\Omega)$ is a *weak solution* of (4.13) if (4.14) holds for all $v \in H^1_0(\Omega)$. Clearly, if $u$ is a classical solution then it is a weak solution.

Now, for $f \in L^2(\Omega)$, the map $F : H^1_0(\Omega) \to \mathbb{C}$ given by $v \mapsto (\overline{f}, v)_{L^2}$ is a bounded linear operator, hence we can apply Riesz representation theorem for the Hilbert space $H^1_0(\Omega)$ to deduce that there exists a unique $\tilde{u} \in H^1_0(\Omega)$ such that $F(v) = (u, v)_{H^1}$ for all $v \in H^1_0(\Omega)$. This is precisely the solution we seek! In conclusion, then, we have shown:

**Lemma 4.24.** *Given $f \in L^2(\Omega)$ there exists a unique $u \in H^1_0(\Omega)$ solving (4.13) in the sense that (4.14) holds for all $v \in H^1_0(\Omega)$.*

We note that setting $v = u$ in (4.14), and using Cauchy-Schwarz we have:

$$\|u\|^2_{H^1} = (\overline{f}, u)_{L^2} \leqslant \|f\|_{L^2} \|u\|_{L^2} \leqslant \|f\|_{L^2} \|u\|_{H^1},$$

so that

$$\|u\|_{H^1} \leqslant \|f\|_{L^2}.$$

We will now show that we can improve the regularity of $u$, at least in the interior of $\Omega$, provided we make some assumptions on $f$. For this, we introduce the space (here $k \in \mathbb{Z}_{\geqslant 0}$)

$$H^k_{loc.}(\Omega) = \left\{ u : \Omega \to \mathbb{C} |\ \chi u \in H^k(\mathbb{R}^n), \text{ for all } \chi \in C^\infty_c(\Omega) \right\}$$

Fix a compact $K \subset \Omega$ and suppose that the real function $\chi \in C^\infty_c(\Omega)$ satisfies $\chi(x) = 1$ for $x \in K$. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$, then since $\chi\phi \in C^\infty(\Omega)$, we can set $v = \chi\phi$ in (4.15):

$$\int_\Omega \overline{Du} \cdot D(\chi\phi) + \overline{u}\chi\phi dx = \int_\Omega \overline{f}\chi\phi dx$$

rearranging, we have:

$$\int_\Omega \overline{D(\chi u)} \cdot D\phi + \overline{Du} \cdot (D\chi)\phi - D\phi \cdot (D\chi)\overline{u} + \overline{u}\chi\phi dx = \int_\Omega \overline{f}\chi\phi dx$$

and hence:

$$\int_\Omega -\overline{(\chi u)}\Delta\phi + 2\overline{Du} \cdot (D\chi)\phi + \phi(\Delta\chi)\overline{u} + \overline{u}\chi\phi dx = \int_\Omega \overline{f}\chi\phi dx$$

So that $v = \chi u$ satisfies:

$$\int_{\mathbb{R}^n} \overline{v}(-\Delta\phi + 1) dx = \int_{\mathbb{R}^n} \overline{g}\phi dx,$$

where

$$g = -2Du \cdot (D\chi) - u\Delta\chi + f\chi \in L^2(\mathbb{R}^n).$$

We have deduced that $v \in H^1(\mathbb{R}^n)$ satisfies:

$$-\Delta v + v = g$$

in the sense of $\mathscr{S}'(\mathbb{R}^n)$. Now, by the results of the previous section, we deduce $v \in H^2(\mathbb{R}^n)$ with:

$$\|v\|_{H^2} \leqslant C \|g\|_{L^2}.$$

Further, $v(x) = u(x)$ for all $x \in K$. Suppose $\tilde{\chi} \in C_c^\infty(\Omega)$, then by applying the above argument with the compact set $K = \operatorname{supp} \tilde{\chi}$ we deduce that $\tilde{\chi}u = \tilde{\chi}\chi u \in H^2(\mathbb{R}^n)$. Thus $u \in H_0^1 \cap H^2_{loc.}(\Omega)$.

