M3/4P18: Fourier Analysis and Theory of Distributions

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Abstract

Syllabus:

- Spaces of test functions and distributions,
- Fourier Transform (discrete and continuous),
- Bessel's, Parseval's Theorems,
- Laplace transform of a distribution,
- Solution of classical PDE's via Fourier transform,
- Basic Sobolev Inequalities, Sobolev spaces.

Contents

Acknowledgements

In preparing this course, I have benefitted from the notes of previous lecturers, in particular Prof. Ari Laptev, Timothy Candy and Salvador Rodriguez-Lopez.

The textbooks that I have found most useful are:

- "Distributions, Theory and Applications", Duistermaat and Kolk;
- "Introduction to the theory of distributions", Friedlander and Joshi;
- "Fourier Analysis", Körner;
- "Functional Analysis", Rudin;
- "Topology", Munkres.

Material marked with (M) is examinable only in the mastery question, while material marked with (U) is not examinable.

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Introduction

Fourier analysis is a fundamental part of modern mathematics. Loosely speaking, Fourier analysis is the study of general functions as a superposition of harmonic 'waves' of different frequencies. It has important applications across mathematics, science and engineering, including (but not limited to):

- Number theory
- PDE theory
- Probability
- Numerical analysis
- Quantum mechanics
- Fluids
- Radio transmission
- Acoustics
- Optics
- Signal processing

Much modern technology would be impossible without our understanding of Fourier analysis. In this course we shall study Fourier analysis, but before we do this, we are first going to need to set up some important analytical machinery, that of distribution theory. Distributions are a generalisation of the notion of a function. Distribution theory is an important topic in its own right, and we shall spend quite some time on it. One of the main motivations for our study will be that distributions are the 'right' objects on which to study Fourier analysis, in the same way that measurable functions are the 'right' objects on which to study integration.

A motivational example

In order to give some motivation for why we introduce distributions, let us look at a simple problem in differential equations. We seek a function $\psi \in C^2(\mathbb{R}; \mathbb{C})$ which is 2π

periodic[1], $\psi(\theta + 2\pi) = \psi(\theta)$ and which satisfies the equation:

$$-\frac{d^2\psi}{d\theta^2} + \psi = f \tag{1}$$

where $f(\theta) = f(\theta + 2\pi)$ is a given function with the same period. Let us suppose that there exist numbers $f_n, \psi_n \in \mathbb{C}$ such that

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \tag{2}$$

$$\psi(\theta) = \sum_{n=-\infty}^{\infty} \psi_n e^{in\theta} \tag{3}$$

Periodic sums of this type are known as *Fourier series*. Clearly the sum on the right hand side has the correct periodicity. It is not a priori obvious that f, ψ can be represented by such a sum. Assuming however that (2) holds, we can calculate the coefficients f_n. To do this, we note that

$$\int_0^{2\pi} e^{-im\theta} e^{in\theta} d\theta = \begin{cases} \left[\frac{e^{i(n-m)\theta)}}{i(n-m)}\right]_0^{2\pi} = 0 & n \neq m \\ 2\pi & n = m \end{cases}$$

So that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-im\theta} d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} f_n e^{in\theta}\right) e^{-im\theta} d\theta \\ &= \sum_{n=-\infty}^{\infty} f_n \left(\frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta} e^{in\theta} d\theta\right) \\ &= f_m. \end{aligned}$$

In passing from the second line to the third, we have interchanged two limits, which should ring alarm bells. Let's suspend our disbelief for the time being and plough on regardless.

