MA4K5: Introduction to Mathematical Relativity

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Abstract

One of the crowning achievements of modern physics is Einstein's theory of general relativity, which describes the gravitational field to a very high degree of accuracy. As well as being an astonishingly accurate physical theory, the study of general relativity is also a fascinating area of mathematical research, bringing together aspects of differential geometry and PDE theory. In this course, I will introduce the basic objects and concepts of general relativity without assuming a knowledge of special relativity. The ultimate goal of the course will be a discussion of the Cauchy problem for the vacuum Einstein equations, including a statement of the relevant well-posedness theorems and a discussion of their relevance. We will take a 'field theory' approach to the subject, emphasising the deep connection between Lorentzian geometry and hyperbolic PDE. In contrast to the course PX436 General Relativity offered by the department of physics, we concentrate on the mathematical structure of the theory rather than its physical implications.

By the end of the module, students should be able to:

- Understand how the Minkowski geometry and Lorentz group arise from considerations of signal propagation for the scalar wave equation.
- Understand the basics of Lorentzian geometry: the metric; causal classification of vectors; connection and curvature; hypersurface geometry; conformal compactifications; the d'Alembertian operator.
- Be able to state the well-posedness theorems for the Cauchy problem for the Einstein equations and sketch the proof of local well posedness.

Contents

Acknowledgements

Introduction

A brief history of light

Light moves fast. Famously fast. So fast, in fact, that it's extremely difficult to observe that its speed, c, is finite. For this reason many people through history have, understandably, concluded that the speed of light is infinite. Debate on the matter dates back at least to the ancient Greeks, and the finiteness of c was only really established towards the end of the 17^{th} century.

The first definitive experimental proof that light has a finite speed was given by Rømer in 1676. His argument was based on measurements of Jupiter's moon Io. At some times in the year, Earth is moving towards Jupiter, while at others it is moving away. Rømer observed what we would now call a Doppler shift in the orbital frequency of Io. When the Earth moves towards Jupiter, the frequency of its orbit appears to increase, while when Earth is moving away from Jupiter the frequency decreases. By combining his measurements of the frequency change with estimates of the diameter of Earth's orbit, Rømer argued for a value (in modern units)

$$c \simeq 2.2 \times 10^8 \ ms^{-1},$$

which is pretty good considering our modern value of $c = 3.00 \times 10^8 \ ms^{-1}$. Although controversial at first, further measurements confirmed Rømer's demonstration that c is indeed finite.

At the time, there were two competing theories regarding the nature of light. Isaac Newton favoured the 'corpuscular theory of light', according to which light consists of particles. The competing theory of Huygens instead described light as a wave, propagating through the 'luminiferous æther'. It was not until 1804 that Young undertook his famous 'twin slits' experiment and demonstrated the wave character of light.

The next step in understanding light came as part of one of the great achievements of 19^{th} century physics. This was the unification in the early 1860s, by Maxwell, of the electrical and magnetic forces. Maxwell's equations in a vacuum have the form:

$$\nabla \cdot \boldsymbol{E} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3}$$

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} - \frac{1}{\mu_0} \nabla \times \boldsymbol{B} = 0. \tag{4}$$

Here $\boldsymbol{E}, \boldsymbol{B}$ are the electric and magnetic fields and μ_0, ϵ_0 are two constants, known as the permeability and permittivity of free space. From these constants, we can form a combination, $(\epsilon_0 \mu_0)^{-\frac{1}{2}}$, which somewhat suggestively has dimensions of speed. One of Maxwell's great contributions was to show that the system of PDEs (1–4) admits propagating wave solutions, with speed $c = (\epsilon_0 \mu_0)^{-\frac{1}{2}}$, and to identify these electromagnetic waves with light.

Exercise($*$). Consider a system of particles of mass m_i at positions $\boldsymbol{r}_i(t) \in \mathbb{R}^3$. Suppose that the particle j exerts a force $\boldsymbol{F}_{ij}$ on the particle i, where $\boldsymbol{F}_{ij} = \boldsymbol{F}_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j)$ depends only on the relative separation of the particles. Show that Newton's equation of motion for the system:

$$m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = \sum_j \boldsymbol{F}_{ij}$$

is invariant under the Galilean boost $\boldsymbol{r}_i \to \boldsymbol{r}_i(t) + \boldsymbol{v}t$.

