Appendix A

Background Material: Functional Analysis

A.1 Topological vector spaces

This section is intended to recap some of the basic material in Linear Analysis, and to give a bit more detail on the functional analytical underpinnings of some of the more exotic spaces we consider in particular when constructing distributions. The material is not examinable, but is included here to justify various assertions earlier in the course.

In the Linear Analysis course, the principle objects of study are Hilbert or Banach spaces. These are vector spaces which are given the additional structure of an inner product or a norm respectively. This additional structure allows us to make sense of ideas such as convergence of a sequence, or continuity of a real valued map. Unfortunately, some of the vector spaces that we require for this course (for example $\mathscr{D}(U)$, $\mathscr{S}$ and $\mathscr{E}(U)$) are *not* Hilbert or Banach spaces. We need to add to the vector spaces some additional structure, which permits us to discuss the notions of convergence and continuity, but which is not as restrictive as assuming the presence of a norm or inner product. The extra structure that we shall require is of course a topology, but we shall require the topology to be in some sense *consistent* with the vector space structure. We are therefore led to the idea of *topological vector spaces.*

A.1.1 Vector spaces and normed spaces

In order to fix notation, let's recall a few standard definitions.

Definition A.1 (Field axioms)**.** *A field Φ is a set together with two operations, addition $+$ and multiplication $\cdot$ which satisfy the following axioms:*

i) *Φ is* closed *under addition and multiplication: for all $a, b \in \Phi$, we have $a + b \in \Phi$ and $a \cdot b \in \Phi$.*

ii) *Both addition and multiplication are* associative*: the following identities hold for all $a, b, c \in \Phi$:*

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c, \qquad a + (b + c) = (a + b) + c.$$

iii) *Both addition and multiplication are* commutative*: the following identities hold for all* $a, b \in \Phi$*:*

$$a \cdot b = b \cdot a, \qquad a + b = b + a.$$

iv) *There exist unique, distinct, additive and multiplicative* identity elements*: there exist* $0 \in \Phi$ *and* $1 \in \Phi$ *with* $0 \neq 1$ *such that for all* $a \in \Phi$ *we have:*

$$a \cdot 1 = a, \qquad a + 0 = a.$$

v) *There exist additive and multiplicative* inverses*. For every* $a \in \Phi$*, there exists an element* $(-a) \in \Phi$ *such that*

$$a + (-a) = 0.$$

Moreover, for every $a \in \Phi$ *with* $a \neq 0$*, there exists an element* a^{-1} *such that*

$$a \cdot a^{-1} = 1.$$

vi) *The multiplication operation is* distributive *over addition: the following identity holds for all* $a, b \in \Phi$*:*

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

> **Exercise(∗).** Show that $\mathbb{R}$, $\mathbb{C}$ and the integers modulo p, $\mathbb{Z}_p$ form fields with the usual definition of addition and multiplication.

The standard examples of fields that you should keep in mind for our purposes are $\mathbb{R}$ and $\mathbb{C}$. With the definition of a field in hand, we can now define a vector space.

Definition A.2 (Vector space axioms)**.** *Let* Φ *be a field, which we call the* scalar field*, and we call elements of* Φ scalars*. A vector space* X *over* Φ *is a set whose elements are called* vectors *together with two operations:*

i) ***Addition:*** *to every pair of vectors* $x, y \in X$ *is associated a unique vector* $x + y \in X$ *such that for all* $x, y, z \in X$*:*

$$x + y = y + x, \qquad \text{and} \qquad (x + y) + z = x + (y + z).$$

Moreover, there exists a unique element $0 \in X$ *such that for all* $x \in X$*:*

$$x + 0 = x.$$

Finally, for each $x \in X$*, there exists a unique vector* $(-x)$ *such that:*

$$x + (-x) = 0.$$

ii) ***Scalar multiplication:*** *to every pair* (a, x) *with* $a \in \Phi$ *and* $x \in X$ *is associate a unique vector* $ax \in X$ *in such a way that*

$$1x = x, \qquad a(bx) = (a \cdot b)x,$$

and such that the distributive laws:

$$a(x + y) = ax + ay, \qquad (a + b)x = ax + bx$$

hold for every $x, y \in X$ *and* $a, b \in \Phi$*.*

Note that the same symbols have different meanings in different contexts: $+$ can mean either scalar or vector multiplication, while 0 refers to the zero element of both the field and the vector space.

It is useful to extend the operations of vector addition and scalar multiplication to act on sets as follows. If $a \in X$, $\lambda \in \Phi$ $U_1, U_2 \subset X$, then we define:

$$\begin{aligned} a + U_1 &= \{a + x : x \in U\}, \\ U_1 + U_2 &= \{x + y : x \in U_1, y \in U_2\}, \\ \lambda U_1 &= \{\lambda x : x \in U_1\}. \end{aligned}$$

Note that $0 + U = U$, $1U = U$, $2U \subset U + U$, but that in general $2U \neq U + U$.

Definition A.3. *Suppose X is a vector space over Φ, where Φ is either $\mathbb{R}$ or $\mathbb{C}$. We say that a subset $U \subset X$ is* convex *if*

$$x, y \in U \quad \Longrightarrow \quad tx + (1-t)y \in U \text{ for all } t \in [0,1].$$

We say that U is balanced *if $\lambda U \subset U$ for all $\lambda \in \Phi$ with $|\lambda| \leqslant 1$.*

> **Exercise A.1.** Suppose that $\lambda_1 \lambda_2 \geqslant 0$ and that $U \subset X$ is a convex subset of a vector space X. Show that:
>
> $$\lambda_1 U + \lambda_2 U = (\lambda_1 + \lambda_2) U.$$

Finally, we shall define a norm on a vector space

Definition A.4. *A norm on a vector space X over Φ, where Φ is either $\mathbb{R}$ or $\mathbb{C}$ is a map:*

$$\|\cdot\| : X \to \mathbb{R}$$

such that

i) We have $\|x\| \geqslant 0$ for all $x \in X$, with equality if and only if $x = 0$.

ii) The triangle identity holds for all $x, y \in X$:

$$\|x + y\| \leqslant \|x\| + \|y\| .$$

iii) For any $a \in \Phi$ and $x \in X$ we have:

$$\|ax\| = |a| \, \|x\| .$$

A more general notion of distance than a norm is often useful. We define a metric space as follows:

Definition A.5. *A metric space (S, d) is a set S, together with a function $d : S \times S \to \mathbb{R}$, called the metric, which satisfies:*

i) The metric is symmetric*:*

$$d(x,y) = d(y,x), \qquad \textit{for all } x,y \in S.$$

ii) The metric is positive definite*:*

$$0 \leqslant d(x,y), \qquad \textit{for all } x,y \in S,$$

with equality if and only if $x = y$.

iii) The triangle inequality *holds:*

$$d(x,y) \leqslant d(x,z) + d(z,y), \qquad \textit{for all } x,y,z \in S.$$

To see that this is a more general notion than a normed space, we have the following result:

Lemma A.1. *If $(X, \|\cdot\|)$ is a normed vector space, then it is naturaly a metric space, with the metric:*

$$d(x,y) := \|x-y\|$$

Proof. We simply have to verify the three conditions on d. We find:

i) Noting that $|-1| = 1$, we have:

$$d(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1|\,\|y-x\| = \|y-x\| = d(y,x),$$

so the metric is symmetric.

ii) Since we know that $\|x\| \geqslant 0$, with equality if and only if $x = 0$, clearly

$$d(x,y) = \|x-y\| \geqslant 0$$

with equality if and only if $x - y = 0$, which holds if and only if $x = y$.

iii) Recall the triangle inequality for norms $\|x+y\| \leqslant \|x\| + \|y\|$. We calculate:

$$\begin{aligned} d(x,y) &= \|x-y\| = \|(x-z)-(y-z)\| \\ &\leqslant \|x-z\| + \|y-z\| \\ &= d(x,z) + d(z,y). \end{aligned}$$

Thus d satisfies the conditions to be a metric. □

A.1.2 Topological spaces

The definitions above are purely algebraic in nature. In particular, we have not introduced any notions of convergence, completeness or continuity for these spaces. The natural setting in which to do this is that of topology. Let us recall briefly a few definitions and facts.

Definition A.6 (Topology axioms)**.** *A topological space is a set S in which a collection of subsets τ (called* open sets*) has been specified, with the following properties:*

i) The empty set is open: $\emptyset \in \tau$

ii) The whole space is open: $S \in \tau$,

iii) If $U_1, U_2 \in \tau$ are open sets, then their intersection is open:

$$U_1 \cap U_2 \in \tau.$$

iv) If $\mathcal{U} \subset \tau$ is any *collection of open sets, then their union is open:*

$$\bigcup \mathcal{U} \in \tau.$$

Note that by repeatedly applying *iii*), we can easily see that any finite intersection of open sets is open. Let's recall some standard nomenclature associated with topological concepts. A set $E \subset S$ is closed if its complement $E^c = S \setminus E$ is open. The closure $\overline{E}$ of any set E is the intersection of all closed sets containing E. The interior E° of any set E is the union of all open sets contained in E. Note that the closure is always closed and the interior is always open. A *neighbourhood* of a point $p \in S$ is an open set containing p. A *limit point* of a set $E \subset S$ is a point $p \in S$ (not necessarily with $p \in E$) such that every neighbourhood of p intersects E in some point other than p itself.

Lemma A.2. *Suppose (S, τ) is a topological space. If $U \subset S$ is open then $U = U^\circ$. If $E \subset S$ is closed, then $\overline{E} = E$ and E contains all of its limit points.*

Proof. The fact that $U^\circ \subset U$ follows from the definition of the interior as U° is a union over sets contained in U. Since U is itself an open set contained in U, we also have $U \subset U^\circ$. Similarly, $E \subset \overline{E}$ from the definition of the closure. Since E is itself a closed set containing E, we have $\overline{E} \subset E$. Now suppose that p is a limit point, and assume for contradiction that $p \in E^c$. Then since E is closed, E^c is open and hence a neighbourhood of p. By the definition of a limit point we have $E \cap E^c$ is non-empty, a contradiction. Thus $p \in E$. □

A *base*, β for the topology τ is a collection of open sets, $\beta \subset \tau$ such that any open set in τ can be written as a union of elements of β. A collection γ of neighbourhoods of p is a *local base at p* if every neighbourhood of p contains a member of γ.

A set $K \subset S$ is compact if every open cover of K has a finite subcover. That is to say that from any collection $\{U_i\}_{i \in \mathcal{I}}$ of open sets such that $K \subset \cup_{i \in \mathcal{I}} U_i$, we can extract a finite collection $\{U_{i_k}\}_{k=1}^n$ such that $K \subset \cup_{k=1}^n U_{i_k}$. A topological space is *Hausdorff* if any two distinct points have disjoint neighbourhoods.

