Chapter 4

Sobolev Spaces and PDE (M)

4.1 Sobolev spaces

4.1.1 The spaces $W^{k,p}(\Omega)$

Suppose $\Omega \subset \mathbb{R}^n$ is an open set. For $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq p \leq \infty$, we say that $f \in L^p(\Omega)$ belongs to the Sobolev space $W^{k,p}(\Omega)$ if for any $|\alpha| \leq k$ there exists $f^\alpha \in L^p(\Omega)$ with:

$$D^\alpha T_f = T_{f^\alpha}.$$

We call f^α the weak, or distributional derivative of f and write $D^\alpha f := f^\alpha$. We can equip $W^{k,p}(\Omega)$ with the norm:

$$||f||_{W^{k,p}(\Omega)} := \sum_{|\alpha|\leq k} ||D^\alpha f||_{L^p(\Omega)}\,.$$

With this norm, $W^{k,p}(\Omega)$ is complete, and hence a Banach space. The Sobolev spaces are particularly well suited to the study of PDE, and form the starting point for many modern PDE investigations.

We can think of k as telling us how *differentiable* our function is, while p tells us how *integrable* our function is. Roughly speaking spaces with larger k contain smoother functions, while spaces with larger p contain less 'spiky' functions. We shall see that (roughly speaking) one can trade smoothness for integrability: a function that belongs to $W^{k,p}(\mathbb{R}^n)$ belongs to certain $W^{l,q}(\mathbb{R}^n)$ where $l < k$ and $p > q$. If k and p are large enough we can even conclude that the function must be classically differentiable.

We will frame the result as concerning the embedding of $W^{k,p}(\mathbb{R}^n)$ spaces. Recall that a Banach space $(X, ||\cdot||_X)$ is said to embed continuously into the Banach space $(Y, ||\cdot||_Y)$ if $X \subset Y$ and there exists a constant C such that:

$$||x||_Y \leq C\,||x||_X\,, \qquad \text{for all } x \in X.$$

Theorem 4.1 (Sobolev embedding theorem). *Suppose $k > l$ and $1 \leq p < q < \infty$ satisfy $(k-l)p < n$ and:*

$$\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n}.$$

Then $W^{k,p}(\mathbb{R}^n)$ embeds continuously into $W^{l,q}(\mathbb{R}^n)$.

If $kp > n$, then $W^{k,p}(\mathbb{R}^n)$ embeds continuously into the Hölder space $C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\mathbb{R}^n)$, where $[x]$ is the largest integer less than or equal to x, and

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] - 1 + \frac{n}{p} & \frac{n}{p} \notin \mathbb{Z}, \\ \textit{any element of } (0,1) & \frac{n}{p} \in \mathbb{Z}. \end{cases}$$

Here we have introduced the Hölder space $C^{m,\kappa}(\mathbb{R}^n)$ which consists of $f \in C^m(\mathbb{R}^n)$ such that:

$$||f||_{C^{m,\kappa}(\mathbb{R}^n)} := \sum_{\alpha \leq m} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| + \sum_{\alpha = m} \sup_{x,y \in \mathbb{R}^n} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\kappa} < \infty.$$

We shan't attempt to prove the general Sobolev embedding theorems, but will establish a special case later on.

4.1.2 The space $H^s(\mathbb{R}^n)$

We shall immediately specialise to the case $p = 2$ and $\Omega = \mathbb{R}^n$. This is an important special case for two reasons. Firstly, $W^{k,2}(\Omega)$ is a Hilbert space (in addition to being a Banach space), and so carries additional structure. Secondly, $W^{k,2}(\mathbb{R}^n)$ is very well adapted to the Fourier transform. To see this, we recall that if $f \in L^2(\mathbb{R}^n)$, then:

$$\widehat{T_f} = T_{\hat{f}}$$

where $\hat{f} \in L^2(\mathbb{R}^n)$ is the Fourier-Plancherel transform of f. We immediately obtain an alternative characterisation of the space $W^{k,2}(\mathbb{R}^n)$. A function $f \in L^2(\mathbb{R}^n)$ belongs to $W^{k,2}(\mathbb{R}^n)$ if and only if:

$$\int_{\mathbb{R}^n} (1 + |\xi|)^{2k} \left|\hat{f}(\xi)\right|^2 d\xi < \infty.$$

