Exercise 3.1. Suppose that S is a $(1,1)-$tensor and R is a $(0,2)-$tensor, both at least C^1-regular, which with respect to a local basis $\{e_\mu\}$ may be written:

$$S = S^\mu{}_\nu e_\mu \otimes e^\nu, \qquad R = R_{\sigma\tau} e^\sigma \otimes e^\tau.$$

Suppose ∇ is the Levi-Civita connection.

i) With the notation of Lemma 2.5, show that:

$$\nabla_\kappa \left(S^\mu{}_\nu R_{\sigma\tau} \right) = \left(\nabla_\kappa S^\mu{}_\nu \right) R_{\sigma\tau} + S^\mu{}_\nu \left(\nabla_\kappa R_{\sigma\tau} \right).$$

ii) how also that

$$\nabla_\kappa \delta^\mu{}_\sigma = 0, \qquad \nabla_\kappa g_{\mu\nu} = 0, \qquad \nabla_\kappa g^{\mu\nu}.$$

Conclude that ∇, thought of as a map from $(p,q)-$tensors to $(p,q+1)-$tensors commutes with contractions and raising and lowering of indices.

Exercise 3.2. In a local coordinate basis, $\left\{ e_\mu = \frac{\partial}{\partial x^\mu} \right\}$, show that we can write

$$H_f(X,Y) = X^\mu Y^\nu \left(\frac{\partial^2 f}{\partial x^\mu \partial x^\nu} - \Gamma^\sigma{}_{\mu\nu} \frac{\partial f}{\partial x^\sigma} \right),$$

Deduce that for a local coordinate basis

$$\nabla_\mu \nabla_\nu f = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} - \Gamma^\sigma{}_{\mu\nu} \frac{\partial f}{\partial x^\sigma}$$

Exercise 3.3. For $V \in \mathfrak{X}_r(\mathcal{M})$, the divergence of V, written $\mathrm{div}_g V \in C^{r-1}(\mathcal{M};\mathbb{R})$ is defined with respect to a local basis by:

$$\mathrm{div}_g V := \nabla_\mu V^\mu = e^\mu \left(\nabla_{e_\mu} V \right).$$

Show that with respect to a local coordinate basis we have:

$$\mathrm{div}_g V = \frac{1}{\sqrt{|g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{|g|} V^\mu \right).$$

Deduce the expression for the wave/Laplace operator with respect to local coordinates, using:

$$\Box_g f = \mathrm{div}_g \left(\mathrm{grad}_g f \right).$$

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Exercise 3.4. Consider the sphere $S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = 1\}$, and set

$$\mathcal{U} = S^2 \setminus \{(-\sqrt{1-t^2}, 0, t) : t \in [-1,1]\},$$

i.e. the sphere with a line of longitude from north to south pole removed. We define a coordinate chart $(\mathcal{U}, \varphi)$, where φ is defined in terms of its inverse by:

$$\begin{array}{cccc} \varphi^{-1} & : & (0,\pi)\times(-\pi,\pi) & \to & \mathcal{U} \\ & & (\theta,\phi) & \mapsto & \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \end{array}$$

a) Show that with the standard identification of tangent vectors in S^2 with vectors in $\mathbb{R}^3$ that we have

$$\left.\frac{\partial}{\partial\theta}\right|_{\varphi^{-1}(\theta,\phi)} \simeq \boldsymbol{e}_\theta := \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix}, \qquad \left.\frac{\partial}{\partial\phi}\right|_{\varphi^{-1}(\theta,\phi)} \simeq \boldsymbol{e}_\phi := \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix}.$$

b) Show that

$$\begin{aligned} \langle \boldsymbol{e}_\theta, \boldsymbol{e}_\theta\rangle|_{\varphi^{-1}(\theta,\phi)} &= 1, \\ \langle \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\rangle|_{\varphi^{-1}(\theta,\phi)} &= 0, \\ \langle \boldsymbol{e}_\phi, \boldsymbol{e}_\phi\rangle|_{\varphi^{-1}(\theta,\phi)} &= \sin^2\theta. \end{aligned}$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^3$.

c) Deduce that in the θ, ϕ coordinates:

$$g_{S^2} = d\theta^2 + \sin^2\theta d\phi^2.$$

d) Show that in these coordinates:

$$\Delta_{g_{S^2}} f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

e) Suppose f is a smooth function on S^2. Show that:

$$\int_{\theta=0}^{\pi}\int_{\phi=-\pi}^{\pi} f\Delta_{g_{S^2}} f \sin\theta d\theta d\phi = -\int_{\theta=0}^{\pi}\int_{\phi=-\pi}^{\pi}\left[\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2\right]\sin\theta d\theta d\phi$$

Exercise 3.5. In the case that $(\mathcal{M}, g)$ is the Minkowski spacetime and that $\{e_\mu\}$ is an inertial frame, show that (2.18) agrees with the previous defninition of the energy momentum tensor.