Lemma A.3. *Suppose (S,τ) is a Hausdorff topological space, and that $K \subset S$ is compact. Then K is closed.*

Proof. Let us fix $p \in K^c$, and consider an arbitrary $q \in K$. By the Hausdorff property of S, we know that there exist U_q, V_q open, with $q \in U_q$, $p \in V_q$ and $U_q \cap V_q = \emptyset$. Now, $\{U_q : q \in K\}$ is an open cover of K, hence by the compactness of K there is a finite subcover, i.e. $q_1, \dots q_N$ such that $K \subset U = U_{q_1} \cup \dots \cup U_{q_N}$. Consider $V = V_{q_1} \cap \dots \cap V_{q_N}$. We have that $V_p \cup U = \emptyset$, so that $V \subset K^c$. Moreover, as a finite intersection of open sets V is open. Writing K^c as the union of the sets V for all $p \in K^c$, we see that K^c is open and thus K is closed. □

Lemma A.4. *Suppose (S,τ) is a Hausdorff topological space, and $E \subset S$. Then p is a limit point of E if and only if every neighbourhood of p contains infinitely many elements of E.*

Proof. If every neighbourhood of p contains infinitely many elements of E, it certainly intersects E in some point other than p, thus p is a limit point. Conversely, suppose that p is a limit point and suppose that U is some neighbourhood intersecting E in only finitely many points, say $\{x_1, \dots, x_N\}$. by the Hausdorff property, we know that there exist open sets U_i, V_i such that $x_i \in U_i$, $p \in V_i$ and $U_i \cap V_i = \emptyset$. Then $\cap_{i=1}^N V_i \cap U$ is open, contains p and doesn't contain any other points of E. This contradicts the assumption that p is a limit point. □

A sequence $(x_n)_{n \in \mathbb{N}}$ in a Hausdorff space *converges* to a point x if every neighbourhood of x contains all but finitely many of the points x_n. If (S_1, τ_1) and (S_2, τ_2) are two topological spaces, then we say that $f : S_1 \to S_2$ is continuous if $f^{-1}(U) \in \tau_1$ for all $U \in \tau_2$. A *homeomorphism* $f : S_1 \to S_2$ is a bijective continuous map whose inverse is also continuous.

If τ_1, τ_2 are two different topologies on the same set S such that $\tau_1 \subset \tau_2$, then we say that τ_1 is a *coarser* topology than τ_2, or alternatively that τ_2 is a *finer* topology than τ_1. A finer topology has 'more open sets'. Note that if a sequence converges in τ_2 then it necessarily converges in τ_1 but that the converse does not hold. The coarsest topology on any set S is the trivial topology, whose only open sets are the empty set and S itself. The finest topology on any set is the discrete topology, for which any subset of S is declared to be open.

Exercise A.2. a) Suppose that (S,τ) is a topological space, and that β is a base for τ. Show that:

i) If $x \in S$, then there exists some $B \in \beta$ with $x \in B$.

ii) If B_1, $B_2 \in \beta$, then for every $x \in B_1 \cap B_2$ there exists $B \in \beta$ with:

$$x \in B \qquad B \subset B_1 \cap B_2.$$

b) Conversely, suppose that one is given a set S and a collection β of subsets of S satisfying *i*), *ii*) above. Define τ by:

$$U \in \tau \iff \text{for all } x \in U, \text{ there exists } B \in \beta \text{ such that } x \in B \text{ and } B \subset U.$$

i.e. τ is the set of all unions of elements of β. Show that (S,τ) is a topological space, with base β. We say that τ is the topology generated by β

c) Suppose that β, β' both satisfy conditions $i)$, $ii)$ above and generate topologies τ, τ' respectively. Moreover, suppose that if $B \in \beta$ then for every $x \in B$ there exists $B' \in \beta'$ satisfying

$$x \in B', \qquad \text{and} \qquad B' \subset B$$

Then $\tau \subset \tau'$.

If $E \subset S$ is any subset of a topological space (S,τ), then E inherits a topology, $\tau|_E$, called the subspace topology given by:

$$\tau|_E = \{E \cap U : U \in \tau\}.$$

If (S_1,τ_1) and (S_2,τ_2) are two topological spaces, then $S_1 \times S_2$ inherits a topology τ called the product topology, which is generated by the base

$$\beta = \{U_1 \times U_2 : U_i \in \tau_i, i = 1, 2\}$$

In other words, a set U is open in the product topology if it is the union of sets of the form $U_1 \times U_2$ with $U_i \in \tau_i$, $i = 1, 2$.

Exercise A.3. Suppose (S_1,τ_1), (S_2,τ_2) and (S_3,τ_3) are topological spaces, and that $f : S_1 \times S_2 \to S_3$ is a continuous map. Show that for each $a \in S_1$ and $b \in S_2$, the maps

$$f_a : \begin{array}{l} S_2 \to S_3, \\ y \mapsto f(a,y), \end{array} \qquad f^b : \begin{array}{l} S_1 \to S_3, \\ x \mapsto f(x,b), \end{array}$$

are continuous.

The condition that f is continuous with respect to the product topology is sometimes called *joint continuity*, while the continuity of f_a, f^b is called *separate continuity.* Thus joint continuity implies separate continuity. The converse is not true.

Theorem A.5. *Let (S_1,τ_1) and (S_2,τ_2) be two topological spaces, and let β_1 respectively β_2 be a base. Then the set*

$$\beta = \{B_1 \times B_2 : B_1 \in \beta_1, B_2 \in \beta_2\},$$

is a base for the product topology $(S_1 \times S_2, \tau)$.

Proof. Suppose $U \in \tau$, and let $x = (x_1, x_2) \in U$. By the definition of the product topology, there exist $U_1 \in \tau_1$ and $U_2 \in \tau_2$ with $x \in U_1 \times U_2$ and $U_1 \times U_2 \subset U$. Since β_1 is a base for (S_1,τ_1), and $U_1 \in \tau_1$, there exists $B_1 \in \beta_1$ with $x_1 \in B_1$ and $B_1 \subset U_1$. Similarly there exists $B_2 \in \beta_2$ such that $x_2 \in B_2$ and $B_2 \subset U_2$. Thus $x \in B_1 \times B_2$ and $B_1 \times B_2 \subset U_1 \times U_2 \subset U$. Considering these sets as x ranges over U, we see that U may be written as a union of elements of β and we're done. □

Example 19. *The real numbers* $\mathbb{R}$ *carry a topology, called the order topology, generated by the base:*

$$\beta_{\mathbb{R}} = \{(a,b) : a, b \in \mathbb{R},\ a < b\}\,.$$

This induces the product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$*. This is called the standard topology on* $\mathbb{R}^n$*.*

> **Exercise A.4.** Show that the base
>
> $$\beta_{\mathbb{Q}} = \{(p,q) : p, q \in \mathbb{Q},\ p < q\}\,,$$
>
> generates the standard topology on $\mathbb{R}$.

With the result of this exercise, we can establish the following very useful fact about open sets in $\mathbb{R}^n$.

Lemma A.6. *Suppose* $\Omega \subset \mathbb{R}^n$ *is open. Then there exists an* exhaustion of Ω by compact sets*. That is to say a family* $(K_i)_{i=1}^{\infty}$ *of compact sets* $K_i \subset \Omega$ *such that*

$$K_i \subset (K_{i+1})^{\circ}\,, \qquad \bigcup_{i=1}^{\infty} K_i = \Omega.$$

Proof.

1. Recall that by the definition of the product topology, a base for the standard topology of $\mathbb{R}^n$ is given by:

   $$\beta = \{I_1 \times \cdots \times I_n : I_k \in \beta_{\mathbb{Q}}\}$$

   For any $B \in \beta$ we have $B = \bigcup\{B' \in \beta : \overline{B'} \subset B\}$ since, for example

   $$(p,q) = \bigcup_{n=N}^{\infty} \left(p + \frac{1}{n}, q - \frac{1}{n}\right)$$

   for some $N > [2(q-p)]^{-1}$, and taking products of such sets the result follows.

2. Let

   $$\beta' = \left\{B \in \beta : \overline{B} \subset \Omega\right\}.$$

   Since β is a base, $\bigcup\{B \in \beta : B \subset \Omega\} = \Omega$, thus in view of the discussion above $\Omega = \bigcup \beta'$. Moreover, since β can be put into one-to-one correspondence with a subset of $\mathbb{Q}^{2n}$, we have that β and hence β' is countable.

3. Let us take an enumeration

   $$\beta' = \{B_1, B_2, \ldots\}.$$

   We define K_i inductively as follows. Pick $K_1 = \overline{B_1}$. This is a closed box in $\mathbb{R}^n$, so is compact. Now suppose that $K_1, \ldots K_n$ have been chosen. Since $K_n \subset \Omega$, β' is an open cover of K_n and so admits a finite subcover. Therefore there exists i_n such that $K_n \subset B_1 \cup \ldots \cup B_{i_n}$. We define $K_{n+1} = \overline{B_1} \cup \ldots \cup \overline{B_{i_n}}$. This is a union of closed boxes, hence is compact.

4. By construction we have $K_i \subset (K_{i+1})^\circ$. Moreover $K_{n+1} \not\subset B_1 \cup \ldots \cup B_{i_n}$, so $i_{n+1} > i_n$, and thus $i_n \to \infty$ as $n \to \infty$. Pick $x \in \Omega$. Then $x \in B_i$ for some i. Since $i_n \to \infty$, $x \in K_n$ for sufficiently large n, thus $\bigcup_{i=1}^\infty K_i = \Omega$.

□

For another example of a topological space, we return to the vector space setting.

Example 20. *Let (S, d) be a metric space. The open ball of radius $r > 0$ about $x \in S$ is defined to be:*

$$B_r(x) := \{y \in S : d(x, y) < r\}.$$

The metric topology *is the topology induced by the base:*

$$\beta = \{B_r(x) : x \in S, r \in \mathbb{R}_+\}.$$

We say that a general topological space (S, τ) is metrizable *if there exists some metric d on S such that the metric topoplogy of (S, d) coincides with τ.*

> **Exercise A.5.** Suppose that (S, d) is a metric space. Show that S is Hausdorff with respect to the metric topology.

An important feature of metric spaces is that the notions of compactness and *sequential compactness* are equivalent. We say that a topological space (S, τ) is sequentially compact if every sequence $(x_n)_{n=1}^\infty$ with $x_n \in S$ admits a subsequence $(x_{n_i})_{i=1}^\infty$ such that x_{n_i} converge to $x \in S$ as $i \to \infty$.

Theorem A.7. *Let (S, d) be a metric space endowed with the metric space topology. Then S is compact if and only if it is sequentially compact.*

Proof. 1. First suppose S is compact and consider the sequence $(x_n)_{n=1}^\infty$. We must exhibit a convergent subsequence. Let us consider the set $A = \{x_n\}_{n=1}^\infty$. If A is finite, then x_n must take at least one value an infinite number of times, so has a subsequence converging to that value, and we're done.

Now suppose A is infinite. We claim that A has a limit point. Suppose not. In particular, this means that each $y \in S$ has a neighbourhood U_y such that $U_y \cap A \subset \{y\}$. The collection $\{U_y : y \in S\}$ is an open cover of S, hence admits a finite subcover, say $\{U_{y_1}, \ldots U_{y_N}\}$. Note that we have

$$\begin{aligned} A = S \cap A &= (U_{y_1} \cup \ldots \cup U_{y_N}) \cap A \\ &= (U_{y_1} \cap A) \cup \ldots \cup (U_{y_N} \cap A) \subset \{y_1, \ldots, y_N\} \end{aligned}$$

Since A is infinite, this contradicts the assumption that A has no limit points.

Let x be a limit point of A. Since any metric space is Hausdorff, every neighbourhood of x must contain infinitely many points in A. Define a subsequence as follows. We pick n_1 such that $x_{n_1} \in B_1(x)$. Suppose we have $x_{n_{k-1}}$. We define n_k by requiring $x_{n_k} > x_{n_{k-1}}$ and $x_{n_k} \in B_{k^{-1}}(x)$. This can always be done. By construction the subsequence $\{x_{n_k}\}_{k=1}^\infty$ converges to x.

2. Now suppose that (S, d) is sequentially compact. We first claim that if $\mathscr{U}$ is any open cover of S, then there exists $\delta > 0$ with the property that for each $x \in S$, there exists $U \in \mathscr{U}$ with $B_\delta(x) \subset U$. Note that while U will depend on x, δ does not.

