Exercise A.1. a) Suppose that a tensor $\boldsymbol{T} \in T^1{}_1(V)$ has components $T^i{}_j = \lambda \delta^i{}_j$ with respect to a given basis B. Show that $\boldsymbol{T}$ has the same components with respect to any other basis. Deduce that the Kronecker delta is invariant under a change of coordinates.

b) Suppose that $\boldsymbol{v} \in V$, with components v^i and $\omega \in V^*$ with components ω_j. Show that the numbers

$$T^i{}_j := v^i \omega_j,$$

transform as the components of a $(1,1)-$tensor. In this way we can build tensors of higher rank from lower rank tensors.

c) Suppose that $\boldsymbol{T} \in T^0{}_2(V)$ has components T_{ij}. Show that the numbers

$$\overline{T}_{ij} := T_{ji},$$

transform as a $(0,2)-$tensor. Deduce that

$$T_{(ij)} := \frac{1}{2}\left(T_{ij} + T_{ji}\right), \quad \text{and} \quad T_{[ij]} := \frac{1}{2}\left(T_{ij} - T_{ji}\right),$$

transform as the components of $(0,2)-$tensors. We call the tensors with components $T_{(ij)}$ and $T_{[ij]}$ the symmetric and antisymmetric part of $\boldsymbol{T}$ respectively.

Exercise A.2. Verify that if the numbers g^{ij} are the unique solution of (A.5) then they indeed transform as the components of a $(2,0)-$tensor. This tensor is called the cometric.

Exercise A.3. Adapt the Gram-Schmidt process to show that there always exists a basis such that (A.6) holds.

Exercise A.4. Consider $\mathbb{R}^3$ with its canonical basis. Find an orthonormal basis for the metric whose components are given by

$$g_{12} = g_{21} = g_{33} = 1, \qquad \text{all other components } 0.$$

and write down the signature of the metric.

Exercise A.5. a) Suppose that we consider a new basis $\{\boldsymbol{e}'_i\}$ for $\mathbb{R}^3$. Show that the set of numbers

$$\frac{\partial f}{\partial x^i},$$

transform in the same way as the components of a *co-vector.*

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b) Fix $\boldsymbol{x} \in U$ and suppose that $\boldsymbol{V}, \boldsymbol{W} \in T_{\boldsymbol{x}}U$ are two vectors such that for any $f \in C^1(U)$ we have

$$\boldsymbol{V}[f](\boldsymbol{x}) = \boldsymbol{W}[f](\boldsymbol{x}).$$

Show that $\boldsymbol{V} = \boldsymbol{W}$.

Exercise A.6. a) Show that the identity map on a C^k-manifold, $\mathcal{M}$, always belongs to $C^k(\mathcal{M}; \mathcal{M})$.

b) Show that if $f, g \in C^k(\mathcal{M}; \mathbb{R})$, we have $fg \in C^k(\mathcal{M}; \mathbb{R})$, where we define the product pointwise $fg(p) = f(p)g(p)$.

c) Suppose $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are all at least C^k regular, and that $f \in C^k(\mathcal{M}_1; \mathcal{M}_2)$, $g \in C^k(\mathcal{M}_2; \mathcal{M}_3)$. Show that $g \circ f \in C^k(\mathcal{M}_1; \mathcal{M}_3)$.

d) Let $\mathcal{M}$ be a C^k-manifold of dimension n and let $\pi^i : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the i^{th} coordinate. Suppose that $(\mathcal{U}_\alpha, \varphi_\alpha)$ is a chart. Show that

$$\varphi^i_\alpha = \pi^i \circ \varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}$$

is C^k-smooth.