Notice that in this characterisation there is no need to restrict k to be an integer. This motivates the following definition. For $s \geq 0$ we say that $f \in L^2(\mathbb{R}^n)$ belongs to the space $H^s(\mathbb{R}^n)$ provided:

$$||f||_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{2s} \left|\hat{f}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}} < \infty.$$

$H^s(\mathbb{R}^n)$ is complete, and moreover is a Hilbert space. We see that if $k \in \mathbb{Z}_{\geq 0}$ then $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$. Also $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

Exercise 5.8. Suppose $s \geq 0$.

a) Show that $\mathscr{S} \subset H^s(\mathbb{R}^n)$.

b) Suppose $f \in H^s(\mathbb{R}^n)$. Show that given $\epsilon > 0$ there exists $f_\epsilon \in \mathscr{S}$ with:

$$||f - f_\epsilon||_{H^s(\mathbb{R}^n)} < \epsilon.$$

Hint: First find $g_\epsilon \in \mathscr{S}$ such that

$$\left|\left|(\hat{f} - g_\epsilon)(1+|\xi|)^s\right|\right|_{L^2(\mathbb{R}^n)} < \epsilon.$$

c) Show that

$$||f||_{H^s(\mathbb{R}^n)} \leq ||f||_{H^t(\mathbb{R}^n)}$$

for $t \geq s$. Deduce that:

$$||f||_{L^2(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^n} ||f||_{H^s(\mathbb{R}^n)}$$

Hint: Use Parseval's formula

d) Show that the derivative D^α is a bounded linear map from $H^{s+k}(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$, where $k = |\alpha|$.

An important feature of the Sobolev spaces $H^s(\mathbb{R}^n)$ is that for s sufficiently large, they embed into $C^k(\mathbb{R}^n)$. More previsely:

Theorem 4.2. *Suppose that $f \in H^s(\mathbb{R}^n)$ for some $s > k + \frac{n}{2}$, then (possibly after redefinition on a set of measure zero) $f \in C^k(\mathbb{R}^n)$. That is, we have:*

$$H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n).$$

Proof. First suppose $f \in \mathscr{S}$. Then by the Fourier inversion theorem we have for $|\alpha| \leq k$:

$$D^\alpha f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{f}(\xi) d\xi.$$

We estimate with the Cauchy-Schwarz inequality:

$$\begin{aligned} |D^\alpha f(x)| &= \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{f}(\xi) d\xi \right| \\ &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| \xi^\alpha \hat{f}(\xi) \right| d\xi \\ &\leq \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2s} \left|\hat{f}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{|\xi^\alpha|^2}{(1+|\xi|)^{2s}} d\xi \right)^{\frac{1}{2}} \end{aligned}$$

Now, since $|\xi^\alpha|^2 \leq c_k (1+|\xi|)^{2k}$ for some $c_k > 0$, we have that:

$$\frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \frac{|\xi^\alpha|^2}{(1+|\xi|)^{2s}} d\xi \right)^{\frac{1}{2}} \leq \frac{c_k}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{2s-2k}} d\xi \right)^{\frac{1}{2}} =: C_{n,k,s} < \infty$$

where we have used $s > k + \frac{n}{2}$ in order to ensure that the integral converges. We thus have that:

$$\sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| \leq C_{n,k,s} ||f||_{H^s(\mathbb{R}^n)}. \tag{4.1}$$