   Suppose not. Then for each n, there exists $x_n \in S$ such that $B_{\frac{1}{n}}(x_n)$ is not contained in any element of $\mathscr{U}$. By the assumption of sequential compactness, we can choose a subsequence $x_{n_i} \to a$ for some $a \in S$. Now, since $\mathscr{U}$ is a open cover, there exists $U \in \mathscr{U}$ with $a \in U$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subset U$. Now pick i sufficiently large that $n_i^{-1} < \epsilon/2$ and $x_{n_i} \in B_{\epsilon/2}(a)$. Then we have $B_{1/n_i}(x_{n_i}) \subset B_\epsilon(a) \subset U$, a contradiction.

3. Next, we show that if (S, d) is sequentially compact, then for each $\epsilon > 0$ there exists a finite covering of S by balls of radius ϵ. Suppose not, then S cannot be covered by finitely many balls of radius ϵ. Construct a sequence as follows: take $x_1 \in S$ to be arbitrary. Given $x_1, \ldots x_n$, choose

$$x_{n+1} \in (B_\epsilon(x_1) \cup \ldots \cup B_\epsilon(x_n))^c$$

   which is always possible. Now, by construction (x_n) has no convergent subsequence, since $B_{\epsilon/2}(x)$ contains at most one element of (x_n), and we have a contradiction with the assumption of sequential compactness.

4. Finally we are ready to show that if (S, d) is sequentially compact, then it is compact. Let $\mathscr{U}$ be an open cover of S. Then by 2. above, there exists $\delta > 0$ such that for any $x \in S$, $B_\delta(x)$ is contained in an element of $\mathscr{U}$. By 3. we know that we can choose $x_1, \ldots x_N$ such that the sets $B_\delta(x_i)$ for $i = 1, \ldots, N$ cover S. Let $U_i \in \mathscr{U}$ be such that $B_\delta(x_i) \subset U_i$. Then we must have

$$S = \bigcup_{i=1}^{N} B_\delta(x_i) \subset \bigcup_{i=1}^{N} U_i,$$

   so by construction, $\{U_i\}_{i=1}^N$ is a finite subcover of $\mathscr{U}$. □

Exercise A.6. Let us take $X = \mathbb{R}^n$, thought of as a vector space over $\mathbb{R}$ and define:

$$\|(x_1, \ldots, x_n)\|_p = (|x_1|^p + \ldots |x_n|^p)^{\frac{1}{p}}, \qquad p \geq 1.$$

a) Show that $(\mathbb{R}^n, \|\cdot\|_p)$ is a normed vector space:

   i) First check that the positivity and homogeneity property are satisfied.

   ii) Establish the triangle inequality for the special case $p = 1$.

   iii) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

   Hint: set $t = p^{-1}$, *consider the function* $\log\left[ta^p + (1-t)b^q\right]$ *and use the concavity of the logarithm*

iv) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $\|x\|_p = 1$ and $\|y\|_q = 1$ then

$$\sum_{i=1}^{n} |x_i y_i| \leqslant 1.$$

Deduce Hölder's inequality:

$$\sum_{i=1}^{n} |x_i y_i| \leqslant \|x\|_p \|y\|_q, \qquad \text{for all } x, y, \in \mathbb{R}^n.$$

v) Show that

$$\|x+y\|_p^p \leqslant \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$$

vi) Apply Hölder's inequality to deduce:

$$\|x+y\|_p^p \leqslant \left(\|x\|_p + \|y\|_p \right) \|x+y\|_p^{p-1}$$

and conclude

$$\|x+y\|_p \leqslant \|x\|_p + \|y\|_p .$$

b) Show that the metric topology of $(\mathbb{R}^n, \|\cdot\|_p)$ agrees with the standard topology. *Hint: Use part c) of Exercise A.2*

Exercise A.7 ($\star$)**.** Let $X = C[0,1]$, the set of continuous functions on the closed interval $[0,1]$. For $f \in X$, $p \geqslant 0$ define:

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

a) Show that X is a vector space over $\mathbb{R}$, where scalar multiplication and vector addition are defined pointwise.

b) Establish Hölder's inequality:

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q$$

for $p, q > 1$ with $p^{-1} + q^{-1} = 1$.

c) Show that $(X, \|\cdot\|_p)$ is a normed space.

d) Suppose $p \leqslant p'$. Show that:

$$\|f\|_p \leqslant \|f\|_{p'}$$

e) Let τ_p be the metric topology of $(X, \|\cdot\|_p)$. Show that if $p \leqslant p'$:

$$\tau_p \subset \tau_{p'}.$$

f) Consider the sequence of functions:

$$f_n(x) = \begin{cases} n^{\gamma-1} & 0 \leqslant x < \frac{1}{n} \\ \frac{1}{n} x^{-\gamma} & \frac{1}{n} \leqslant x \leqslant 1 \end{cases}$$

where $n = 1, 2, \ldots$

i) Show that $f_n \in C[0,1]$ and

$$\lim_{n\to\infty} \|f_n\|_p = \begin{cases} 0 & \gamma < \frac{p+1}{p} \\ \left(\frac{p+1}{p}\right)^{\frac{1}{p}} & \gamma = \frac{p+1}{p} \\ \infty & \gamma > \frac{p+1}{p} \end{cases}$$

ii) By choosing γ carefully, show that if $p < p'$ then

$$\tau_{p'} \not\subset \tau_p.$$

Hint: in parts b), c) follow the same steps as for the finite dimensional case in Exercise A.6.

Exercise A.8. Verify that if (S, d) is a metric space, then the metric topology defines the same notions of convergence and continuity as the standard definitions for a metric space.

A.1.3 Topological vector spaces

Having briefly introduced the concept of vector spaces and topological spaces, we are now ready to define a topological vector space.

Definition A.7 (Topological vector space axioms)**.** *A vector space X over a field Φ, where Φ is either $\mathbb{R}$ or $\mathbb{C}$ is called a* topological vector space *if X is endowed with a topology τ such that:*

i) Every point of X is a closed set,

ii) The vector space operations are continuous with respect to τ.

To put a bit of flesh on the bones of this definition, the first condition implies that for any $x \in X$, the set $\{x\}$ is closed, or equivalently $X \setminus \{x\}$ should be open. The second condition should be understood as follows. We require firstly that the map:

$$\begin{aligned} + \quad : \quad & X \times X \to X \\ & (x, y) \mapsto x + y \end{aligned}$$

is continuous, where $X \times X$ inherits the product topology from X. Secondly, we require that the map:

$$\begin{aligned} \cdot \ : \quad & \Phi \times X \to X \\ & (a, x) \mapsto ax \end{aligned}$$

Where $\Phi \times X$ is endowed with the product topology, and we take the topology on Φ to be the standard topology on $\mathbb{R}$ or $\mathbb{C} \simeq \mathbb{R}^2$ as appropriate.

We say that a subset, E, of a topological vector space is *bounded* if for every neighbourhood V of 0 we can find $s > 0$ such that $E \subset tV$ whenever $t > s$.

A useful source of topological vector spaces are the normed spaces that we previously introduced. We can verify that these indeed satisfy the topological vector space axioms:

Theorem A.8. *If $(X, \|\cdot\|)$ is normed vector space, endowed with the metric topology, then X is a topological vector space. A set $E \subset X$ is bounded if and only if $\sup_{x \in E} \|x\| < \infty$.*

Proof.

1. First we note that for each $x \in X$, the set $\{x\}$ is closed. To see this, suppose that $y \neq x$, and set $r = \frac{1}{2}\|y - x\|$. Then the open ball $B_r(y) = \{z \in X : \|z - y\| < r\}$ does not contain x, thus we have shown that $X \setminus \{x\}$ is open.

2. Now suppose U is an open set in X and let $x, y \in X$ be such that $z = x + y \in U$. By the openness of U in the norm topology, there exists $r > 0$ such that the set $B_r(z) \subset U$. Let $W = B_{\frac{r}{2}}(x) \times B_{\frac{r}{2}}(y)$, and suppose $(x', y') \in W$. Clearly

$$\left\|x' + y' - z\right\| = \left\|x' - x + y' - y\right\| \leqslant \left\|x' - x\right\| + \left\|y' - y\right\| < r$$

so that $x' + y' \in B_r(z) \subset U$. Thus the set $W \subset (+)^{-1}(U) \subset X \times X$. However, W is open in $X \times X$ by the definition of the product topology. Since $(x, y) \in (+)^{-1}(U)$ was arbitrary, we deduce that $(+)^{-1}(U)$ is open and so $+ : X \times X \to X$ is continuous.

3. Finally, suppose that U is an open set in X and let $x \in X$, $a \in \Phi$ be such that $z = ax \in U$. By the openness of U in the norm topology, there exists $r > 0$ such that the set $B_r(z) \subset U$. Let $W = \{b \in \Phi : |a - b| < r_1\} \times B_{r_2}(x)$, and suppose $(a', x') \in W$. Then we have:

$$\begin{aligned} \left\|a'x' - z\right\| &= \left\|a'x' - ax\right\| = \left\|(a' - a)x' - a(x - x')\right\| \\ &\leqslant \left\|(a' - a)x'\right\| + \left\|a(x - x')\right\| \\ &< r_1 \left(\|x\| + r_2\right) + |a|\, r_2 \end{aligned}$$

Setting $r_1 = r_2 = \frac{\min\{r,1\}}{4(1+\|x\|+|a|)}$, we have

$$r_1 \left(\|x\| + r_2\right) + |a|\, r_2 \leqslant \frac{3r}{4},$$

and so $a'x' \in B_r(z) \subset U$. Thus the set $W \subset (\cdot)^{-1}(U) \subset \Phi \times X$. However, W is open in $\Phi \times X$ by the definition of the product topology. Since $(a, x) \in (\cdot)^{-1}(U)$ was arbitrary, we deduce that $(\cdot)^{-1}(U)$ is open and so $\cdot : \Phi \times X \to X$ is continuous.

4. Now suppose $E \subset X$ is bounded. Then in particular, since $B_1(0)$ is an open neighbourhood of 0, we have that $E \subset tB_1(0)$ for some $t > 0$. However, $tB_1(0) = B_t(0) = \{x \in X : \|x\| < t\}$, so necessarily we must have $\sup_{x\in E} \|x\| < t < \infty$. Conversely, suppose that $\sup_{x\in E} \|x\| = M < \infty$, and let V be any neighbourhood of 0. Since V is open in the metric topology, there exists $\epsilon > 0$ such that $B_\epsilon(0) \subset V$. Let $t = 2M\epsilon^{-1}$. We have $tB_\epsilon(0) = B_{2M}(0) = \{x \in X : \|x\| < 2M\}$, thus $E \subset tB_\epsilon(0) \subset tV$. □

Remark. *One has to be careful with various notions of boundedness for sets. While for a normed space the notion of boundedness introduced for topological vector spaces above is equivalent to the set having finite diameter, this is not true for general metric spaces. See the remark after Theorem A.18.*

We now prove a simple but useful consequence of the topological vector space definition.

Lemma A.9. *Let X be a topological vector space. For any $a \in X$ and $\lambda \in \Phi$ with $\lambda \neq 0$, define the maps:*

$$T_a : X \to X, \quad x \mapsto x + a. \qquad M_\lambda : X \to X, \quad x \mapsto \lambda x.$$

These are homeomorphisms of X to itself.

Proof. These maps are manifestly bijective, with inverses given by $(T_a)^{-1} = T_{-a}$ and $(M_\lambda)^{-1} = M_{\lambda^{-1}}$. All four maps are continuous by the definition of the topological vector space, since joint continuity implies separate continuity (see Exercise A.3). □

Lemma A.9 tells us that a set $E \subset X$ is open if and only if all of the translates $a + E$ are open. In particular this means that the topology of a topological vector space is determined by a *local* base at the origin.