We can insert (2) and (3) into (1), to find that if the equation is satisfied we have:

$$\begin{aligned} 0 &= -\frac{d^2\psi}{d\theta^2} + \psi - f \\ &= -\frac{d^2}{d\theta^2}\left(\sum_{n=-\infty}^{\infty} \psi_n e^{in\theta}\right) + \sum_{n=-\infty}^{\infty} \psi_n e^{in\theta} - \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} \left(-\psi_n \frac{d^2}{d\theta^2} e^{in\theta} + \psi_n e^{in\theta} - f_n e^{in\theta}\right) \\ &= \sum_{n=-\infty}^{\infty} \left[(n^2+1)\psi_n - f_n\right] e^{in\theta}. \end{aligned}$$

[1] alternatively, one can think of ψ and f as mapping $S^1 \to \mathbb{C}$

Again, we've interchanged limiting operations, but again we'll ignore this for the time being. Our previous argument which shows us how to extract the numbers f_n from the function $f(\theta)$ suggests that the sum can only vanish if each term separately vanishes, so we expect that ψ will solve the equation if:

$$\psi_n = \frac{f_n}{n^2+1}.$$

We hope then that a solution to (1) is given by:

$$\psi(\theta) = \sum_{n=-\infty}^{\infty} \frac{f_n}{1+n^2} e^{in\theta} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{1+n^2} \left(\int_0^{2\pi} f(\alpha) e^{-in\alpha} d\alpha \right)$$

making another interchange of limits, we have:

$$\begin{aligned}\psi(\theta) &= \int_0^{2\pi} f(\alpha) \frac{1}{2\pi} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\alpha)}}{1+n^2} \right) d\alpha \\ &= \int_0^{2\pi} f(\alpha) G(\theta-\alpha) d\alpha,\end{aligned}$$

where we introduce a new function in the last line:

$$G(\theta) := \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{1+n^2}. \tag{4}$$

By the Weierstrass M-test, the sum defining G converges uniformly to a 2π-periodic continuous function.In fact, it's possible to perform this sum explicitly, and we find:

$$G(\theta) = \frac{\cosh(\pi-\theta)}{2\sinh\pi}, \qquad 0 \le \theta < 2\pi. \tag{5}$$

We can plot this function, and it is shown in Figure 1.

Exercise 1.1. Consider the meromorphic function

$$F(z) = \frac{1}{1+z^2} \frac{\cos[(\pi-\theta)z]}{2\sin\pi z}$$

a) Show that the poles of $F(z)$ occur at $z = \pm i$ and $z = n$, for $n \in \mathbb{Z}$, with:

$$\mathrm{Res}[F, \pm i] = -\frac{\cosh(\pi-\theta)}{4\sinh\pi}, \qquad \mathrm{Res}[F, n] = \frac{\cos(n\theta)}{2\pi(1+n^2)}.$$

b) For $N \in \mathbb{N}$, let $\Gamma_i^{(N)}$ be the curves

$$\begin{aligned}\Gamma_1^{(N)}(t) &= \left(N + \frac{1}{2}\right)(1+it), &\qquad -1 \le t \le 1 \\ \Gamma_2^{(N)}(t) &= \left(N + \frac{1}{2}\right)(-t+i), &\qquad -1 \le t \le 1 \\ \Gamma_3^{(N)}(t) &= \left(N + \frac{1}{2}\right)(-1-it), &\qquad -1 \le t \le 1 \\ \Gamma_4^{(N)}(t) &= \left(N + \frac{1}{2}\right)(t-i), &\qquad -1 \le t \le 1\end{aligned}$$

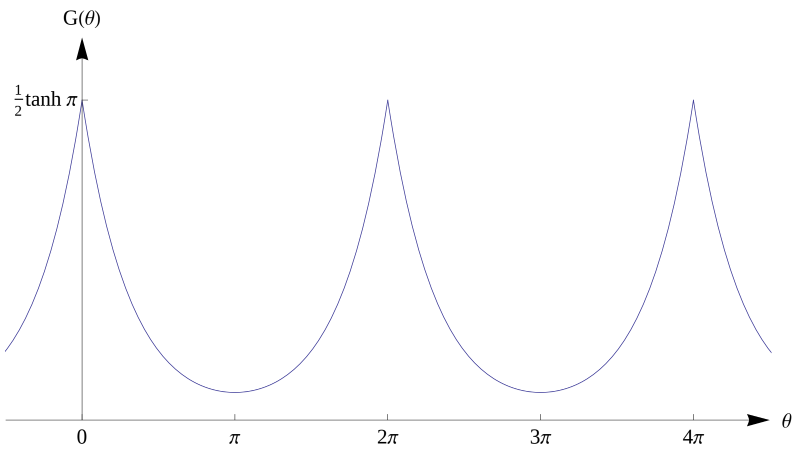

Figure 1 The function $G(\theta)$

and let $\Gamma^{(N)}$ be the closed contour which results from following Γ_1, Γ_2, Γ_3 and Γ_4 in turn. Sketch $\Gamma^{(N)}$, together with the locations of the poles of F in the complex plane.