Now suppose that $f \in L^2 \cap H^1_{loc.}(\Omega)$. Repeating the above argument, we notice that $g \in H^1(\mathbb{R}^n)$, and so $v \in H^3(\mathbb{R}^n)$, and as a consequence we can conclude $u \in H_0^1 \cap H^3_{loc.}(\mathbb{R}^n)$. Iterating, we find:

**Theorem 4.25.** *Suppose $\Omega \subset \mathbb{R}^n$ is open and $f \in L^2 \cap H^k_{loc.}(\Omega)$. Then there exists a unique $u \in H_0^1 \cap H^{k+2}_{loc.}(\Omega)$ solving (4.13) in the weak sense. In particular, by Sobolev embedding if $f \in L^2 \cap C^\infty(\Omega)$, then $u \in H_0^1 \cap C^\infty(\Omega)$.*

Now, if $u \in C^\infty(\Omega)$, then we can see that the equation $-\Delta u + u = f$ must hold in $\Omega$ in the classical sense. If we assume more regularity of the boundary (and $f$), then we can also show that $u$ extends to the boundary as a continuous function, and the boundary condition holds classically also. Discussing boundary regularity would take us beyond the remit of this course however.

We note, that our proof shows that the elliptic regularity phenomenon that we observed above for an equation on $\mathbb{R}^n$ is in fact localisable: if $(-\Delta u + u)$ is smooth in the interior of some open set, then $u$ is smooth in that set. This is certainly not true for (for example) the wave operator $-\partial_t^2 + \Delta$. It is straightforward (try it!) to find a function that satisfies the wave equation in one dimension, hence $u_{tt} - u_{xx} = 0 \in C^\infty(\mathbb{R}^2)$, but for which $u \notin C^\infty(\mathbb{R}^2)$.

### Spectral theory for elliptic boundary value problems

We now assume that $\Omega \subset \mathbb{R}^n$ is both open and *bounded.* Let us represent by $A$ the map which takes $f \in L^2(\Omega)$ to the unique solution $u \in H_0^1(\Omega)$ to (4.13). We can check that $A$ is linear, since if $u = Af$ and $w = Ag$ for some $f, g \in L^2(\Omega)$ and $a \in \mathbb{C}$, then for any $v \in H_0^1(\Omega)$ we have:

$$(u + aw, v)_{H^1} = (u, v)_{H^1} + \overline{a}(w, v)_{H^1} = (f, v)_{L^2} + \overline{a}(g, v)_{L^2} = (f + ag, v)_{L^2}$$

so that $Af + aAg = A(f + ag)$. Moreover, $A : L^2(\Omega) \to L^2(\Omega)$ is Hermitian. Suppose $u = Af$ and $w = Ag$ for some $f, g \in L^2(\Omega)$. Then

$$(f, Ag)_{L^2} = (f, w)_{L^2} = (u, w)_{H^1} = \overline{(w, u)_{H^1}} = \overline{(g, u)_{L^2}} = (Af, g)_{L^2}.$$

Finally, by Corollary 4.23 we have that $A : L^2(\Omega) \to L^2(\Omega)$ is compact. Thus by the spectral theorem for compact operators (see Linear Analysis), the spectrum of $A$ takes the form $\sigma(A) = \{0, \mu_1, \mu_2, \ldots\}$, where $\mu_k \in \mathbb{R}$, $\mu_k \to 0$. Further, there exists an orthonormal basis for $L^2(\Omega)$ consisting of eigenvectors of $A$. An eigenvector of $A$ satisfies $Aw = \mu w$ for $\mu \in \mathbb{R}$, and thus for $v \in H_0^1(\Omega)$:

$$(w, v)_{L^2} = (Aw, v)_{H^1} = \mu(w, v)_{H^1} \tag{4.15}$$

Setting $v = w$ we deduce $\mu > 0$, so in particular $\mu \neq 0$, and we deduce that $w$ solves:

$$-\Delta w + w = \frac{1}{\mu} w$$

in the weak sense. This means that we can test the equation against elements of $H^1_0(\Omega)$ (alternatively, we can understand the equation as holding in $\mathscr{D}'(\Omega)$). Now, since $\mu^{-1}w \in H^1_0(\Omega)$, we conclude from our previous work that $w \in H^1_0 \cap H^3(\Omega)$. Hence $w \in H^1_0 \cap H^5(\Omega)$, etc. We conclude, after Sobolev embedding that $w \in C^\infty(\Omega)$.