The result follows by the density of $\mathscr{S}$ in $H^s(\mathbb{R}^n)$. □

[Details of density argument. Unexaminable] Suppose that $f \in H^s(\mathbb{R}^n)$. We can find $f_j \in \mathscr{S}$ for $j = 1, 2, \ldots$ such that:

$$||f - f_j||_{H^s(\mathbb{R}^n)} < \frac{1}{j}.$$

Since $\{f_j\}_{j=1}^{\infty}$ converges in $H^s(\mathbb{R}^n)$, it is necessarily Cauchy, so given $\epsilon > 0$ there exists J such that for all $j, l > J$ we have:

$$||f_j - f_l||_{H^s(\mathbb{R}^n)} < \epsilon.$$

Applying the estimate (4.1), we deduce that:

$$\sup_{x \in \mathbb{R}^n} |D^\alpha f_j(x) - D^\alpha f_l(x)| \leq C_{n,k,s}\epsilon$$

Thus $\{D^\alpha f_j\}_{j=1}^{\infty}$ is a Cauchy sequence in the Banach space of bounded continuous functions with the sup norm. We deduce that there exist $\tilde{f}^\alpha \in C^0(\mathbb{R}^n)$ such that $D^\alpha f_j \to \tilde{f}^\alpha$ uniformly. By a result in analysis, $\tilde{f}^\alpha = D^\alpha \tilde{f}$ for $\tilde{f} = \tilde{f}^0$. We've established that $f \to \tilde{f}$ uniformly, for some $\tilde{f} \in C^k(\mathbb{R}^n)$. We also know that $f_j \to f$ in $L^2(\mathbb{R}^n)$. We claim that $f = \tilde{f}$ almost everywhere. To see this, suppose $\psi \in C_0^\infty(\mathbb{R}^n)$ and consider:

$$I_j := \int_{\mathbb{R}^n} (f - f_j)(x)\psi(x)dx.$$

By Cauchy-Schwarz we have:

$$|I_j| \leq ||f - f_j||_{L^2(\mathbb{R}^n)} \, ||\psi||_{L^2(\mathbb{R}^n)}$$

so $I_j \to 0$ as $j \to \infty$. On the other hand, since $f_j \to \tilde{f}$ uniformly, we have:

$$I_j \to \int_{\mathbb{R}^n} (f(x) - \tilde{f}(x))\psi(x)dx.$$

Thus:

$$\int_{\mathbb{R}^n} (f(x) - \tilde{f}(x))\psi(x)dx = 0$$

for any $\psi \in C_0^\infty(\mathbb{R}^n)$, which implies $f = \tilde{f}$ almost everywhere.

4.2 PDE Examples

4.2.1 Helmholtz equation

Recall that the inhomogeneous Helmholtz equation is:

$$\Delta u + k^2 u = f$$

where f is given and we wish to find u. Suppose that $f \in H^s(\mathbb{R}^n)$ for some $s \geq 0$. We claim that there is a unique solution $u \in H^{s+2}(\mathbb{R}^n)$. The assumptions permit us to take the Fourier transform of Helmholtz equation so we find:

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 + k^2}.$$

Since $|\xi|^2 + k^2 \geq C(1 + |\xi|^2)$ we deduce that:

$$||u||_{H^{s+2}(\mathbb{R}^n)} \leq C\, ||f||_{H^s(\mathbb{R}^n)}\,.$$

Thus we indeed have that $u \in H^{s+2}(\mathbb{R}^n)$. Uniqueness follows from the injectivity of the Fourier transform. Note that if $s > \frac{n}{2}$ then $f \in C^0(\mathbb{R}^n)$ and $u \in C^2(\mathbb{R}^n)$, so that we in fact have a classical solution to the PDE.