c) Show that if $0 \leq \theta \leq 2\pi$, then:

$$\left| \int_{\Gamma^{(N)}} F(z) dz \right| \leq \frac{C}{N},$$

where C is a fixed constant, independent of N.

d) By applying Cauchy's residue theorem, show that for $0 \leq \theta \leq 2\pi$:

$$\left| \frac{\cosh(\pi - \theta)}{2 \sinh \pi} - \sum_{n=-N}^{N} \frac{e^{in\theta}}{1+n^2} \right| \leq \frac{C}{N},$$

and conclude that

$$\frac{\cosh(\pi - \theta)}{2 \sinh \pi} = \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{1+n^2},$$

with the sum converging uniformly in θ.

It is straightforward to verify that G has the following properties:

i) It is 2π-periodic: $G(\theta + 2\pi) = G(\theta)$.

ii) It is everywhere continuous

iii) Restricted to the open interval $(0, 2\pi)$, G is smooth and satisfies:

$$-G''(\theta) + G(\theta) = 0.$$

iv) The first derivative $G'(\theta)$ has finite jump discontinuities at $\theta \in 2\pi\mathbb{Z}$, and[2]:

$$G'(0_+) - G'(0_-) = G'(0_+) - G'(2\pi_-) = -1.$$

These properties are sometimes summarised in the formula:

$$-G''(\theta) + G(\theta) \quad \text{“}=\text{”} \sum_{n=-\infty}^{\infty} \delta(\theta - 2\pi n). \tag{6}$$

Here $\delta(\theta)$ is the so called "Dirac delta function" whose defining properties are often given as: $\delta(\theta) = 0$ when $\theta \neq 0$, and:

$$\int_{-\epsilon}^{\epsilon} \delta(\theta) d\theta = 1, \qquad \epsilon > 0.$$

With this property understood, we can recover the jump condition on the derivative of G by integrating (6). Obviously such an object doesn't make sense within our usual notions of a function. Assuming that we can insert our series representation (4) for G into (6) and commute the derivative and the sum, we arrive at:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta} \quad \text{“}=\text{”} \sum_{n=-\infty}^{\infty} \delta(\theta - 2\pi n). \tag{7}$$

Showing that we can give a meaning to this expression will occupy quite a substantial part of the course.

Now, let us return to our candidate solution to (1):

$$\psi(\theta) = \int_0^{2\pi} f(\alpha) G(\theta - \alpha) d\alpha,$$

And let us put to once side our discussion involving Fourier series and just take G as being defined by (5) together with the assumption that it is periodic. Clearly, since G is 2π-periodic, so is ψ. Suppose $\theta \in [0, 2\pi]$. Then we can write

$$\begin{aligned}
\psi(\theta) &= \int_0^{\theta} f(\alpha) G(\theta - \alpha) d\alpha + \int_{\theta}^{2\pi} f(\alpha) G(\theta - \alpha) d\alpha \\
&= \int_0^{\theta} f(\alpha) G(\theta - \alpha) d\alpha + \int_{\theta}^{2\pi} f(\alpha) G(\theta - \alpha + 2\pi) d\alpha \\
&= \int_0^{\theta} f(\alpha) \frac{\cosh(\pi - \theta + \alpha)}{2\sinh\pi} d\alpha + \int_{\theta}^{2\pi} f(\alpha) \frac{\cosh(-\pi - \theta + \alpha)}{2\sinh\pi} d\alpha
\end{aligned}$$

