Exercise 1.1. Let $S_T := \mathbb{R}^3 \times (-T, T)$ and $\Sigma_t = \mathbb{R}^3 \times \{t\}$. Fix $k \in \mathbb{N}$. Suppose that $u \in C^{2+k}(S_T)$, solves the wave equation (1.1) in S_T, and that there exists R such that $u(x,t) = 0$ for $|x| > R$. Define $u_0 := u|_{\Sigma_0}$ and $u_1 := u_t|_{\Sigma_0}$.

a) By deriving an equation for $\nabla_i u$ for $i = 1, 2, 3$ show that[1]

$$\frac{1}{2}\int_{\Sigma_t} \left(|\boldsymbol{\nabla} u_t|^2 + \left|\boldsymbol{\nabla}^2 u\right|^2 \right) d\sigma = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\boldsymbol{\nabla} u_1|^2 + \left|\boldsymbol{\nabla}^2 u_0\right|^2 \right) dx$$

for $-T < t < T$.

b) Deduce that:

$$\frac{1}{2}\int_{\Sigma_t} \left(\left|\boldsymbol{\nabla}^k u_t\right|^2 + \left|\boldsymbol{\nabla}^{k+1} u\right|^2 \right) d\sigma = \frac{1}{2}\int_{\mathbb{R}^3} \left(\left|\boldsymbol{\nabla}^k u_1\right|^2 + \left|\boldsymbol{\nabla}^{k+1} u_0\right|^2 \right) dx.$$

for $-T < t < T$.

Exercise 1.2. Let $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$, $S_{*,T} := \mathbb{R}^3_* \times (-T, T)$ and $|x| = r$. You may assume the result that if $u = u(r,t)$ is radial, we have

$$\Delta u(|x|, t) = \Delta u(r, t) = \frac{\partial^2 u}{\partial r^2}(r,t) + \frac{2}{r}\frac{\partial u}{\partial r}(r,t)$$

a) Suppose $u(x,t) = \frac{1}{r} v(r,t)$ for some function v. Show that u solves the wave equation on $\mathbb{R}^3_* \times (0, T)$ if and only if v satisfies the one-dimensional wave equation

$$-\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} = 0$$

on $(0, \infty) \times (-T, T)$.

b) Suppose $f, g \in C^2_c(\mathbb{R})$. Deduce that

$$u(x,t) = \frac{f(r+t)}{r} + \frac{g(r-t)}{r}$$

is a solution of the wave equation on $S_{*,T}$ which vanishes for large $|x|$.

c) Show that if $f \in C^3_c(\mathbb{R})$ is an odd function (i.e. $f(s) = -f(-s)$ for all s) then

$$u(x,t) = \frac{f(r+t) + f(r-t)}{2r}$$

extends as a C^2 function which solves the wave equation on S_T, with

$$u(0,t) = f'(t).$$

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[1] Here $\left|\boldsymbol{\nabla}^2 u\right|^2 = \sum_{i,j} \nabla_i \nabla_j u \nabla_i \nabla_j u$, etc.

*d) By considering a suitable sequence of functions f, or otherwise, deduce that there exists no constant C independent of u such that the estimate

$$\sup_{S_T} (|u| + |u_t|) \leq C \sup_{\Sigma_0} (|u| + |u_t|)$$

holds for all solutions $u \in C^2(S_T)$ of the wave equation which vanish for large $|x|$.

Exercise 1.3. Show that if $\Lambda^\nu{}_\mu$ is the matrix of a Lorentz transformation and

$$x'^\nu = \Lambda^\nu{}_\mu x^\mu, \qquad y'^\nu = \Lambda^\nu{}_\mu y^\mu.$$

then

$$x'^\mu y'^\mu \eta_{\mu\nu} = x^\mu y^\mu \eta_{\mu\nu}.$$

Exercise 1.4. Using $\mathbb{P}, \mathbb{T}$ and the transformations of Examples 1, 2 or otherwise:

a) Suppose that $\vec{X} \in \mathbb{E}^{1,3}$ is a unit timelike vector, i.e. $\eta\left(\vec{X}, \vec{X}\right) = -1$. Show that there exists an inertial frame $\{\vec{e}_\mu\}_{\mu=0,\ldots,3}$, such that writing $\vec{X} = x^\mu \vec{e}_\mu$, we have

$$x^\mu = (1, 0, 0, 0).$$

Deduce that if $\vec{X}$ is timelike and $\vec{Y} \neq 0$ satisfies $\eta\left(\vec{X}, \vec{Y}\right) = 0$, then $\vec{Y}$ is spacelike.

b) Suppose that $\vec{X} \in \mathbb{E}^{1,3}$ is a null vector, i.e. $\eta\left(\vec{X}, \vec{X}\right) = 0$. Show that there exists an inertial frame $\{\vec{e}_\mu\}_{\mu=0,\ldots,3}$ such that writing $\vec{X} = x^\mu \vec{e}_\mu$, we have

$$x^\mu = \lambda(1, 1, 0, 0).$$

for some $\lambda > 0$. Deduce that if $\vec{X}$ is null and $\vec{Y} \neq 0$ satisfies $\eta\left(\vec{X}, \vec{Y}\right) = 0$, then either $\vec{Y}$ is either spacelike or parallel to $\vec{X}$.

c) Suppose that $X \in \mathbb{E}^{1,3}$ is a unit spacelike vector, i.e. $\eta\left(\vec{X}, \vec{X}\right) = 1$. Show that there exists an inertial frame $\{\vec{e}_\mu\}_{\mu=0,\ldots,3}$ compatible with the time orientation such that writing $\vec{X} = x^\mu \boldsymbol{e}_\mu$, we have

$$x^\mu = (0, 1, 0, 0).$$

Deduce that if $\vec{X}$ is spacelike and $\vec{Y} \neq 0$ satisfies $\eta\left(\vec{X}, \vec{Y}\right) = 0$, then $\vec{Y}$ can be timelike, null or spacelike.

Exercise 1.5. Show that the relation $\sim$ between timelike vectors defined in (1.7) is an equivalence relation. [Hint: the final part of Exercise 1.4 a) may be useful]