4.2.2 Spaces involving time

For certain PDE problems it's useful to separate out the time direction from the spatial directions. To do this, it's useful to introduce some new function spaces:

Definition 4.1. *Given a Banach space $(X, ||\cdot||_X)$, and an interval $I \subset \mathbb{R}$, the space $C^0(I;X)$ is the space of continuous functions $\boldsymbol{u} : I \to X$.*

If I is open, we define $C^k(I;X)$ for $k \geq 0$ inductively as follows. We say $\boldsymbol{u} \in C^{k-1}(I;X)$ belongs to $C^k(I,X)$ if there exists $\boldsymbol{u}' \in C^{k-1}(I;X)$ such that for each $t \in I$:

$$\left|\left| \frac{\boldsymbol{u}(t+\epsilon) - \boldsymbol{u}(t)}{\epsilon} - \boldsymbol{u}'(t) \right|\right|_X \to 0, \qquad \text{as } \epsilon \to 0.$$

A typical example of X will be one of the space $H^s(\mathbb{R}^n)$ for $s > 0$.

4.2.3 The heat equation

Let us now give another example to show how powerful the Fourier transform can be for solving PDE problems. Let us consider the heat equation on $\mathbb{R}^n$. The problem we shall consider is, given $u_0 : \mathbb{R}^n \to \mathbb{R}$, determine $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$, such that

$$\begin{cases} u_t = \Delta u & \text{in } (0,T) \times \mathbb{R}^n, \\ u = u_0 & \text{on } \{0\} \times \mathbb{R}^n \end{cases} \tag{4.2}$$

We suppose that our solution is a continuous mapping from $(0,T)$ into $H^2(\mathbb{R}^n)$, i.e. for each fixed t we wish $u(t,\cdot) =: \boldsymbol{u}(t)$ to be an element of $H^2(\mathbb{R}^n)$. In terms of the function spaces above $\boldsymbol{u} \in C^0((0,T);H^2(\mathbb{R}^n))$. We will also suppose that u is continuously differentiable as a mapping from $(0,T)$ into $L^2(\mathbb{R}^n)$. In other words, $\boldsymbol{u} \in C^1((0,T);L^2(\mathbb{R}^n))$. Finally, we wish for the initial condition to make sense, so we also require $\boldsymbol{u} \in C^0([0,T);L^2(\mathbb{R}^n)$.

> **Exercise(∗).** Show that if $u \in C^0((0,T);H^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$, then denoting by $\hat{u}$ the Fourier transform of u in the spatial variables:
>
> $$\hat{u}(t,\xi) = \lim_{R\to\infty} \int_{B_R(0)} u(t,x) e^{-ix\cdot\xi} dx,$$
>
> we have $\hat{u} \in C^0((0,T);L^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$.

Let us, then, seek a solution of (4.2) such that

$$\boldsymbol{u} \in C^0([0,T);L^2(\mathbb{R}^n)) \cap C^0((0,T);H^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$$

Under this assumption we can take the Fourier transform of (4.2) for $(t,x) \in (0,T) \times \mathbb{R}^n$ to get:

$$\begin{cases} \hat{u}_t(t,\xi) &= -|\xi|^2 \hat{u}(t,\xi) \qquad (t,\xi) \in (0,T) \times \mathbb{R}^n, \\ \hat{u}(0,\xi) &= \hat{u}_0(\xi) \qquad\qquad \xi \in \mathbb{R}^n \end{cases}$$

Now, the PDE has become an ODE for each fixed ξ! This ODE has a unique solution given for almost every $\xi \in \mathbb{R}^n$ by:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi) e^{-t|\xi|^2}.$$

We note that if $u_0 \in L^2(\mathbb{R}^n)$, then $\hat{u}_0 \in L^2(\mathbb{R}^n)$ and thus $\hat{u} \in C^0([0,T);L^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$. In fact, for $t > 0$, we have that $\hat{u}(t,\xi)$ and $\hat{u}_t(t,\xi)$ are rapidly decaying functions of ξ, in particular they belong to $H^s(\mathbb{R}^n)$ for any $s \geq 0$, so we have that $u(t,x)$ is smooth in x. Since u satisfies the equation $(\partial_t)^n u = (\Delta)^n u$, we have that u is smooth in both t and x. We can recover $u(t,x)$ via the inverse Fourier transform formula:

$$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{-t|\xi|^2} e^{i\xi\cdot x} d\xi. \tag{4.3}$$