Finally, noting that an eigenfunction of $(-\Delta + 1)$ is also an eigenfunction of $-\Delta$, we have shown:

**Theorem 4.26.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ for $L^2(\Omega)$ such that $w_k \in H^1_0 \cap C^\infty(\Omega)$ satisfy*

$$-\Delta w_k = \lambda_k w_k \qquad \text{in } \Omega,$$

*where $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$, and $\lambda_k \to \infty$. (In fact, by Exercise 4.5 we can show that $0 < \lambda_1$).*

**Exercise 4.6.** Let $H$ be the completion of $\mathscr{S}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_H := \left( \int_{\mathbb{R}^n} \left( |Du|^2 + |x|^2 |u|^2 \right) dx \right)^{\frac{1}{2}}$$

a) Show that $H$ is a Hilbert space with the inner product:

$$(u, v)_H := \int_{\mathbb{R}^n} \left( \overline{Du} \cdot Dv + |x|^2 \overline{u} v \right) dx,$$

and show that if $u \in H, \chi \in C^\infty_c(B_R(0))$, then $\chi u \in H^1_0(B_R(0))$, with $\|\chi u\|_{H^1} \leqslant C_{R,\chi} \|u\|_H$ for some constant $C_{R,\chi} > 0$.

b) Show that $H$ embeds compactly into $L^2(\mathbb{R}^n)$, that is $H \subset L^2(\mathbb{R}^n)$ and if $(u_n)_{n=1}^\infty$ is a bounded sequence in $H$ then it admits a subsequence which converges in $L^2(\mathbb{R}^n)$.
*[Hint: take a subsequence converging weakly in both $H$ and $L^2(\mathbb{R}^n)$, and write $u_n = u_n \chi_R + u_n(1 - \chi_R)$, where $\chi_R \in C^\infty_c(B_R(0))$ satisfies $\chi_R(x) = 1$ for $|x| < R - 1$, where $R$ is to be chosen.]*

c) If $f \in L^2(\mathbb{R}^n)$, we say that $u \in H$ is a weak solution of:

$$-\Delta u + |x|^2 u = f \tag{$\dagger$}$$

if

$$(u, v)_H = (f, v)_{L^2} \text{ for all } v \in H. \tag{$\diamond$}$$

Show that if $u, f \in \mathscr{S}(\mathbb{R}^n)$ solve ($\dagger$), then $u$ satisfies ($\diamond$). Show that for any $f \in L^2(\mathbb{R}^n)$, there exists a unique solution $u \in H$ to ($\diamond$).

d) Denote by $Lf$ the unique solution $u \in H$ to ($\diamond$) for $f \in L^2(\mathbb{R}^n)$. Show that the map $f \mapsto Lf$ is a compact, symmetric, linear operator $L : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Deduce that there exists an orthonormal basis $(w_k)_{k=1}^\infty$ for $L^2(\mathbb{R}^n)$ consisting of $w_k \in H$ satisfying:

$$(w_k, v)_H = \lambda_k (w_k, v)_{L^2} \text{ for all } v \in H, \qquad (\flat)$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$, and $\lambda_k \to \infty$.

e) Show that if $w_k \in H$ satisfies ($\flat$), then in fact $w_k \in C^\infty(\mathbb{R}^n)$. Show further that $\hat{w}_k$ will also solve ($\flat$) with the same $\lambda_k$. Deduce that there exists an orthonormal basis for $L^2(\mathbb{R}^n)$, consisting of smooth functions, which diagonalises the Fourier–Plancherel transform.