Theorem A.10. *Suppose that (X, τ) is a topological vector space, and that $\dot\beta$ is a local base at 0. Then the collection*

$$\beta = \left\{ a + B : a \in X, B \in \dot\beta \right\}$$

is a base for τ.

Proof. Recall that a collection of open sets $\dot\beta$ is a local base at the origin if every neighbourhood of the origin contains a member of $\dot\beta$. First note that β is a collection of open sets, since translations of open sets are open. Now suppose that $U \in \tau$ is an open set and pick $x \in U$. We have that $(-x) + U$ is a neighbourhood of the origin, and so there exists $B \in \dot\beta$ such that $B \subset (-x) + U$. Since translation of sets preserves inclusions, we have $x + B \subset x + (-x) + U = U$. Thus for any $U \in \tau$ we have exhibited an element of β contained in U, so β is indeed a base for τ. □

Theorem A.11. *Suppose that X is a topological vector space. Then:*

a) If $U \subset X$ is a neighbourhood of 0 then U contains a balanced neighbourhood of 0.

b) If $U \subset X$ is a convex neighbourhood of 0, then U contains a convex balanced neighbourhood of 0.

Proof. 1. Since scalar multiplication is continuous, there exists $\delta > 0$ and V open such that $\alpha V \subset U$ for all $|\alpha| < \delta$. Let

$$W = \bigcup_{|\alpha|<\delta} \alpha V.$$

Then W is balanced and open, and $U' \subset U$, establishing *a*).

2. Now, suppose that U is convex, set

$$A = \bigcap_{|\alpha|=1} \alpha U$$

and choose W as in the previous paragraph. Since W is balanced, $\alpha^{-1}W = W$ whenever $|\alpha| = 1$, so $W \subset \alpha U$ for all $|\alpha| = 1$ and thus $W \subset A$. Thus A° is a neighbourhood of the origin. Clearly $A^\circ \subset U$. Since U is convex, so is αU for any α and thus A is an intersection of convex sets hence convex. The interior of a convex set is convex, thus A° is convex. Next I claim that A is balanced. Suppose $0 \leqslant r \leqslant 1$ and $|\beta| = 1$. To show A is balanced, it suffices to show that $r\beta A \subset A$. Note

$$r\beta A = \bigcap_{|\alpha|=1} r\beta\alpha U = \bigcap_{|\alpha|=1} r\alpha U.$$

However, since αU is convex and contains 0, we have $r\alpha U \subset \alpha U$, and it follows that A is balanced. It follows that A° is balanced, convex, open, contains 0 and is a subset of U.

□

Lemma A.12. *Suppose (X, τ) is a topological vector space. Then:*

a) τ is Hausdorff.

b) The set $\{x\}$ is bounded for any $x \in X$.

c) If E_1, E_2 are bounded, then so is $E_1 + E_2$. In particular, $x + E_1$ is bounded for any $x \in X$.

d) If $(x_n)_{n=1}^\infty$ is a sequence in X such that $\{x_n\}_{n=1}^\infty$ is bounded and $(a_n)_{n=1}^\infty$ is a sequence of scalars with $a_n \to 0$, then $a_n x_n \to 0$.

Proof. 1. We first show that every neighbourhood, W, of 0 contains a balanced open set U satisfying $U + U \subset W$. To see this, note that $0 + 0 = 0$, so by the continuity of 0, there exist neighbourhoods U_1, U_2 of 0 such that $U_1 + U_2 \subset W$. We let

$$U' = U_1 \cap U_2$$

which satisfies $U' + U' \subset W$. By Theorem A.11, U' has a balanced subset U, and $U + U \subset U' + U' \subset W$.

2. Now consider $x, y \in X$ with $x \neq y$. Since $\{y\}$ is closed and $x \in \{y\}^c$, there exists W, a neighbourhood of x with $y \notin W$. Since $-x + W$ is a neighbourhood of 0, there exists a balanced U with $U + U \subset -x + W$. Thus $x + U + U \subset W$ and in particular $y \notin x + U + U$. I claim $(x + U) \cap (y + U) = \emptyset$. Suppose not, then there exists $a, b \in U$ such that $x + a = y + b$, which implies $y = x + a - b$. But $a, -b \in U$, so $y \in x + U + U$, a contradiction. We have constructed sets $x + U$ and $y + U$ which are open, and contain x, y respectively, thus X is Hausdorff and we have established a).

3. Fix $x \in X$ and consider the map $f_x : \mathbb{R} \to X$ given by $f_x(\lambda) = \lambda x$. This is a continuous map, so ${f_x}^{-1}(W)$ is open in $\mathbb{R}$. Since $0 \in W$, we have $0 \in {f_x}^{-1}(W)$, and thus from the definition of an open set in $\mathbb{R}$, the interval $(-\epsilon, \epsilon) \in {f_x}^{-1}(W)$ for some $\epsilon > 0$. Thus $\lambda x \in W$ for $\lambda \in (0, \epsilon)$, or equivalently $x \in tW$ for $t > \epsilon^{-1}$. Thus we have established b).

4. Let W be any neighbourhood of 0. By paragraph 1. there exists U a neighbourhood or 0 such that $U + U \subset W$. Since E_1, E_2 are both bounded, there exists $s \in \mathbb{R}$ such that $t^{-1}E_i \subset U$ for $t > s$ and $i = 1, 2$. Thus for $t > s$,
$$t^{-1}(E_1 + E_2) = t^{-1}E_1 + t^{-1}E_2 \subset U + U \subset W,$$
or equivalently $E_1 + E_2 \subset tW$ and hence $E_1 + E_2$ is bounded, which is the first part of c). The final part of c) follows by applying the result from b).

5. For part d), suppose W is any neighbourhood of the origin in X. As in part 1., we can take U balanced and open with $U \subset W$. Then there exists $s > 0$ such that $x_n \in tU$ for all $n = 1, 2, \ldots$ and any $t > s$. Since $a_n \to 0$, there exists N such that $|a_n| < s^{-1}$ for all $n \geqslant N$. Since U is balanced, and we have $x_n \in tU$ and $|ta_n| < 1$ for $n \geqslant N$, we deduce that $a_n x_n \in U \subset W$ for all $n \geqslant N$ and we're done. □

Suppose that X is a vector space equipped with a metric d. We say that d is *invariant* if
$$d(x + z, y + z) = d(x, y),$$
for all $x, y, z \in X$. We have the following useful result

Lemma A.13. *Suppose that X is a vector space, equipped with an invariant norm d, and let τ be the induced metric topology. Given a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \to 0$, there exist scalars $\alpha_n \to \infty$ such that $\alpha_n x_n \to 0$.*

Proof. 1. First note that if d is invariant, then
$$d(nx, 0) \leqslant nd(x, 0).$$
This is clearly true if $n = 1$. Suppose it holds for $n = 1, \ldots k - 1$ Then
$$\begin{aligned} d(kx, 0) &\leqslant d(kx, x) + d(x, 0) \\ &= d((k-1)x, 0) + d(x, 0) \\ &\leqslant (k-1)d(x, 0) + d(x, 0) = kd(x, 0) \end{aligned}$$
and we're done by induction.

2. Now note that since $x_n \to 0$, for any $m \in \mathbb{N}$ there exists N_m such that
$$d(x_n, 0) < \frac{1}{m^2} \qquad n \geqslant N_m$$
where we can assume that $N_m < N_{m+1}$. We define $\alpha_n = m$ for $N_m \leqslant n < N_{m+1}$. Suppose that $N_m \leqslant n < N_{m+1}$. Then
$$d(a_n x_n, 0) \leqslant m d(x_n, 0) < \frac{1}{m}.$$
Thus as $n \to \infty$, we have that $a_n x_n \to 0$, however $a_n \to \infty$.

□

A crucially important concept which you may have come across when studying metric spaces is the idea of a Cauchy sequence.

Definition A.8. *i) Suppose (S, d) is a metric space. We say that a sequence $(x_n)_{n=1}^{\infty}$ is d-Cauchy if for every $\epsilon > 0$ we can find an integer N such that*
$$d(x_n, x_m) < \epsilon, \qquad \text{for all } n, m \geqslant N.$$
A metric space is called complete *if every d-Cauchy sequence converges in S.*

ii) Suppose (X, τ) is a topological vector space. We say that a sequence $(x_n)_{n=1}^{\infty}$ is τ-Cauchy if for every neighbourhood, U, of the origin we can find an integer N such that
$$x_n - x_m \in U, \qquad \text{for all } n, m \geqslant N.$$

Exercise A.9. Let (X, τ) be a topological vector space

a) Show that if $(x_n)_{n=1}^{\infty}$ is a τ-Cauchy sequence, then $\{x_n\}_{n=1}^{\infty}$ is bounded.

b) Fix a local base $\dot{\beta}$. Show that a sequence $(x_n)_{n=1}^{\infty}$ is τ-Cauchy if and only if for any $B \in \dot{\beta}$ we can find an integer N such that
$$x_n - x_m \in B, \qquad \text{for all } n, m \geqslant N.$$

Lemma A.14. *Suppose that X is a vector space, equipped with an invariant norm d, and let τ be the induced metric topology. Then a sequence $(x_n)_{n=1}^{\infty}$ is d-Cauchy if and only if it is τ-Cauchy.*

Proof. Suppose that (x_n) is τ-Cauchy. Then for any $\epsilon > 0$, there exists N such that for all $n, m > N$ we have $x_n - x_m \in B_\epsilon(0)$, i.e.
$$\epsilon > d(0, x_n - x_m) = d(x_n, x_m),$$
thus (x_n) is d-Cauchy.

Now suppose (x_n) is d-Cauchy. Let V be any neighbourhood of 0. Since V is open, there exists $\epsilon > 0$ such that $B_\epsilon(0) \subset V$. Since (x_n) is d-Cauchy, there exists N such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$. Thus
$$\epsilon > d(x_n, x_m) = d(0, x_n - x_m),$$
so $x_n - x_m \in B_\epsilon(0) \subset V$ and (x_n) is τ-Cauchy. □

We're now in a position to distinguish various useful classes of topological vector space. There is some difference of opinion on the definitions below, we follow here the conventions of Rudin. Here (X, τ) always refers to a topological vector space:

i) X is a *locally convex* topological vector space if there is a local base $\dot{\beta}$ whose members are convex.

ii) X is *locally bounded* if 0 has a bounded neighbourhood.

iii) X is *locally compact* if 0 has a neighbourhood whose closure is compact.

iv) X is *metrizable* if there exists some metric d on X such that τ is the metric topology induced by d.

v) X is an *F-space* if its topology τ is induced by a complete invariant metric.

vi) X is a *Fréchet space* if it is a locally convex F-space.

vii) X is *normable* if a norm exists on X such that the metric topology of the norm agrees with τ.

viii) A normed space $(X, \|\cdot\|)$, with the metric topology, is *Banach* if the metric induced by the norm is complete.

ix) A space X has the *Heine-Borel property* if every closed and bounded subset of X is compact.

The space $\mathbb{R}^n$ with the norm $\|\cdot\|_p$ introduced in the exercises is an example of a topological vector space which belongs to all of these classes. The spaces that you studied in Functional Analysis were mostly Banach spaces, although not all. For example if X is an infinite dimensional Banach space, then the weak-* topology of X^* is locally convex, but not metrisable.

We note that the converse of the Heine-Borel property is always true for a topological vector space:

Lemma A.15. *Suppose (X, τ) is a topological vector space and that $K \subset X$ is compact. Then K is closed and bounded.*

Proof.