Summarising, we have the following result:

Lemma 4.3. *Suppose $u_0 \in L^2(\mathbb{R}^n)$. Then (4.2) admits a unique solution u such that*

$$\boldsymbol{u} \in C^0([0,T);L^2(\mathbb{R}^n)) \cap C^0((0,T);H^2(\mathbb{R}^n)) \cap C^1((0,T);L^2(\mathbb{R}^n))$$

given by (4.3). In fact,

$$u \in C^\infty((0,T) \times \mathbb{R}^n).$$

Even with very rough initial data, the heat equation instantaneously gives a smooth solution. This is an example of what is known as *parabolic regularity.*

Exercise 5.9. Suppose that $u_0 \in L^2(\mathbb{R}^n)$ and that $u(t,x)$ is the solution of the heat equation with initial data u_0. Explicitly, u is given by:

$$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{-t|\xi|^2} e^{i\xi\cdot x} d\xi,$$

for $t > 0$.

a) Show that:

$$||u(t,\cdot)||_{L^2(\mathbb{R}^n)} \leq ||u_0||_{L^2(\mathbb{R}^n)},$$

b) Show that:

$$u(t,x) = u_0 \star K_t(x)$$

where the *heat kernel* is given by:

$$K_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

c) Suppose that $u_0 \geq 0$. Show that $u \geq 0$, and:

$$||u(t, \cdot)||_{L^1(\mathbb{R}^n)} = ||u_0||_{L^1(\mathbb{R}^n)} .$$

[Hint: Lemma 1.9 may be useful]

Exercise 5.10. Consider the Schrödinger equation:

$$\begin{cases} u_t &= i\Delta u & \text{in } (0,T) \times \mathbb{R}^n, \\ u &= u_0 & \text{on } \{0\} \times \mathbb{R}^n \end{cases} \tag{4.4}$$

Suppose $u_0 \in L^2(\mathbb{R}^n)$.

a) Show that (4.4) admits a unique solution u such that

$$\boldsymbol{u} \in C^0([0,T); H^2(\mathbb{R}^n)) \cap C^1((0,T); L^2(\mathbb{R}^n)),$$

whose spatial Fourier-Plancherel transform is given by:

$$\hat{u}(t, \xi) = \hat{u}_0(\xi) e^{-it|\xi|^2}.$$

b) Show that:

$$||u(t, \cdot)||_{H^2(\mathbb{R}^n)} = ||u_0||_{H^2(\mathbb{R}^n)}$$

*c) For $t > 0$, let $K_t \in L^1_{loc.}(\mathbb{R}^n)$ be given by:

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{\frac{i|x|^2}{4t}},$$

where for n odd we take the usual branch cut so that $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. For $\epsilon > 0$ set $K^\epsilon_t(x) = e^{-\epsilon|x|^2} K_t(x)$.

i) Show that $T_{K^\epsilon_t} \to T_{K_t}$ in $\mathscr{S}'$ as $\epsilon \to 0$.

ii) Show that if $\Re(\sigma) > 0$, then:

$$\int_{\mathbb{R}} e^{-\sigma x^2 - ix\xi} dx = \sqrt{\frac{\pi}{\sigma}} e^{-\frac{\xi^2}{4\sigma}}.$$

iii) Deduce that

$$\widehat{K^\epsilon_t}(\xi) = \left(\frac{1}{1 + 4it\epsilon}\right)^{\frac{n}{2}} e^{\frac{-it|\xi|^2}{1+4it\epsilon}}$$

iv) Conclude that:

$$\widehat{T_{K_t}} = T_{\tilde{K}_t},$$

where $\tilde{K}_t = e^{-it|\xi|^2}$.