Exercise 1.2. Suppose that $f \in C^0(\mathbb{R})$ is a continuous function with period 2π, i.e. $f(\theta) = f(\theta + 2\pi)$. For $\theta \in [0, 2\pi)$, define:

$$\psi(\theta) := \int_0^{\theta} f(\alpha) \frac{\cosh(\pi - \theta + \alpha)}{2\sinh\pi} d\alpha + \int_{\theta}^{2\pi} f(\alpha) \frac{\cosh(-\pi - \theta + \alpha)}{2\sinh\pi} d\alpha$$

and extend ψ to a function on $\mathbb{R}$ by periodicity: $\psi(\theta) = \psi(\theta + 2\pi)$.

[2]The notation $f(x_+)$ means $\lim_{\epsilon \searrow 0} f(x + \epsilon)$. Similarly, $f(x_-) = \lim_{\epsilon \searrow 0} f(x - \epsilon)$.

a) Show that $\psi \in C^0(\mathbb{R})$.

b) By directly differentiating the formula, show that:

$$\psi'(\theta) = -\int_0^\theta f(\alpha)\frac{\sinh(\pi - \theta + \alpha)}{2\sinh\pi}d\alpha - \int_\theta^{2\pi} f(\alpha)\frac{\sinh(-\pi - \theta + \alpha)}{2\sinh\pi}d\alpha$$

and show that $\psi \in C^1(\mathbb{R})$.

c) Differentiating again, show that

$$\psi''(\theta) = -f(\theta) + \psi(\theta)$$

Conclude that $\psi \in C^2(\mathbb{R})$ is a solution to

$$-\psi''(\theta) + \psi(\theta) = f(\theta), \qquad \theta \in \mathbb{R}.$$

Exercise 1.3. Let $\phi \in C^2(\mathbb{R})$ be 2π-periodic, i.e. $\phi(\theta) = \phi(\theta+2\pi)$, and suppose that ϕ satisfies:

$$-\phi''(\theta) + \phi(\theta) = 0, \qquad \theta \in \mathbb{R}.$$

a) Show that if ϕ attains a maximum at $\theta_0 \in \mathbb{R}$, then $\phi(\theta_0) \leq 0$.

b) Show that if ϕ attains a minimum at $\theta_0 \in \mathbb{R}$, then $\phi(\theta_0) \geq 0$.

c) Show that $\phi \equiv 0$.

d) Conclude that there is at most one $\psi \in C^2(\mathbb{R})$ satisfying $\psi(\theta) = \psi(\theta + 2\pi)$ and solving:

$$-\psi''(\theta) + \psi(\theta) = f(\theta), \qquad \theta \in \mathbb{R},$$

where $f \in C^0(\mathbb{R})$ is a given 2π-periodic function.

The conclusion of these two exercises is the following result

Lemma 0.1. *Suppose that $f \in C^0(\mathbb{R})$ is a 2π-periodic function, and define:*

$$\psi(\theta) = \int_0^{2\pi} f(\alpha)G(\theta - \alpha)d\alpha.$$

Where $G : \mathbb{R} \to \mathbb{R}$ is the 2π-periodic function given by:

$$G(\theta) = \frac{\cosh(\pi - \theta)}{2\sinh\pi} \quad \text{for} \quad 0 \leq \theta < 2\pi.$$

Then $\psi \in C^2(\mathbb{R})$ is the unique 2π-periodic function satisfying:

$$-\psi'' + \psi = f.$$

Some remarks are in order at this stage. In the derivation of our solution ψ, we performed several manipulations which are not justified without further assumptions. In particular, in order to justify the interchange of various limiting operations, we would expect that we require the Fourier series of f to converge at least uniformly. However, there are C^0 functions with period 2π whose Fourier series do *not* converge pointwise everywhere, let alone uniformly [see the Functional Analysis course]. On the other hand, we have verified that the result that we arrived at is indeed correct. Part of the aim of this course will be to provide a framework in which we can rigorously carry out the manipulations in this example.

We also introduced the "Dirac delta function". Making sense of this and the formula (6) in which it appears will require us to consider a class of objects which generalise our familiar idea of smooth functions.