f) (**) Show that $w \in H \cap C^\infty(\mathbb{R}^n)$ satisfies:

$$-\Delta w + |x|^2 w = \lambda w$$

for some $\lambda \in \mathbb{R}$ if and only if:

$$w(x) = H_{k_1}(x_1) \cdots H_{k_n}(x_n) e^{-\frac{1}{2}|x|^2},$$

where $x = (x_1, \ldots, x_n)$, $H_k(t)$ are the Hermite polynomials, and $\lambda = n + 2k_1 + \ldots 2k_n$.
*[Hint: treat the case $n = 1$ first. You may wish to look up the simple harmonic oscillator in a textbook on quantum mechanics.]*

### 4.7.3 Spaces involving time

For certain PDE problems it's useful to separate out the time direction from the spatial directions. To do this, it's useful to introduce some new function spaces:

**Definition 4.3.** *Given a Banach space $(X, \|\cdot\|_X)$, and an interval $I \subset \mathbb{R}$, the space $C^0(I; X)$ is the space of continuous functions $\boldsymbol{u} : I \to X$.*

*If $I$ is open, we define $C^k(I; X)$ for $k \geqslant 0$ inductively as follows. We say $\boldsymbol{u} \in C^{k-1}(I; X)$ belongs to $C^k(I, X)$ if there exists $\boldsymbol{u}' \in C^{k-1}(I; X)$ such that for each $t \in I$:*

$$\left\| \frac{\boldsymbol{u}(t+\epsilon) - \boldsymbol{u}(t)}{\epsilon} - \boldsymbol{u}'(t) \right\|_X \to 0, \qquad \text{as } \epsilon \to 0.$$

A typical example of $X$ will be one of the space $H^s(\mathbb{R}^n)$ for $s > 0$.

### 4.7.4 The heat equation

Let us now give another example to show how powerful the Fourier transform can be for solving PDE problems. Let us consider the heat equation on $\mathbb{R}^n$. The problem we shall consider is, given $u_0 : \mathbb{R}^n \to \mathbb{R}$, determine $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$, such that

$$\left\{ \begin{array}{rcll} u_t & = & \Delta u & \text{in } (0,T) \times \mathbb{R}^n, \\ u & = & u_0 & \text{on } \{0\} \times \mathbb{R}^n \end{array} \right. \qquad (4.16)$$

We suppose that our solution is a continuous mapping from $(0,T)$ into $H^2(\mathbb{R}^n)$, i.e. for each fixed $t$ we wish $u(t,\cdot) =: \boldsymbol{u}(t)$ to be an element of $H^2(\mathbb{R}^n)$. In terms of the function spaces above $\boldsymbol{u} \in C^0((0,T);H^2(\mathbb{R}^n))$. We will also suppose that $u$ is continuously differentiable as a mapping from $(0,T)$ into $L^2(\mathbb{R}^n)$. In other words, $\boldsymbol{u} \in C^1((0,T);L^2(\mathbb{R}^n))$. Finally, we wish for the initial condition to make sense, so we also require $\boldsymbol{u} \in C^0([0,T);L^2(\mathbb{R}^n)$.

**Exercise(∗).** Show that if $u \in C^0((0,T);H^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$, then denoting by $\hat{u}$ the Fourier transform of $u$ in the spatial variables:

$$\hat{u}(t,\xi) = \lim_{R\to\infty} \int_{B_R(0)} u(t,x)e^{-ix\cdot\xi}dx,$$

we have $\hat{u} \in C^0((0,T);L^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$.

Let us, then, seek a solution of (4.16) such that

$$\boldsymbol{u} \in C^0([0,T);L^2(\mathbb{R}^n)) \cap C^0((0,T);H^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$$

Under this assumption we can take the Fourier transform of (4.16) for $(t,x) \in (0,T)\times\mathbb{R}^n$ to get:

$$\begin{cases} \hat{u}_t(t,\xi) &= -|\xi|^2\,\hat{u}(t,\xi) & (t,\xi)\in(0,T)\times\mathbb{R}^n, \\ \hat{u}(0,\xi) &= \hat{u}_0(\xi) & \xi\in\mathbb{R}^n \end{cases}$$

Now, the PDE has become an ODE for each fixed $\xi$! This ODE has a unique solution given for almost every $\xi \in \mathbb{R}^n$ by:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)e^{-t|\xi|^2}.$$