Exercise 3.6. a) Suppose $\omega \in \mathfrak{X}^*_{k-1}(\mathcal{M})$ and $X, Y, Z \in \mathfrak{X}_{k-1}(\mathcal{M})$. By considering (2.21) with $f = \omega[Z]$, and recalling that $X(\omega[Z]) = (\nabla_X \omega)[Z] + \omega[\nabla_X Z]$, show that:

$$\omega[R(X,Y)Z] + (R(X,Y)\omega)\,Z = 0.$$

b) Deduce that in a local basis, the action of $R(X,Y)$ on a one-form is given by:

$$R(X,Y)\omega = -R^\tau{}_{\sigma\mu\nu} X^\mu Y^\nu \omega_\tau e^\sigma$$

c) Show that if S_1, S_2 are $C^{k-1}-$regular tensor fields, then

$$R(X,Y)\left[S_1 \otimes S_2\right] = \left[R(X,Y)S_1\right] \otimes S_2 + S_1 \otimes \left[R(X,Y)S_2\right].$$

d) Deduce that in a local basis, the action of $R(X,Y)$ on an arbitrary $(p,q)-$tensor is given by:

$$\begin{aligned}[R(X,Y)S]^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q} = \Bigg[& R^{\mu_1}{}_{\sigma\mu\nu} S^{\sigma\mu_2\ldots\mu_p}{}_{\nu_1\ldots\nu_q} + \cdots + R^{\mu_p}{}_{\sigma\mu\nu} S^{\mu_1\ldots\mu_{p-1}\sigma}{}_{\nu_1\ldots\nu_q} \\ & - R^\sigma{}_{\nu_1\mu\nu} S^{\mu_1\ldots\mu_p}{}_{\sigma\nu_2\ldots\nu_q} - \cdots - R^\sigma{}_{\nu_q\mu\nu} S^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_{q-1}\sigma} \Bigg] X^\mu Y^\nu\end{aligned}$$

Exercise 3.7. Recall the Schwarzschild metric in Painlevé-Gullstrand coordinates (Examples 7, 9, 12) is a Lorentzian metric on $\mathcal{M} = \mathbb{R} \times (0,\infty) \times S^2$ given by:

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + 2\sqrt{\frac{2m}{r}} dt dr + dr^2 + r^2 g_{S^2},$$

And let us choose the local basis of vector fields $\{e_\mu\}$, as in Examples 7, 9. Let $\Sigma = (0,\infty) \times S^2 \simeq \mathbb{R}^3 \setminus \{0\}$. Consider the map

$$\begin{array}{rccl} \imath \ : & \Sigma & \hookrightarrow & \mathcal{M} \\ & (\rho, \boldsymbol{X}) & \mapsto & (0, \rho, \boldsymbol{X}). \end{array}$$

In other words, the suface $\imath(\Sigma)$ is the surface $\{t = 0\}$.

a) Show that the future directed unit normal of $\imath(\Sigma)$ is given by:

$$N = e_0 = \frac{\partial}{\partial t} - \sqrt{\frac{2m}{r}} \frac{\partial}{\partial r}.$$

b) Show that the induced metric h is the canonical flat metric on $\mathbb{R}^3 \setminus \{0\}$:

$$h = d\rho^2 + \rho^2 g_{S^2}.$$

c) Show that:

$$k = -\frac{1}{2\rho}\sqrt{\frac{2m}{\rho}} d\rho^2 + \rho\sqrt{\frac{2m}{\rho}} g_{S^2}.$$

[Hint: consider $k(b_i, b_j)$ for a suitable basis of vector fields on Σ such that $\imath^* b_i = e_i$, and use (2.13)] d*) Under a change of coordinates $\boldsymbol{x} = \rho \boldsymbol{X}$ from polar to Cartesian coordinates, you are given that h and k become:

$$h = \delta_{ij} dx^i dx^j,$$
$$k = \sqrt{\frac{2m}{|\boldsymbol{x}|^3}} \left(\delta_{ij} - \frac{3}{2} \frac{x_i x_j}{|\boldsymbol{x}|^2} \right) dx^i dx^j.$$

Show that:

$$|k|_h^2 - (\mathrm{Tr}_h k)^2 = 0,$$

and

$$\mathrm{div}_h k - d\,(\mathrm{Tr}_h k) = 0.$$

[Hint: Note that since h is the canonical metric on $\mathbb{R}^3$ in Cartesian coordinates, $(\mathrm{div}_h k)_j = \partial_i k_{ij}$ and $(df)_i = \partial_i f$.]