1. The fact that K is closed follows immediately from Lemmas A.3, A.12.
2. Next, suppose U is a neighbourhood of 0. By the continuity of scalar multiplication, there exists $\delta > 0$ and a neighbourhood V of the origin in X such that $\alpha V \subset U$ for any $|\alpha| < \delta$. Define W to be the union of these sets as α varies over $\{|\alpha| < \delta\}$. Then $W \subset V$ is an open, balanced, neighbourhood of 0.
3. Now I claim that
$$\bigcup_{n=1}^{\infty} nW = X.$$

To see this, fix $x \in X$. Since the map $\alpha \mapsto \alpha x$ is continuous, the set of all α with $\alpha x \in W$ is open and contains 0, hence contains n^{-1} for sufficiently large n. Thus $n^{-1}x \in W$, or $x \in nW$ for large enough n. Note that since W is balanced, in particular $sW \subset tW$ for $s < t$.

4. Finally, since $\mathscr{U} = \{nW\}_{n=1}^{\infty}$ is an open cover of X, it is also an open cover of K. Thus there exist $n_1, \ldots, n_N$ such that

$$K \subset \bigcup_{i=1}^{N} n_i W = n_N W \subset n_N U.$$

Thus K is bounded.

□

A.2 Locally convex spaces

We shall now specialise somewhat, to the case of *locally convex* topological vector spaces. These can be given a nice description in terms of a family of semi-norms. When that family is countable, the topology is equivalent to that induced by an invariant metric, which if it is complete gives a Fréchet space. These can be thought of as generalisations of the Banach spaces that you may be familiar with from functional analysis. The canonical example of a Fréchet space that is not a Banach space is $C^{\infty}(\Omega)$, the space of smooth functions on an open set Ω. The finite regularity spaces on a compact set $C^k(K)$ are Banach spaces in a natural way, but this is not true of $C^{\infty}(\Omega)$.

A.2.1 Semi-norms

A very useful way to construct the topology for a locally convex topological vector space is via a family of semi-norms.

Definition A.9. *A* seminorm *on a vector space X over Φ (with Φ being either $\mathbb{R}$ or $\mathbb{C}$) is a map $p : X \to \mathbb{R}$ satisfying:*

i) p is subadditive. *For all $x, y \in X$ we have:*

$$p(x + y) \leqslant p(x) + p(y)$$

ii) For all $\lambda \in \Phi$ and $x \in X$ we have:

$$p(\lambda x) = |\lambda|\, p(x)$$

A family of seminorms $\mathscr{P}$ is said to be separating *if for every $x \in X$ with $x \neq 0$, there is at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.*

From the definition we can immediately deduce some useful properties:

Lemma A.16. *Let X be a vector field over $\mathbb{R}$ or $\mathbb{C}$, and let $p : X \to \mathbb{R}$ be a seminorm. Then:*

a) $p(0) = 0$

b) $|p(x) - p(y)| \leqslant p(x - y)$

c) $p(x) \geqslant 0$

d) $\{x : p(x) = 0\}$ *is a vector subspace of* X*.*

e) The set $B = \{x : p(x) < 1\}$ *is convex and balanced.*

Proof. a) Applying property *ii*) from the definition of a seminorm with $\lambda = 0$ we immediately have $p(0) = 0$.

b) From the subadditivity property we have

$$p(x) = p(x - y + y) \leqslant p(x - y) + p(y)$$

so $p(x) - p(y) \leqslant p(x - y)$. Similarly $p(y) - p(x) \leqslant p(y - x)$, but $p(x - y) = p(y - x)$ and the result follows.

c) Applying $a), b)$ with $y = 0$ gives $|p(x)| \leqslant p(x)$ which implies $p(x) \geqslant 0$.

d) Suppose $p(x) = p(y) = 0$ and $\lambda, \mu \in \Phi$. Applying $c)$ we have:

$$0 \leqslant p(\lambda x + \mu y) \leqslant |\lambda|\, p(x) + |\mu|\, p(y) = 0,$$

so that $p(\lambda x + \mu y) = 0$ and thus $\{x : p(x) = 0\}$ is a vector subspace.

e) It is clear that B is balanced by property *ii*). To see that B is convex, suppose that $x, y \in B$ and $0 < t < 1$. Then

$$p(tx + (1 - t)y) \leqslant tp(x) + (1 - t)p(y) < 1,$$

so $tx + (1 - t)y \in B$ and B is convex. □

Note that these results are already enough to show that a seminorm p with the property that $p(x) \neq 0$ whenever $x \neq 0$, is in fact a norm.

We are now ready to prove an important result that shows that a family of seminorms specifies a locally convex topology on a vector space. The proof is quite long, and you may wish to omit it on a first read through. The argument is similar to the proof of Theorem A.8, which in fact could be understood as a corollary of this result.

Theorem A.17. *Suppose that* $\mathscr{P}$ *is a separating family of seminorms on a vector space* X*. Associate to each* $p \in \mathscr{P}$ *and* $n \in \mathbb{N}$ *the set:*

$$V(p, n) = \left\{ x \in X : p(x) < \frac{1}{n} \right\}.$$

Let $\dot{\beta}$ *be the collection of all finite intersections of the sets* $V(p, n)$*. Then* $\dot{\beta}$ *is a convex, balanced, local base for a topology* τ *on* X*, which turns* X *into a locally convex topological vector space such that:*

a) every $p \in \mathscr{P}$ is continuous

b) a set $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.

Proof. 1. Let us define β by

$$\beta = \left\{ x + B : x \in X, B \in \dot{\beta} \right\}.$$

Since $0 \in B$ for any $B \in \dot{\beta}$, we immediately have that for any $x \in X$ there is an element of β containing x. Now suppose that $B_1, B_2 \in \beta$. We may write

$$B_1 = y + \bigcap_{i=1}^{N} V(p_i, n_i), \qquad B_2 = z + \bigcap_{j=1}^{M} V(q_i, m_j),$$

for $y, z \in X$, $p_i, q_j \in \mathscr{P}$ and $N, M, n_i, m_j \in \mathbb{N}$. Fix $x \in B_1 \cap B_2$. Clearly $x \in B$ and $B \in \beta$. I claim that $B \subset B_1 \cap B_2$ for

$$B = x + \left(\bigcap_{i=1}^{N} V(p_i, n_i') \right) \cap \left(\bigcap_{j=1}^{M} V(q_j, m_j') \right),$$

provided that n_i', m_i' are chosen sufficiently large. Since $x \in B_1 \cap B_2$, we have:

$$p_i(x-y) < \frac{1}{n_i'}, \quad q_i(x-z) < \frac{1}{m_j'}, \qquad \text{for all } i = 1, \dots, N, \quad j = 1, \dots, M$$

For each i, j, pick n_i', m_j' sufficiently large that

$$p_i(x-y) + \frac{1}{n_i'} < \frac{1}{n_i}, \quad q_j(x-z) + \frac{1}{m_j'} < \frac{1}{m_j}$$

Now suppose $w \in B$. Then we have that:

$$p_i(w-x) < \frac{1}{n_i'}, \quad q_i(w-x) < \frac{1}{m_j'}, \qquad \text{for all } i = 1, \dots, N, \quad j = 1, \dots, M$$

Using the subadditivity of p_i we have that for each i:

$$p_i(w-y) \leqslant p_i(w-x) + p_i(x-y) < \frac{1}{n_i'} + p_i(x-y) < \frac{1}{n_i},$$

thus $w \in B_1$. Similarly, we have for each j that:

$$q_j(w-z) \leqslant q_j(w-x) + q_j(x-z) < \frac{1}{m_j'} + q_j(x-y) < \frac{1}{m_i},$$

Thus the collection β satisfies the conditions of Exercise A.2 and thus defines a topology τ on X. Moreover, $\dot{\beta}$ is a local base for τ and each element of $\dot{\beta}$ is convex and balanced.

2. Suppose that $x, y \in X$ with $x \neq y$. Then since $x - y \neq 0$ and $\mathscr{P}$ is separating, there exists $p \in \mathscr{P}$ such that $p(x-y) > 0$. Thus, there exists $n \in \mathbb{N}$ such that $np(x-y) > 1$. For this n we have that $x \notin (y + V(p,n))$. Thus we may write $\{x\}^c$ as a union of sets which are open in τ, hence $\{x\}$ is closed.

3. Next we must show that addition is continuous. Suppose U is an open set in X, and pick $z \in U$. Then
$$\bigcap_{i=1}^{N} V(p_i, n_i) \subset -z + U$$
for some $p_i \in \mathscr{P}$, $n_i \in \mathbb{N}$. Suppose that $(x,y) \in (+)^{-1}(z)$, i.e. $x + y = z$. Let
$$V_1 = x + \bigcap_{i=1}^{N} V(p_i, 2n_i), \qquad V_2 = y + \bigcap_{i=1}^{N} V(p_i, 2n_i)$$
and suppose $(w_1, w_2) \in V_1 \times V_2$. Then for all i we have
$$p_i(w_1 + w_2 - z) = p_i(w_1 - x + w_2 - y) \leqslant p_i(w_1 - x) + p_i(w_2 - y) < \frac{1}{2n_i} + \frac{1}{2n_i} < \frac{1}{n_i}$$
so that $V_1 + V_2 \subset U$, or alternatively $V_1 \times V_2 \subset (+)^{-1}(U)$. Thus we can write $(+)^{-1}(U)$ as a union of sets which are open in the product topology. This proves that addition is continuous.

4. Next we must show that scalar multiplication is continuous. Suppose U is an open set in X, and pick $z \in U$. Then
$$\bigcap_{i=1}^{N} V(p_i, n_i) \subset -z + U$$
for some $p_i \in \mathscr{P}$, $n_i \in \mathbb{N}$. Suppose that $(\alpha, x) \in (\cdot)^{-1}(z)$, i.e. $\alpha y = z$. Let
$$V = x + \bigcap_{i=1}^{N} V(p_i, n_i'), \qquad D = \{\beta \in \Phi : |\alpha - \beta| < \epsilon\}$$
Suppose $(\beta, y) \in D \times V$ Then for each i we have:
$$\begin{aligned} p_i(\beta y - \alpha x) &= p_i(\beta(y-x) - (\alpha - \beta)x) \\ &< \frac{|\beta|}{n_i'} + \epsilon p_i(x) \\ &\leqslant \frac{|\alpha| + \epsilon}{n_i'} + \epsilon p_i(x). \end{aligned}$$
Taking $\epsilon < (2n_i p_i(x))^{-1}$ and $n_i' > 2(|\alpha| + (2n_i p_i(x))^{-1})$ for each i, we conclude that
$$(\beta y - z) \in \bigcap_{i=1}^{N} V(p_i, n_i),$$
which implies that $D \times V \in (\cdot)^{-1}(U)$. Thus scalar multiplication is continuous, and we have established that (X, τ) is a locally convex topological space.

5. To see that $p \in \mathscr{P}$ is continuous, we must show that $p^{-1}(a,b)$ is open, where $a < b$. Suppose $x \in p^{-1}(a,b)$, so that $a < p(x) < b$. Consider
$$U = x + V(p,n)$$
Suppose $y \in V$. Then
$$|p(y) - p(x)| \leqslant p(y-x) < \frac{1}{n},$$
by part *c*) of Lemma A.16. For n sufficiently large, we have $p(y) \in (a,b)$ so that $V \subset p^{-1}(a,b)$ and we're done.

6. It remains to show that $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E. First suppose E is bounded and fix $p \in \mathscr{P}$. Since $V(p,1)$ is a neighbourhood of the origin, from the definition of boundedness we have that $E \subset kV(p,1)$ for some $k < \infty$. But $x \in kV(p,1)$ implies $p(x) < k$, so that p is bounded on E.