We note that if $u_0 \in L^2(\mathbb{R}^n)$, then $\hat{u}_0 \in L^2(\mathbb{R}^n)$ and thus $\hat{u} \in C^0([0,T);L^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$. In fact, for $t > 0$, we have that $\hat{u}(t,\xi)$ and $\hat{u}_t(t,\xi)$ are rapidly decaying functions of $\xi$, in particular they belong to $H^s(\mathbb{R}^n)$ for any $s \geqslant 0$, so we have that $u(t,x)$ is smooth in $x$. Since $u$ satisfies the equation $(\partial_t)^n u = (\Delta)^n u$, we have that $u$ is smooth in both $t$ and $x$. We can recover $u(t,x)$ via the inverse Fourier transform formula:

$$u(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{u}_0(\xi)e^{-t|\xi|^2}e^{i\xi\cdot x}d\xi. \tag{4.17}$$

Summarising, we have the following result:

**Lemma 4.27.** *Suppose $u_0 \in L^2(\mathbb{R}^n)$. Then (4.16) admits a unique solution $u$ such that*

$$\boldsymbol{u} \in C^0([0,T);L^2(\mathbb{R}^n)) \cap C^0((0,T);H^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$$

*given by (4.17). In fact,*

$$u \in C^\infty((0,T)\times\mathbb{R}^n).$$

Even with very rough initial data, the heat equation instantaneously gives a smooth solution. This is an example of what is known as *parabolic regularity.*

**Exercise 4.7.** Suppose that $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and that $u(t,x)$ is the solution of the heat equation with initial data $u_0$. Explicitly, $u$ is given by:

$$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{-t|\xi|^2} e^{i\xi\cdot x} d\xi,$$

for $t > 0$.

a) Show that:

$$\|u(t,\cdot)\|_{L^2} \leqslant \|u_0\|_{L^2},$$

b) Show that:

$$u(t,x) = u_0 \star K_t(x)$$

where the *heat kernel* is given by:

$$K_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

c) Suppose that $u_0 \geqslant 0$. Show that $u \geqslant 0$, and:

$$\|u(t,\cdot)\|_{L^1} = \|u_0\|_{L^1}.$$

**Exercise 4.8.** Consider the free Schrödinger equation:

$$\begin{cases} u_t = i\Delta u & \text{in } (0,T)\times\mathbb{R}^n, \\ u = u_0 & \text{on } \{0\}\times\mathbb{R}^n \end{cases} \tag{$*$}$$

Suppose $u_0 \in H^2(\mathbb{R}^n)$.

a) Show that $(*)$ admits a unique solution $u$ such that

$$\boldsymbol{u} \in C^0([0,T); H^2(\mathbb{R}^n)) \cap C^1((0,T); L^2(\mathbb{R}^n)),$$

whose spatial Fourier-Plancherel transform is given by:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi) e^{-it|\xi|^2}.$$

b) Show that:

$$\|u(t,\cdot)\|_{H^2(\mathbb{R}^n)} = \|u_0\|_{H^2(\mathbb{R}^n)}$$

*c) For $t > 0$, let $K_t \in L^1_{loc.}(\mathbb{R}^n)$ be given by:

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{\frac{i|x|^2}{4t}},$$

where for $n$ odd we take the usual branch cut so that $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. For $\epsilon > 0$ set $K_t^\epsilon(x) = e^{-\epsilon|x|^2} K_t(x)$.

i) Show that $T_{K_t^\epsilon} \to T_{K_t}$ in $\mathscr{S}'$ as $\epsilon \to 0$.

ii) Show that if $\Re(\sigma) > 0$, then:

$$\int_{\mathbb{R}} e^{-\sigma x^2 - ix\xi} dx = \sqrt{\frac{\pi}{\sigma}} e^{-\frac{\xi^2}{4\sigma}}.$$

iii) Deduce that

$$\widehat{K_t^\epsilon}(\xi) = \left( \frac{1}{1 + 4it\epsilon} \right)^{\frac{n}{2}} e^{\frac{-it|\xi|^2}{1+4it\epsilon}}$$

iv) Conclude that:

$$\widehat{T_{K_t}} = T_{\tilde{K}_t},$$

where $\tilde{K}_t = e^{-it|\xi|^2}$.