*d) Suppose that $u \in \mathscr{S}(\mathbb{R}^n)$. Show that for $t > 0$:

$$u(t, x) = \int_{\mathbb{R}^n} u_0(y) K_t(x - y) dy,$$

and deduce:

$$\sup_{t>0, x\in\mathbb{R}^n} |u(t, x)| \le \frac{1}{(4\pi t)^{\frac{n}{2}}} ||\hat{u}_0||_{L^1(\mathbb{R}^n)} .$$

This type of estimate which shows us that (locally) solutions to the Schrödinger equation decay in time is known as a *dispersive estimate*.

4.2.4 The wave equation

Now let us consider the wave equation on $\mathbb{R}^n$. The problem we shall consider is, given $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}$, determine $u : \mathbb{R}^n \times (-T, T) \to \mathbb{R}$, such that

$$\left\{ \begin{array}{rcll} u_{tt} & = & \Delta u & \text{in } (-T, T) \times \mathbb{R}^n, \\ u & = & u_0 & \text{on } \{0\} \times \mathbb{R}^n \\ u_t & = & u_1 & \text{on } \{0\} \times \mathbb{R}^n \end{array} \right. \tag{4.5}$$

We will seek a solution in the space:

$$X_s := C^0((-T, T), H^{s+2}(\mathbb{R}^n)) \cap C^2((-T, T) \times H^s(\mathbb{R}^n)).$$

Fourier transforming in the spatial variable, we have:

$$\left\{ \begin{array}{rcll} \hat{u}_{tt}(t, \xi) & = & -|\xi|^2 \hat{u}(t, \xi) & (t, \xi) \in (-T, T) \times \mathbb{R}^n, \\ \hat{u}(0, \xi) & = & \hat{u}_0(\xi) & \xi \in \mathbb{R}^n \\ \hat{u}_t(0, \xi) & = & \hat{u}_1(\xi) & \xi \in \mathbb{R}^n \end{array} \right.$$

Again, this is an ODE for each fixed ξ, and we deduce:

$$\hat{u}(t, \xi) = \hat{u}_0(\xi) \cos(|\xi| t) + \hat{u}_1(\xi) \frac{\sin(|\xi| t)}{|\xi|}$$

Notice that if $u_0 \in H^{s+2}(\mathbb{R}^n)$ and $u_1 \in H^{s+1}(\mathbb{R}^n)$, then we conclude $\hat{u} \in X_s$. Thus (after taking the inverse Fourier transform) we have found the unique solution of the wave equation in X_s.

Let's specialise to $\mathbb{R}^3$. We'd like to write this solution as some sort of convolution, at least for initial data in the Schwarz class. For this we need to find the (inverse) Fourier transform of $\cos(|\xi| t)$ and $\frac{\sin(|\xi|t)}{|\xi|}$, where we have to understand these functions as tempered distributions. Let us define, for $t > 0$ the distribution:

$$U_t[\phi] = \frac{1}{4\pi t} \int_{\partial B_t(0)} \phi(y) d\sigma_y$$

for all $\phi \in \mathscr{S}'$, where $d\sigma_y$ is the surface measure on the sphere $\partial B_t(0)$. This is a distribution of compact support, so we can invoke Theorem 3.13 to find the Fourier transform:

$$\widehat{U_t} = T_{\hat{v}_t}$$

where:

$$\hat{v}_t(\xi) = U_t[e_{-\xi}] = \frac{1}{4\pi t}\int_{\partial B_t(0)} e^{-i\xi\cdot y} d\sigma_y$$

We can perform this integral by choosing spherical polar coordinates for y with the axis aligned with the vector ξ. Doing so, the integral becomes:

$$\begin{aligned}
\hat{v}_t(\xi) &= \frac{1}{4\pi t}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} e^{-i|\xi|t\cos\theta} t^2 \sin\theta d\theta d\phi \\
&= \frac{t}{2}\int_{-1}^{1} e^{-i|\xi|tz} dz = \frac{t}{2}\left(\frac{e^{-i|\xi|t}}{-i\,|\xi|\,t} - \frac{e^{i|\xi|t}}{-i\,|\xi|\,t}\right) \\
&= \frac{\sin(|\xi|\,t)}{|\xi|}.
\end{aligned}$$