   Now suppose that every $p \in \mathscr{P}$ is bounded on E. Let U be a neighbourhood of the origin. Then
$$\bigcap_{i=1}^{N} V(p_i, n_i) \subset U$$
for some $p_i \in \mathscr{P}$, $n_i \in \mathbb{N}$. By our assumption, there exist $M_i < \infty$ such that $p_i < M_i$ on E of $1 \leqslant i \leqslant N$. If $n > M_i n_i$ for all i, then $E \subset nU$, since if $p_i(x) < M_i$, we have
$$p_i\,(x) < M_i < \frac{n}{n_i} \qquad i = 1, \ldots, N$$
so that
$$p_i\left(\frac{1}{n}x\right) < \frac{1}{n_i} \qquad i = 1, \ldots, N$$
and $n^{-1}x \in U$. □

Thus we have seen that a separating family of seminorms gives rise to a locally convex topological space. In fact, the converse is true: given a locally convex topological space, we can find a (not necessarily unique) separating family of seminorms which generates the topology in the manner of the previous theorem.

In the case where the separating family of seminorms $\mathscr{P}$ is *countable*, we have an alternative means of describing the topology.

Theorem A.18. *Let*
$$\mathscr{P} = \{p_i\}_{i=1}^{\infty}$$
be a countable separating family of seminorms on a vector space X, and let τ be the topology induced by this family as described in Theorem A.17. Then the locally convex topological vector space (X,τ) is metrizable, *and the topology τ agrees with that induced by the invariant metric:*
$$d(x,y) = \sum_{i=1}^{\infty} 2^{-i} \frac{p_i(x-y)}{1 + p_i(x-y)}$$

Proof. 1. We first verify that d indeed defines an invariant metric. It is clearly symmetric since $p(-x) = p(x)$ for a seminorm. We note that the map

$$F : t \mapsto \frac{t}{1+t}$$

is smooth, monotone increasing, concave and takes $[0,\infty)$ to $[0,1)$. Thus $d(x,y)$ is a sum of non-negative terms, so $d(x,y) \geqslant 0$. Equality occurs if and only if $p_i(x-y) = 0$ for all i, which by the fact that $\mathscr{P}$ is separating implies $x = y$. Next we claim that F is subadditive. To see this, we note that by the convexity of F, together with $F(0) = 0$ we have for $t \geqslant 0$ and $0 < \lambda < 1$:

$$F(\lambda t) = F(\lambda t + (1-\lambda)0) \geqslant \lambda F(t) + (1-\lambda)F(0) = \lambda F(t).$$

Then for $t, s \geqslant 0$:

$$\begin{aligned} F(t) + F(s) &= F\left((t+s)\frac{t}{t+s}\right) + F\left((t+s)\frac{s}{t+s}\right) \\ &\geqslant \frac{t}{t+s}F(t+s) + \frac{s}{t+s}F(t+s) = F(t+s). \end{aligned}$$

Now, since p is a seminorm we have

$$\begin{aligned} & p(x-y) \leqslant p(x-z) + p(z-y) \\ \implies\quad & F[p(x-y)] \leqslant F\left[p(x-z) + p(z-y)\right] \qquad \text{(Monotonicity of } F) \\ \implies\quad & F[p(x-y)] \leqslant F\left[p(x-z)\right] + F[p(z-y)] \qquad \text{(Subadditivity of } F) \end{aligned}$$

Thus we conclude

$$\begin{aligned} d(x,y) &= \sum_{i=1}^{\infty} 2^{-i} \frac{p_i(x-y)}{1+p_i(x-y)} \\ &\leqslant \sum_{i=1}^{\infty} 2^{-i} \left(\frac{p_i(x-z)}{1+p_i(x-z)} + \frac{p_i(z-y)}{1+p_i(z-y)}\right) \\ &= d(x,z) + d(z,y), \end{aligned}$$

so d is indeed a metric on X. It is manifestly invariant.

2. Now we need to show that the topology induced by d, which we denote τ_d, agrees with the topology τ induced by the family of seminorms $\mathscr{P}$. Recall that the open sets of τ_d are precisely those sets which can be written as a union of the open balls $B_r(x) = \{y \in X : d(x,y) < r\}$.

3. From the definition of τ, we have that each p_i is τ-continuous on X. Since $|F(t)| < 1$, we conclude by the Weierstrass M-test that the sum in $d(x,y)$ converges uniformly. Hence d is continuous as a real valued function on $X \times X$ with the product topology coming from τ. In particular, the map $d_x : X \to \mathbb{R}$ given by $d_x(y) = d(x,y)$ is continuous, and thus $d_x^{-1}(-r,r) = B_r(x)$ is an open set in the τ topology. Thus $\tau_d \subset \tau$.

4. Now suppose that W is an open set of τ, and that $x \in W$. By the definition of τ, there exists B such that $x + B \subset W$ and B has the form

$$B = \bigcap_{k=1}^{N} V\left(p_{i_k}, n_k\right)$$

for some $p_{i_k} \in \mathscr{P}$ and $n_k \in \mathbb{N}$, where we recall $V(p, n) = \{y \in X : p(y) < n^{-1}\}$. Now suppose $d(x, y) < \epsilon 2^{-M}$. Then in particular, for $0 \leqslant i \leqslant M$ we have

$$\frac{p_i(x-y)}{1+p_i(x-y)} \leqslant \epsilon,$$

so that if $\epsilon < \frac{1}{2}$:

$$p_i(x-y) \leqslant \frac{\epsilon}{1-\epsilon} \leqslant 2\epsilon.$$

Thus if we take $M \geqslant i_k$ and $\epsilon < (n_k)^{-1}$ for all $k = 1, \ldots, N$ we deduce that if $y \in B_{\epsilon 2^{-M}}(x)$ then $y - x \in B$ and thus $y \in W$. Since x was arbitrary we can write W as a union of open balls for the metric d, and thus $\tau \subset \tau_d$. □

Remark. *1. Note that while a countable separating family of seminorms gives rise to a metrizable locally convex topology, it need not be the case that the metric balls $B_r(0)$ are themselves convex. The sets $V(p, n)$ however are.*

2. It is straightforward to see that $d(x, y) < 1$ for any $x, y \in X$, so that any *subset of X has finite diameter. On the other hand, it does not follow that all subsets of X are bounded in the sense introduced above for a topological vector space.*

A.3 The test function spaces

A.3.1 $\mathscr{E}(\Omega)$ and $\mathscr{D}_K$

Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. Recall that for a function $f : \Omega \to \mathbb{C}$, we say $f \in C^\infty(\Omega)$ if $D^\alpha f$ is a continuous function in Ω for all multiindices α. Clearly $C^\infty(\Omega)$ is a vector space over $\mathbb{C}$, with addition and scalar multiplication defined pointwise: if $f, g \in C^\infty(\Omega)$, $\lambda \in \mathbb{C}$, we define the maps $f + g$, λf by

$$\begin{aligned} f+g \;&:\; \Omega \to \mathbb{C}, \\ & \quad x \mapsto f(x)+g(x), \end{aligned} \qquad\qquad \begin{aligned} \lambda f \;&:\; \Omega \to \mathbb{C}, \\ & \quad x \mapsto \lambda f(x). \end{aligned}$$

Then $f + g, \lambda g \in C^\infty(\Omega)$.

We shall endow $C^\infty(\Omega)$ with a topology which makes it into a Fréchet space with the Heine-Borel property. By the exhaustion lemma, Lemma A.6, we can find a sequence of compact sets $(K_i)_{i=0}^\infty$ such that $K_i \subset \Omega$, $K_i \subset (K_{i+1})^\circ$ and $\bigcup_i K_i = \Omega$. We define a family of seminorms by:

$$p_n(f) = \max\left\{|D^\alpha f(x)| : x \in K_n, |\alpha| \leqslant n\right\}. \tag{A.1}$$

The family $\mathscr{P} = \{p_n : n \in \mathbb{N}\}$ is separating. If $f \neq 0$, then $f(x) \neq 0$ at some point $x \in \Omega$. For n sufficiently large, $x \in K_n$, and thus $p_n(f) > 0$. Thus the family of seminorms $\mathscr{P}$ induces a topology, τ, on $C^\infty(\Omega)$ which is locally convex and metrizable by Theorem A.18. When $C^\infty(\Omega)$ is endowed with the topology τ, we use the notation $\mathscr{E}(\Omega)$. A local base is given by the sets

$$V_N = \left\{ f \in C^\infty(\Omega) : p_N(f) < \frac{1}{N} \right\}, \qquad N = 1, 2, \ldots.$$

It's useful to categorise convergence in this space in terms of more familiar concepts as follows:

Lemma A.19. *A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{E}(\Omega)$ converges to f if and only if $D^\alpha f_n \to D^\alpha f$ uniformly on compact sets for each multiindex α.*

Proof. By the translation invariance of the topology, we can assume w.l.o.g. that $f = 0$. The sequence (f_n) tends to 0 in $\mathscr{E}(\Omega)$ if and only if for each N there exists m_N such that $f_n \in V_N$ for all $n \geqslant m_N$.

First suppose $f_n \to 0$ in $\mathscr{E}(\Omega)$. Fix α and let $K \subset \Omega$ be any compact subset. For any ϵ, there exists N such that $N > \max\{|\alpha|, \epsilon^{-1}\}$, $K \subset K_N$. If $n \geqslant m_N$ then $f_n \in V_N$, which implies

$$\sup_K |D^\alpha f_n| \leqslant \sup_{K_N} |D^\alpha f_n| \leqslant p_N(f_n) \leqslant \frac{1}{N} < \epsilon,$$

Thus $D^\alpha f_n \to 0$ uniformly on K.

Conversely, suppose that for each multiindex α and compact set K we have that $D^\alpha f_n \to 0$ uniformly on K. Fix N. Then for each α with $|\alpha| \leqslant N$ we have $D^\alpha f_n \to 0$ uniformly on K_N. In particular, for each α there exists m_α such that if $n \geqslant m_\alpha$, we have that

$$\sup_{K_N} |D^\alpha f_n| \leqslant \frac{1}{N}.$$

Thus if $m = \max_{|\alpha|\leqslant N} m_\alpha$ then for all $n > m$ we have $f_n \in V_N$ and thus $f_n \to 0$ in $\mathscr{E}$. □

Theorem A.20. *The topological vector space $\mathscr{E}(\Omega)$ is a Fréchet space with the Heine-Borel property.*

Proof.
1. Since we already have that $\mathscr{E}(\Omega)$ is locally convex and inherits its topology from an invariant metric, in order to show that $\mathscr{E}(\Omega)$ is Fréchet, we simply have to show completeness. A sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathscr{E}(\Omega)$ is Cauchy if for any fixed N, there exists M such that for all $i, j \geqslant M$ we have $f_i - f_j \in V_N$. Thus

$$\sup_{K_N} |D^\alpha f_i - D^\alpha f_j| < \frac{1}{N}, \qquad \text{for all } |\alpha| \leqslant N.$$

Since K_N exhaust Ω, this implies that there exist continuous functions g^α such that $D^\alpha f_n \to g^\alpha$ uniformly on compact subsets of Ω. By a standard result, this implies that there exists a smooth function f such that $f_n \to f$ and $D^\alpha f_n \to D^\alpha f$ uniformly on compact subsets of Ω. Thus every Cauchy sequence has a limit and $\mathscr{E}(\Omega)$ is complete.

2. Now suppose that $E \subset \mathscr{E}(\Omega)$ is closed and bounded. We need to show that E is compact. By Theorem A.7 it suffices to show that any sequence in E has a convergent subsequence. By Theorem A.17, the boundedness of E is equivalent to the existence of M_N such that $p_N(f) < M_N$ for all $f \in E$.