*d) Suppose that $u \in \mathscr{S}(\mathbb{R}^n)$. Show that for $t > 0$:

$$u(t,x) = \int_{\mathbb{R}^n} u_0(y) K_t(x-y) dy,$$

and deduce that for $t > 0$:

$$\sup_{x \in \mathbb{R}^n} |u(t,x)| \leqslant \frac{1}{(4\pi t)^{\frac{n}{2}}} \|\hat{u}_0\|_{L^1}.$$

This type of estimate which shows us that (locally) solutions to the Schrödinger equation decay in time is known as a *dispersive estimate.*

### 4.7.5 The wave equation

Now let us consider the wave equation on $\mathbb{R}^n$. The problem we shall consider is, given $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}$, determine $u : \mathbb{R}^n \times (-T,T) \to \mathbb{R}$, such that

$$\left\{ \begin{array}{rcll} u_{tt} &=& \Delta u & \text{in } (-T,T) \times \mathbb{R}^n, \\ u &=& u_0 & \text{on } \{0\} \times \mathbb{R}^n \\ u_t &=& u_1 & \text{on } \{0\} \times \mathbb{R}^n \end{array} \right. \tag{4.18}$$

We will seek a solution in the space:

$$X_s := C^0((-T,T), H^{s+2}(\mathbb{R}^n)) \cap C^2((-T,T) \times H^s(\mathbb{R}^n)).$$

Fourier transforming in the spatial variable, we have:

$$\left\{ \begin{array}{rcll} \hat{u}_{tt}(t,\xi) &=& -|\xi|^2 \hat{u}(t,\xi) & (t,\xi) \in (-T,T) \times \mathbb{R}^n, \\ \hat{u}(0,\xi) &=& \hat{u}_0(\xi) & \xi \in \mathbb{R}^n \\ \hat{u}_t(0,\xi) &=& \hat{u}_1(\xi) & \xi \in \mathbb{R}^n \end{array} \right.$$

Again, this is an ODE for each fixed $\xi$, and we deduce:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi) \cos(|\xi| t) + \hat{u}_1(\xi) \frac{\sin(|\xi| t)}{|\xi|}$$

Notice that if $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$, then we conclude $\hat{u} \in X_s$. Thus (after taking the inverse Fourier transform) we have found the unique solution of the wave equation in $X_s$.

Let's specialise to $\mathbb{R}^3$. We'd like to write this solution as some sort of convolution, at least for initial data in the Schwarz class. For this we need to find the (inverse) Fourier transform of $\cos(|\xi|\, t)$ and $\frac{\sin(|\xi| t)}{|\xi|}$, where we have to understand these functions as tempered distributions. Let us define, for $t > 0$ the distribution:

$$U_t[\phi] = \frac{1}{4\pi t}\int_{\partial B_t(0)} \phi(y) d\sigma_y$$

for all $\phi \in \mathscr{S}'$, where $d\sigma_y$ is the surface measure on the sphere $\partial B_t(0)$. This is a distribution of compact support, so we can invoke Theorem 4.13 to find the Fourier transform:

$$\widehat{U_t} = T_{\hat{v}_t}$$

where:

$$\hat{v}_t(\xi) = U_t[e_{-\xi}] = \frac{1}{4\pi t}\int_{\partial B_t(0)} e^{-i\xi\cdot y} d\sigma_y$$

We can perform this integral by choosing spherical polar coordinates for $y$ with the axis aligned with the vector $\xi$. Doing so, the integral becomes:

$$\begin{aligned}\hat{v}_t(\xi) &= \frac{1}{4\pi t}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} e^{-i|\xi| t\cos\theta} t^2 \sin\theta d\theta d\phi \\ &= \frac{t}{2}\int_{-1}^{1} e^{-i|\xi| tz} dz = \frac{t}{2}\left(\frac{e^{-i|\xi| t}}{-i\,|\xi|\, t} - \frac{e^{i|\xi| t}}{-i\,|\xi|\, t}\right) \\ &= \frac{\sin(|\xi|\, t)}{|\xi|}.\end{aligned}$$