As a means of describing the nature of light, Maxwell's equations were a triumph. Their interpretation gave rise to certain puzzles, however. There is a definite speed, c, present in Maxwell's equations. What should this speed be measured relative to? It had been known since Galileo and Newton that the laws of mechanics do not define a definite frame of reference: two identical mechanical systems moving at a constant speed relative to one another cannot be distinguished. Not so for Maxwell's equations: changing to a different frame moving at a constant speed by the Galilean boost which leaves Newton's mechanics invariant does *not* preserve the Maxwell equations.

One possible resolution to this issue was to postulate the existence of the 'aether', some fluid-like substance through which light moves (and relative to which the speed of light is c). This however gives rise to more questions. If light propagates as waves in some aether, what are its properties? How does it interact with moving bodies? Is it like a fluid, flowing around solid objects and dragged along by them when they move? Or does it flow through matter without interacting with it?

Several experiments were made towards the end of the 19th century to try and answer these questions. The most well known of these was the Michelson-Morley experiment, which aimed to measure the speed of the earth relative to the aether. To do this, they measured the speed of light at various times of day and at different times of the year.

If light does behave like water waves on some background fluid, then one would expect to see directional and seasonal variation in the speed of light, with it apparently moving faster in the direction of the 'aether wind'. No such effect was observed by Michelson and

Morley. Several people helped resolve this paradox, with particular contributions due to Fitzgerald, Lorentz and Poincaré, who between them showed that it's possible to make a change of coordinates preserving the form of Maxwell's equations and representing a shift to a frame in uniform motion, but only if one transforms both time and space variables simultaneously. In 1905, Einstein was able to derive these transformations from the relativity postulate (that physics is the same in two frames which are in uniform relative motion) and the constancy of the speed of light. The resulting theory of space and time is known as Special Relativity.

Gravity and General Relativity

At the start of the 20th century, the prevailing theory of gravitation was that of Newton. In this theory, the gravitational field is represented by a function, Φ, which solves the Poisson equation:

$$\Delta\Phi = 4\pi G\rho, \tag{5}$$

with $\rho(x,t)$ the density of matter. The gravitational force on a particle of mass m is then give by:

$$\boldsymbol{F} = -m\boldsymbol{\nabla}\Phi. \tag{6}$$

This theory of gravitation successfully describes almost all of the gravitational phenomena that are observable in our solar system. However, it is not compatible with Einstein's theory of Special Relativity. The reason for this is that the field Φ exhibits an infinite speed of propagation: a change in ρ instantly causes Φ to change *everywhere in space simultaneously*. Coupling this type of theory to Special Relativity introduces many paradoxes.

Finding a way to reconcile his theory of Special Relativity with Newton's theory of gravity took Einstein 10 years. The crucial observation that permitted him to find a relativistic theory of gravitation is the *principle of equivalence*. All bodies in a gravitational field accelerate at the same rate *regardless of their mass*. Einstein realised that this phenomenon could be explained if freely falling bodies follow the geodesics of a curved geometry with metric g. This postulate replaces the equation (6).

The curved metric g encodes the gravitational field. Einstein's field equations:

$$Ric_g - \frac{1}{2}Rg = \frac{8\pi G}{c^2}T \tag{7}$$

relate the curvature of the geometry to the density of matter, encoded in T, and are the replacement for (5).

In this course, we shall study some of the mathematical underpinnings of the theories of special and general relativity, as they are now understood. We will start off by studying the linear wave equation. For our purposes, this is a slightly simplified version of Maxwell's equations. Through the study of the wave equation, we will be lead to a study of the Lorentzian geometry of Minkowski space. We will then introduce the concepts of curvature, which will allow us to formulate Einstein's equations. We will finally move on to discuss the solvability of Einstein's equations.