3. In particular, we have that

$$|D^\alpha f| < M_N \qquad \text{for } |\alpha| = N$$

holds for all $f \in E$ on K_N. This in particular implies that for each β with $|\beta| < N-1$ the set $\{D^\beta f : f \in E\}$ is equicontinuous on K_{N-1}, and it is trivially pointwise bounded by the condition $p_N(f) < M_N$.

4. Suppose that $(f_n)_{n\in\mathbb{N}}$ is any sequence in E. By Arzelà-Ascoli we can extract a subsequence $(f_{n_k^1})_{k\in\mathbb{N}}$ such that $f_{n_k^1}$ converges uniformly on K_0. Suppose now that an increasing sequence of integers n_k^N are given with the property that $(D^\beta f_{n_k^N})_{k\in\mathbb{N}}$ converges uniformly on K_{N-1} for all $|\beta| \leqslant N-1$. Consider the sequence $(f_{n_k^N})_{k\in\mathbb{N}}$. Since this is a sequence in E, we know that for each β with $|\beta| \leqslant N$ the set $\{D^\beta f_{n_k^N} : f \in E, k \in \mathbb{N}\}$ is equicontinuous and pointwise bounded on K_N. Thus we can extract a subsequence $(f_{n_k^{N+1}})_{k\in\mathbb{N}}$ such that $(D^\beta f_{n_k^{N+1}})_{k\in\mathbb{N}}$ converges uniformly on K_N for all $|\beta| \leqslant N$. Thus by induction, we can find n_k^N with the required property for all N.

5. Consider the sequence $(F_k)_{k\in\mathbb{N}}$ with $F_k = f_{n_k^k}$ where n_k^N are as constructed above. Since n_k^{N+1} is a subsequence of n_k^N, we conclude that $D^\alpha F_k$ converges uniformly on compact subsets for any α, and thus for any sequence in E we have exhibited a convergent subsequence.

□

If $K \subset \mathbb{R}^n$ is a compact set, we denote by $\mathscr{D}_K$ the space of all $f \in C^\infty(\mathbb{R}^n)$ whose support lies in K. If $K \subset \Omega$, then $\mathscr{D}_K$ may be identified with a vector subspace of $C^\infty(\Omega)$. In fact, this subspace is closed with respect to the $\mathscr{E}(\Omega)$ topology. To see this, note that the map $\delta_x : \mathscr{E}(\Omega) \to \mathbb{C}$ given by $f \mapsto f(x)$ is continuous. Thus the set $\delta_x^{-1}(\{0\})$ is a closed set in $\mathscr{E}(\Omega)$. Since we can write:

$$\mathscr{D}_K = \bigcap_{x\in\Omega\setminus K} \delta_x^{-1}(\{0\})$$

and arbitrary intersections of closed sets are closed, we deduce that $\mathscr{D}_K$ is closed (and hence complete in the subspace topology). Thus $\mathscr{D}_K$ is itself a Fréchet space, when equipped with the subspace topology, which we denote τ_K.

A.3.2 $\mathscr{D}(\Omega)$

We have described the spaces $\mathscr{D}_K$, which consist of smooth functions whose support is restricted to a given compact set $K \subset \Omega$. The set $\mathscr{D}(\Omega)$ of test functions is the union over the sets $\mathscr{D}_K$ with $K \subset \Omega$ compact:

$$\mathscr{D}(\Omega) = \bigcup_{K \subset \Omega} \mathscr{D}_K.$$

In other words, $f \in \mathscr{D}(\Omega)$ if f is smooth, and is supported in some compact subset of Ω. It is clear that $\mathscr{D}(\Omega)$ is closed under the natural operations of addition and multiplication by a complex number, and thus $\mathscr{D}(\Omega)$ is a vector space over $\mathbb{C}$. We would like to endow $\mathscr{D}(\Omega)$ with a topology which turns it into a complete, locally convex, topological vector space, such that the subspace topology induced on $\mathscr{D}_K$ agrees with the natural Fréchet topology, τ_K introduced above for each compact $K \subset \Omega$.

One natural possibility is to consider the norms

$$\|f\|_k = \max\{|D^\alpha f(x)| : x \in \Omega, |\alpha| \leqslant k\}.$$

The family $\mathscr{Q} = \{\|\cdot\|_k : k = 0, 1, \ldots\}$ is a countable separating family of seminorms and so defines a locally convex metrizable topology, $\tau_{\mathscr{Q}}$ on $\mathscr{D}(\Omega)$. A local base is given by:

$$W_N = \left\{ f \in C^\infty(\Omega) : \|f\|_N < \frac{1}{N} \right\}, \qquad N = 1, 2, \ldots.$$

The subspace topology induced on $\mathscr{D}_K$ by $\tau_{\mathscr{Q}}$ is indeed τ_K. To see this, recall that τ_K is defined by the family of seminorms introduced in (A.1). Note that for any fixed compact $K \subset \Omega$ there exists N_0 such that $K \subset K_{N_0}$. For $N \geqslant N_0$ we have $\|f\|_N = p_N(f)$ for all $f \in \mathscr{D}_K$. Clearly then:

$$V_N \cap \mathscr{D}_K = W_N \cap \mathscr{D}_K, \qquad N = N_0, N_0 + 1, \ldots.$$

Suppose U is an open set in the subspace topology induced on $\mathscr{D}_K$ by $\tau_{\mathscr{Q}}$ and pick $x \in U$. Then $-x + U$ is a neighbourhood of the origin, and thus there exists n such that

$$W_n \cap \mathscr{D}_K \subset -x + U$$

But $W_{m+1} \subset W_m$ for all m, so without loss of generality we may assume $n \geqslant N_0$. But then we conclude that

$$V_n \cap \mathscr{D}_K = W_n \cap \mathscr{D}_K \subset -x + U$$

and so U is open in τ_K. An identical argument shows the reverse inclusion: i.e. an open set in τ_K is open in the subspace topology induced on $\mathscr{D}_K$ by $\tau_{\mathscr{Q}}$.

Thus the topology $\tau_{\mathscr{Q}}$ is locally convex, and induces the right subspace topology on $\mathscr{D}_K$. However, it is not complete. To see this, consider $\Omega = \mathbb{R}$, and let ϕ be any non-zero function with support in $[0, 1]$. Consider the sequence of functions $(f_m)_{m \in \mathbb{N}}$ with:

$$\phi_m(x) = \phi(x-1) + \frac{1}{2}\phi(x-2) + \frac{1}{3}\phi(x-3) + \ldots + \frac{1}{m}\phi(x-m).$$

This is a Cauchy sequence with respect to the topology $\tau_{\mathscr{Q}}$: if $n < m$, then

$$\|\phi_n - \phi_m\|_k = \frac{1}{n}\|\phi\|_k,$$

thus for any N if $n, m > N \|\phi\|_N$ we have $\phi_n - \phi_m \in W_N$. On the other hand, the sequence has no limit in $\mathscr{D}(\Omega)$, since for sufficiently large m, ϕ_m has support outside any compact set. We thus are led to discard $\tau_{\mathscr{D}}$ as a prospective topology for $\mathscr{D}(\Omega)$.

In fact, the topology $\tau_{\mathscr{D}}$ is too *coarse*: in a sense, the notion of convergence is too loose. This suggests that we should seek a finer topology. The topology that we shall introduce for $\mathscr{D}(\Omega)$ will in fact be the *finest* locally convex topology such that the subspace topology induced on $\mathscr{D}_K$ agrees with the natural Fréchet topology for each compact $K \subset \Omega$.

Definition A.10. *Let Ω be a nonempty open subset of $\mathbb{R}^n$.*

a) For each compact $K \subset \Omega$, τ_K is the Fréchet space topology on $\mathscr{D}_K$ introduced above.

b) β is the collection of all convex balanced sets $W \subset \mathscr{D}(\Omega)$ such that $\mathscr{D}_K \cap W \in \tau_K$ for every compact $K \subset \Omega$.

c) τ is the collection of all (possibly empty) unions of sets of the form $\phi + W$ with $\phi \in \mathscr{D}(\Omega)$ and $W \in \beta$.

Theorem A.21. *a) τ is a topology for $\mathscr{D}(\Omega)$ and β is a local base for τ.*

b) τ makes $\mathscr{D}(\Omega)$ into a locally convex topological vector space.

Proof. 1. It is clear from the definition that $\mathscr{D}(\Omega), \emptyset \in \tau$ and that τ is closed under arbitrary unions. If we can show that for any $V_1, V_2 \in \tau$ and $\phi \in V_1 \cap V_2$, then

$$\phi + W \subset V_1 \cap V_2 \tag{A.2}$$

for some $W \in \beta$, then we can deduce that $V_1 \cap V_2$ is open and so τ is a topology. Moreover, setting $\phi = 0$ and $V_2 = \mathscr{D}(\Omega)$ in (A.2) we deduce that any neighbourhood of 0 contains an element of β and so β is a local base. To show *a)* then, it is enough to establish (A.2).

2. From the definition of τ, there exist $\phi_i \in \mathscr{D}(\Omega)$ and $W_i \in \beta$ such that $\phi \in \phi_i + W_i$ and $\phi_i + W_i \subset V_i$ for $i = 1, 2$. Choose a compact $K \subset \Omega$ such that $\phi, \phi_i \in \mathscr{D}_K$. Since $\mathscr{D}_K \cap W_i$ is open in $\mathscr{D}_K$, we have

$$\phi - \phi_i \in (1 - \delta_i) W_i \tag{A.3}$$

for some $\delta_i > 0$. To see this, recall that $\mathscr{D}_K$ is a topological vector space, so in particular scalar multiplication is continuous. Thus for any $\psi \in \mathscr{D}_K$, the map $F_\psi : \mathbb{R} \to \mathscr{D}_K$ given by $t \mapsto t\psi$ is continuous. Thus the set $A = (F_{\phi - \phi_i})^{-1} [W_i \cap \mathscr{D}_K]$ is open in $\mathbb{R}$. In particular, there exists ϵ_i such that $(1 - 2\epsilon_i, 1 + 2\epsilon_i) \subset A$, which is equivalent to $t(\phi - \phi_i) \in W_i \cap \mathscr{D}_K$ for $t \in (1 - 2\epsilon_i, 1 + 2\epsilon_i)$. But if $(1 + \epsilon_i)(\phi - \phi_i) \in W_i \cap \mathscr{D}_K$, then (A.3) must hold for some $\delta_i > 0$.

3. Since W_i is convex, we can use the result of Exercise A.1 to deduce that

$$\phi - \phi_i + \delta_i W_i \subset (1 - \delta_i) W_i + \delta_i W_i = W_i$$

whence we deduce that

$$\phi + \delta_i W_i \subset \phi_i + W_i \subset V_i.$$

Taking $W = \delta_1\phi_1 \cap \delta_2\phi_2$ we have established (A.2) and thus proven part a) of the theorem.

4. To show that τ makes $\mathscr{D}(\Omega)$ into a locally convex topological space it is enough to show that the topological vector space axioms are satisfied. Since β is a local base, and is convex by construction the result will follow. Suppose that $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$ are distinct, and consider the set:

$$W = \{\phi \in \mathscr{D}(\Omega) : \|\phi\|_0 \leqslant \|\phi_1 - \phi_2\|_0\}$$

This is certainly convex as it is a metric ball. Moreover, since the sets $\{\phi \in \mathscr{D}_K : \|\phi\|_0 < r\}$ (and their translations) are open in τ_K for any r and all compact $K \subset \Omega$, we conclude that $W \in \beta$. Moreover, $\phi_1 \notin \phi_2 + W$. Thus the singleton set $\{\phi_1\}$ is closed in τ.