Now, let us return to our expression for u:

$$\begin{aligned}
\hat{u}(\xi) &= \hat{u}_0(\xi)\cos(|\xi|\,t) + \hat{u}_1(\xi)\frac{\sin(|\xi|\,t)}{|\xi|} \\
&= \frac{\partial}{\partial t}\left(\hat{u}_0(\xi)\frac{\sin(|\xi|\,t)}{|\xi|}\right) + \hat{u}_1(\xi)\frac{\sin(|\xi|\,t)}{|\xi|}
\end{aligned}$$

Suppose $u_0, u_1 \in \mathscr{S}$. Then by Theorem 3.10, we have:

$$\begin{aligned}
u(t,x) &= \frac{\partial}{\partial t} U_t \star u_0(x) + U_t \star u_1(x) \\
&= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B_t(0)} u_0(x-y) d\sigma_y\right) + \frac{1}{4\pi t}\int_{\partial B_t(0)} u_1(x-y) d\sigma_y \\
&= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B_t(x)} u_0(y) d\sigma_y\right) + \frac{1}{4\pi t}\int_{\partial B_t(x)} u_1(y) d\sigma_y \\
&= \frac{\partial}{\partial t}\left(t ⨍_{\partial B_t(x)} u_0(y) d\sigma_y\right) + t ⨍_{\partial B_t(x)} u_1(y) d\sigma_y
\end{aligned} \tag{4.6}$$

Where for a surface Σ with surface measure σ:

$$⨍_{\Sigma} d\sigma := \frac{1}{|\Sigma|}\oint_{\Sigma} d\sigma.$$

Expression (4.6) is known as Kirchoff's formula. While our derivation assumes $u_0, u_1 \in \mathscr{S}$, this assumption can be relaxed. This expression tells us some interesting facts about solutions to the wave equation. First note that the value of $u(x,t)$ depends only on the initial data on the sphere $\partial B_t(x)$. This is known as the strong Huygens principle. In particular this shows us that information is propagated at a *finite speed* by the wave equation. Secondly, note that the value of $u(x,t)$ depends on *derivatives* of u_0. This suggests that C^k−regularity is not propagated in wave evolution, although we have already seen that H^s−regularity is propagated.

Exercise 5.11. Let $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$, $S_{*,T} := (-T, T) \times \mathbb{R}^3_*$ and $|x| = r$. You may assume the result that if $u = u(r, t)$ is radial, we have

$$\Delta u(|x|, t) = \Delta u(r, t) = \frac{\partial^2 u}{\partial r^2}(r, t) + \frac{2}{r}\frac{\partial u}{\partial r}(r, t)$$

a) Suppose $u(x, t) = \frac{1}{r}v(r, t)$ for some function v. Show that u solves the wave equation on $\mathbb{R}^3_* \times (0, T)$ if and only if v satisfies the one-dimensional wave equation

$$-\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} = 0$$

on $(0, \infty) \times (-T, T)$.

b) Suppose $f, g \in C^2_c(\mathbb{R})$. Deduce that

$$u(x, t) = \frac{f(r + t)}{r} + \frac{g(r - t)}{r}$$

is a solution of the wave equation on $S_{*,T}$ which vanishes for large $|x|$.

c) Show that if $f \in C^3_c(\mathbb{R})$ is an odd function (i.e. $f(s) = -f(-s)$ for all s) then

$$u(x, t) = \frac{f(r + t) + f(r - t)}{2r}$$

extends as a C^2 function which solves the wave equation on $S_T := (-T, T) \times \mathbb{R}^3$, with

$$u(0, t) = f'(t).$$

*d) By considering a suitable sequence of functions f, or otherwise, deduce that there exists no constant C independent of u such that the estimate

$$\sup_{S_T} (|u| + |u_t|) \leq C \sup_{\Sigma_0} (|u| + |u_t|)$$

holds for all solutions $u \in C^2(S_T)$ of the wave equation which vanish for large $|x|$.