Now, let us return to our expression for $u$:

$$\begin{aligned}\hat{u}(\xi) &= \hat{u}_0(\xi)\cos(|\xi|\, t) + \hat{u}_1(\xi)\frac{\sin(|\xi|\, t)}{|\xi|} \\ &= \frac{\partial}{\partial t}\left(\hat{u}_0(\xi)\frac{\sin(|\xi|\, t)}{|\xi|}\right) + \hat{u}_1(\xi)\frac{\sin(|\xi|\, t)}{|\xi|}\end{aligned}$$

Suppose $u_0, u_1 \in \mathscr{S}$. Then by Theorem 4.10, we have:

$$\begin{aligned}u(t,x) &= \frac{\partial}{\partial t} U_t \star u_0(x) + U_t \star u_1(x) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B_t(0)} u_0(x-y) d\sigma_y\right) + \frac{1}{4\pi t}\int_{\partial B_t(0)} u_1(x-y) d\sigma_y \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B_t(x)} u_0(y) d\sigma_y\right) + \frac{1}{4\pi t}\int_{\partial B_t(x)} u_1(y) d\sigma_y \\ &= \frac{\partial}{\partial t}\left(t ⨍_{\partial B_t(x)} u_0(y) d\sigma_y\right) + t ⨍_{\partial B_t(x)} u_1(y) d\sigma_y \qquad (4.19)\end{aligned}$$

Where for a surface $\Sigma$ with surface measure $\sigma$:

$$\fint_{\Sigma} d\sigma := \frac{1}{|\Sigma|}\oint_{\Sigma} d\sigma.$$

Expression (4.19) is known as Kirchoff's formula. While our derivation assumes $u_0, u_1 \in \mathscr{S}$, this assumption can be relaxed. This expression tells us some interesting facts about solutions to the wave equation. First note that the value of $u(x,t)$ depends only on the initial data on the sphere $\partial B_t(x)$. This is known as the strong Huygens principle. In particular this shows us that information is propagated at a *finite speed* by the wave equation. Secondly, note that the value of $u(x,t)$ depends on *derivatives* of $u_0$. This suggests that $C^k-$regularity is not propagated in wave evolution, although we have already seen that $H^s-$regularity is propagated.

**Exercise(∗).** Let $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$, $S_{*,T} := (-T,T)\times\mathbb{R}^3_*$ and $|x| = r$. You may assume the result that if $u = u(r,t)$ is radial, we have

$$\Delta u(|x|,t) = \Delta u(r,t) = \frac{\partial^2 u}{\partial r^2}(r,t) + \frac{2}{r}\frac{\partial u}{\partial r}(r,t)$$

a) Suppose $u(x,t) = \frac{1}{r}v(r,t)$ for some function $v$. Show that $u$ solves the wave equation on $\mathbb{R}^3_* \times (0,T)$ if and only if $v$ satisfies the one-dimensional wave equation

$$-\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} = 0$$

on $(0,\infty)\times(-T,T)$.

b) Suppose $f, g \in C^2_c(\mathbb{R})$. Deduce that

$$u(x,t) = \frac{f(r+t)}{r} + \frac{g(r-t)}{r}$$

is a solution of the wave equation on $S_{*,T}$ which vanishes for large $|x|$.

c) Show that if $f \in C^3_c(\mathbb{R})$ is an odd function (i.e. $f(s) = -f(-s)$ for all $s$) then

$$u(x,t) = \frac{f(r+t) + f(r-t)}{2r}$$

extends as a $C^2$ function which solves the wave equation on $S_T := (-T,T)\times\mathbb{R}^3$, with

$$u(0,t) = f'(t).$$

*d) By considering a suitable sequence of functions $f$, or otherwise, deduce that there exists no constant $C$ independent of $u$ such that the estimate

$$\sup_{S_T}(|u| + |u_t|) \leqslant C \sup_{\Sigma_0}(|u| + |u_t|)$$

holds for all solutions $u \in C^2(S_T)$ of the wave equation which vanish for large $|x|$.