5. To establish the τ-continuity of addition, suppose $U \in \tau$ is any open set, and suppose that we have $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$ with $\phi_1 + \phi_2 \in U$. Since β is a local base, $\phi_1 + \phi_2 + W \subset U$ for some $W \in \beta$. I claim $(\phi_1 + \frac{1}{2}W) \times (\phi_2 + \frac{1}{2}W) \subset (+)^{-1}(U)$. To see this, note that by the convexity of W:

$$(\phi_1 + \frac{1}{2}W) + (\phi_2 + \frac{1}{2}W) = \phi_1 + \phi_2 + W \subset U.$$

Thus $(+)^{-1}(U)$ is open in the product topology and addition is τ-continuous.

6. Finally, to show that scalar multiplication is continuous, suppose $U \in \tau$ is any open set, and suppose that we have $\alpha \in \Phi$, $\phi \in \mathscr{D}(\Omega)$ with $\alpha\phi \in U$. Since β is a local base, $\alpha\phi + W \subset U$ for some $W \in \beta$. I claim that for ϵ, δ sufficiently small, we have $\{\alpha' \in \Phi : |\alpha' - \alpha| < \delta\} \times (\phi + \epsilon W) \subset (\cdot)^{-1}(U)$. Note that

$$\alpha'\phi' - \alpha\phi = \alpha'(\phi' - \phi) + (\alpha' - \alpha)\phi$$

Now, by a similar argument to that in paragraph 2. above, the continuity of scalar multiplication restricted to $\mathscr{D}_K$ for a compact K which contains the support of ϕ ensures we can choose $\delta > 0$ such that $\delta\phi \in \frac{1}{2}W$. Let us set $\epsilon = (2(|\alpha| + \delta))^{-1}$. By the fact that W is balanced and convex, we deduce that

$$\alpha'\phi' - \alpha\phi \in \frac{1}{2}W + \frac{1}{2}W = W,$$

so that $\{\alpha' \in \Phi : |\alpha' - \alpha| < \delta\} \times (\phi + \epsilon W) \subset (\cdot)^{-1}(U)$ and scalar multiplication is indeed continuous. □

From now on, whenever we refer to $\mathscr{D}(\Omega)$, we shall assume that it is given the topology τ that has just been constructed. The main results of this section (indeed this chapter) are the following two results which characterise convergence and continuity in $\mathscr{D}(\Omega)$. These results justify the approach taken in lectures to disregard a close study of the topology of $\mathscr{D}(\Omega)$ and focus instead on sequential definitions of continuity.

Theorem A.22. *a) A convex balanced subset V of $\mathscr{D}(\Omega)$ is open if any only if $V \in \beta$.*

b) The Fréchet topology τ_K of any $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ coincides with the subspace topology that $\mathscr{D}_K$ inherits from $\mathscr{D}(\Omega)$.

c) If E is a bounded subset of $\mathscr{D}(\Omega)$ then $E \subset \mathscr{D}_K$ for some $K \subset \Omega$, and there are real numbers $M_N < \infty$ such that every $\phi \in E$ satisfies the inequalities

$$\|\phi\|_N \leqslant M_N, \qquad N = 0, 1, \ldots$$

d) $\mathscr{D}(\Omega)$ has the Heine-Borel property.

e) If $(\phi_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{D}(\Omega)$, then $\{\phi_i\}_{i\in\mathbb{N}} \subset \mathscr{D}_K$ for some compact $K \subset \Omega$, and (ϕ_i) is Cauchy with respect to the norm $\|\cdot\|_N$ for each $N = 0, 1, \ldots$.

f) If $\phi_i \to 0$ in $\mathscr{D}(\Omega)$, then there is a compact $K \subset \Omega$ which contains the support of every ϕ_i , and $D^\alpha \phi_i \to 0$ uniformly, as $i \to \infty$ for every multiindex α.

g) In $\mathscr{D}(\Omega)$, every Cauchy sequence converges.

Proof. 1. Since β is a local base, clearly if $V \in \beta$ then it is open. Now suppose V is an arbitrary convex, balanced, open set. Let K be any compact subset of Ω and pick $\phi \in \mathscr{D}_K \cap V$. Since β is a local base, we have $\phi + W \subset V$ for some $W \in \beta$. Thus

$$\phi + (\mathscr{D}_K \cap W) \subset \mathscr{D}_K \cap V.$$

From the definition of β, we know that $\mathscr{D}_K \cap W \in \tau_K$, so we have shown that $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$. Since β contains all convex, balanced sets whose intersection with each $\mathscr{D}_K$ is open, $V \in \beta$ and we have established *a*).

2. The previous paragraph shows that any element of $\tau|_{\mathscr{D}_K}$ also belongs to τ_K, i.e. any set which is open with respect to the subspace topology is open in the Fréchet topology. Suppose now that $E \in \tau_K$. To show $E \in \mathscr{D}_K \cap W \in \tau_K$, we have to show that $E = \mathscr{D}_K \cap U$ for some $U \in \tau$. Suppose $\phi \in E$. Then from the definition of the topology of τ_K, there exists N, δ such that:

$$\{\psi \in \mathscr{D}_K : \|\psi - \phi\|_N < \delta\} \subset E.$$

Let:

$$W_\phi = \{\psi \in \mathscr{D}(\Omega) : \|\psi\|_N < \delta\}.$$

Then $W_\phi \in \beta$, and moreover

$$\mathscr{D}_K \cap (\phi + W_\phi) = \phi + \mathscr{D}_K \cap W_\phi \subset E$$

Taking:

$$U = \bigcup_{\phi \in E} (\phi + W_\phi)$$

we have $U \in \tau$ and $E = \mathscr{D}_K \cap U$. This establishes *b*).

3. To show the first part of *c*), we prove the contrapositive. Suppose $E \subset \mathscr{D}(\Omega)$ does not lie in any $\mathscr{D}_K$. Let $\{K_m\}_{m\in\mathbb{N}}$ be an exhaustion of Ω. Since $E \not\subset \mathscr{D}_{K_m}$, for each m, we can find $\phi_m \in E$ with supp$\phi_m \not\subset K_m$. In particular, there exists $x_m \in \Omega \setminus K_m$ with $\phi_m(x_m) \neq 0$. Let
$$W = \{\phi \in \mathscr{D}(\Omega) : |\phi(x_m)| < m^{-1} |\phi_m(x_m)|\}.$$
A short calculation shows W is convex and balanced. Suppose $K \subset \Omega$ is compact, then there exists M such that $K \subset K_M$. In particular this implies $x_m \notin K$ for $m \geqslant M$. Thus $\mathscr{D}_K \cap W$ is an intersection of finitely many open sets, and so $W \in \beta$. However, $\phi_m \notin mW$ for any m, so E is not bounded. Thus any bounded set in $\mathscr{D}(\Omega)$ belongs to $\mathscr{D}_K$ for some K. By *b*), E is thus bounded in $\mathscr{D}_K$, and the final part of *c*) follows by Theorem A.17.

4. Statement *d*) follows from *c*), since $\mathscr{D}_K$ has the Heine-Borel property (Theorem A.20). Since Cauchy sequences are bounded (Exercise A.9), *c*) implies that every Cauchy sequence $(\phi_i)_{i\in\mathbb{N}}$ lies in some $\mathscr{D}_K$. By *b*), $(\phi_i)_{i\in\mathbb{N}}$ is Cauchy with respect to τ_K, and *e*) follows. Statement *f*) is a restatement of *e*). Finally, *g*) follows from *e*) together with *b*) and the completeness of $\mathscr{D}_K$.

□

The final major result of this section concerns linear maps from $\mathscr{D}(\Omega)$ into a locally convex space. Before we state the theorem, we introduce the notion of a bounded operator as one which takes bounded sets to bounded sets. That is to say if X, Y are topological vector spaces, then a linear map $\Lambda : X \to Y$ is bounded if $\Lambda(E)$ is bounded in Y whenever E is bounded in X.

Theorem A.23. *Let Y be a locally convex topological vector space. Suppose that $\Lambda : \mathscr{D}(\Omega) \to Y$ is a* linear *mapping. Then the following are equivalent:*

a) Λ is continuous.

b) Λ is bounded.

c) If $\phi_i \to 0$ in $\mathscr{D}(\Omega)$ then $\Lambda\phi_i \to 0$ in Y.

d) The restrictions of Λ to every $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ are continuous.

a) ⇒ *b*) Let E be a bounded set in $\mathscr{D}(\Omega)$ and let W be a neighbourhood of 0 in Y. Since Λ is continuous, there exists a neighbourhood V of 0 in $\mathscr{D}(\Omega)$ such that $\Lambda(V) \subset W$. Since E is bounded, there exists $s > 0$ such that $E \subset tV$ for all $t > s$. Since Λ is linear, $\Lambda(E) \subset \Lambda(tV) = t\Lambda(V) \subset tW$, and hence $\Lambda(E)$ is bounded.

b) ⇒ *c*) By part *e*) of Theorem A.22, if $\phi_i \to 0$ in $\mathscr{D}(\Omega)$, then there exists $K \subset \Omega$ such that $\phi_i \to 0$ in $\mathscr{D}_K$. Since $\mathscr{D}_K$ is metrizable, there exist scalars $\alpha_i \to \infty$ such that $\alpha_i\phi_i \to 0$ in $\mathscr{D}_K$ and hence in $\mathscr{D}(\Omega)$ by part *b*) of Theorem A.22. By the linearity of Λ, we have
$$\Lambda\phi_i = \alpha_i^{-1}\Lambda(\alpha_i\phi_i).$$

Now $(\alpha_i\phi_i)$ is Cauchy in $\mathscr{D}(\Omega)$, and hence bounded. Since Λ is bounded by assumption, $\{\Lambda(\alpha_i\phi_i)\}$ is bounded. As $\alpha_i^{-1} \to 0$, by Lemma A.12, part *d*), $\Lambda\phi_i \to 0$.

c) $\Rightarrow$ *d*) By part *b*) of Theorem A.22, we have that *c*) implies that if $\phi_i \to 0$ in $\mathscr{D}_K$ then $\Lambda\phi_i \to 0$. We work by contradiction. Suppose that the restriction of Λ to $\mathscr{D}_K$ is not continuous. Then there exists a neighbourhood W of 0 in Y such that $\Lambda^{-1}(W)$ contains no neighbourhood of 0 in $\mathscr{D}_K$. Since $\mathscr{D}_K$ is metrisable, pick a metric d which generates τ_K and construct a sequence (x_n) by choosing x_n such that $d(x_n, 0) < n^{-1}$ and $x_n \notin \Lambda^{-1}(W)$. Then $x_n \to 0$ in $\mathscr{D}_K$ and hence in $\mathscr{D}(\Omega)$, but $\Lambda(x_n) \not\to 0$, contradicting *c*).

d) $\Rightarrow$ *a*) Suppose that U is a convex, balanced, neighbourhood of the origin in Y and set $V = \Lambda^{-1}(U)$. Then V is convex and balanced by the linearity of Λ. By part *a*) of Theorem A.22, V is open in $\mathscr{D}(\Omega)$ if $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$ for every $\mathscr{D}_K \subset \mathscr{D}(\Omega)$, but if Λ is continuous when restricted to each $\mathscr{D}_K$, then $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$ from the definition of continuity. Thus V is open. Now suppose that W is any open set in Y, and suppose $\phi \in \Lambda^{-1}(W)$. Since Y is locally convex, there exists U, a convex neighbourhood of 0 in U such that $\Lambda\phi + U \subset W$. By Theorem A.11, we may assume that U is balanced. Since Λ is linear, $\phi + \Lambda^{-1}(U) \subset \Lambda^{-1}(W)$, and $\phi + \Lambda^{-1}(U)$ is open in $\mathscr{D}(\Omega)$, so Λ is continuous.