Exercise 2.1. Let $U \subset \mathbb{E}^{1,3}$ be open. Define an antisymmetric $(0,2)-$tensor field, F on U with components[1]

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

Where $E_i, B_i \in C^1(U)$. Show that the vacuum Maxwell equations for $\boldsymbol{E}, \boldsymbol{B}$ (with units such that $c^2 = \epsilon_0\mu_0 = 1$) hold in U if and only if F satisfies the equations

$$\nabla_\mu F^\mu{}_\nu = 0, \qquad \nabla_{[\mu} F_{\nu\sigma]} = 0,$$

where for any $(0,3)-$tensor $A_{\mu\nu\sigma}$ we define:

$$A_{[\mu\nu\sigma]} = \frac{1}{6}\left(A_{\mu\nu\sigma} + A_{\nu\sigma\mu} + A_{\sigma\mu\nu} - A_{\nu\mu\sigma} - A_{\mu\sigma\nu} - A_{\sigma\nu\mu}\right).$$

Exercise 2.2. Suppose that F is as in Exercise 2.1. Fix an inertial frame $\{\vec{e}_\mu\}$. Define

$$T_{\mu\nu}[F] = F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}\eta_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}.$$

Show that T has the following properties

a) We have a formula for the divergence:

$$\nabla_\mu T^\mu{}_\nu[F] = \left(\nabla_\mu F^\mu{}_\sigma\right)F_\nu{}^\sigma + \frac{3}{2}\left(\nabla_{[\mu}F_{\nu\sigma]}\right)F^{\mu\sigma}$$

b) The $00-$component of T is the local energy density

$$T_{00}[F] = \frac{1}{2}\left[|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right]$$

c) If $\vec{V}$ is any future directed unit timelike vector, then:

$$V^0\left[|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right] \geq V^\mu T_{\mu 0}[F] \geq \frac{1}{4V^0}\left[|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right]$$

Hence, or otherwise, deduce that the electromagnetic field exhibits finite speed of propagation.

Exercise 2.3. Consider the infinite cylinder $\mathbb{R} \times S^1$ and take as coordinates (x, θ) where $\theta \sim \theta + 2\pi$.

Please send any corrections to c.warnick@warwick.ac.uk

[1] The convention is that the first index specifies the row, and the second index the column.

a) Show that when $\mathcal{M}$ is equipped with the Lorentzian metric

$$g = -dx^2 + d\theta^2,$$

it is time-orientable.

b) Now consider the metric

$$g = -\cos\theta dx^2 + 2\sin\theta dx d\theta + \cos\theta d\theta^2$$

i) Show that the vector fields defined for $\theta \in [0, 2\pi)$ by

$$X_0 = \cos\frac{\theta}{2}\partial_x - \sin\frac{\theta}{2}\partial_\theta, \quad X_1 = \sin\frac{\theta}{2}\partial_x + \cos\frac{\theta}{2}\partial_\theta.$$

satisfy

$$g(X_0, X_0) = -1, \quad g(X_0, X_1) = 0, \quad g(X_1, X_1) = 1.$$

Deduce that g is a Lorentzian metric.

ii) Let us denote the point $x = 0, \theta = 0$ by p. Suppose that there exists a nowhere vanishing timelike field T, and without loss of generality assume that $g(X_0, T)|_p < 0$. Show that if $\gamma : [0, 1) \to \mathbb{R} \times [0, 2\pi)$ is any smooth curve with $\gamma(0) = p$ then $g(X_0, T)|_\gamma < 0$.

iii) By considering the curve $\gamma : s \mapsto (0, 2\pi s)$, deduce that $\mathcal{M}$ is not time orientable.

Exercise 2.4. Consider $\mathcal{M} = \mathbb{R}^3$, with a choice of orthonormal basis $\{\boldsymbol{e}_i\}$ with respect to the Euclidean metric $g_{ij} = \delta_{ij}$. Define a smooth connection on vectors by

$${}^{(\tau)}\nabla_{\boldsymbol{e}_i}\boldsymbol{e}_j = \tau\epsilon^k{}_{ij}\boldsymbol{e}_k$$

where ϵ_{ijk} is totally anti-symmetric with $\epsilon_{123} = 1$ and $\tau \in \mathbb{R}$ is a constant. Consider the curve $\gamma : (-1, 1) \to \mathbb{R}^3$ given by $\gamma(t) = t\boldsymbol{e}_3$. Show that the vector fields

$$\begin{aligned}
\boldsymbol{X}_1 &= \boldsymbol{e}_1\cos(\tau x^3) - \boldsymbol{e}_2\sin(\tau x^3) \\
\boldsymbol{X}_2 &= \boldsymbol{e}_1\sin(\tau x^3) + \boldsymbol{e}_2\cos(\tau x^3) \\
\boldsymbol{X}_3 &= \boldsymbol{e}_3
\end{aligned}$$

are all parallely transported by ${}^{(\tau)}\nabla$ along γ.

Exercise 2.5. a) Suppose that $f \in C^1(\mathcal{M}; \mathbb{R})$, show that

$$T(fX, Y) = fT(X, Y), \qquad T(X, Y) = -T(Y, X).$$

Deduce that if $\{e_\mu\}$ is locally a basis with $X = X^\mu e_\mu$, $Y = Y^\mu e_\mu$ then:

$$T(X, Y) = T^\sigma{}_{\mu\nu}X^\mu Y^\nu e_\sigma$$

for some C^r-functions $T^\sigma{}_{\mu\nu} := e^\sigma\left[T(e_\mu, e_\nu)\right]$.

b) Show that for the connection defined in Exercise 2.4, the torsion is given by:

$$T^i{}_{jk} = 2\tau\epsilon^i{}_{jk}.$$

Exercise 2.6. Show that if $f \in C^{k-1}(\mathcal{M})$, then (2.5) implies

$$(\nabla_X\omega)[fY] = f(\nabla_X\omega)[Y]$$

for any $Y \in \mathfrak{X}_{k-1}(\mathcal{M})$. Deduce that $\nabla_X\omega \in \mathfrak{X}^*_r(\mathcal{M})$.

Exercise 2.7. Consider the connection defined in Exercise 2.4. Show that if we define a Riemannian metric on $\mathbb{R}^3$ by $g(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{w} = v^i w^j \delta_{ij}$, then the connection ${}^{(\tau)}\nabla$ satisfies

$$\boldsymbol{x}[g(\boldsymbol{y}, \boldsymbol{z})] = g\left({}^{(\tau)}\nabla_{\boldsymbol{x}}\boldsymbol{y}, \boldsymbol{z}\right) + g\left(\boldsymbol{y}, {}^{(\tau)}\nabla_{\boldsymbol{x}}\boldsymbol{z}\right).$$

Deduce that ${}^{(0)}\nabla$ is the Levi-Civita connection of g.