Chapter 3

The Fourier Transform

3.1 The Fourier transform on $L^1(\mathbb{R}^n)$

The Fourier transform is an extremely powerful tool across the full range of mathematics. Loosely speaking, the idea is to consider a function on $\mathbb{R}^n$ as a superposition of plane waves with different frequencies. For $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by:

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

Sine $\left|f(x)e^{-ix\cdot\xi}\right| \leq |f(x)|$, the integral is absolutely convergent, and $\hat{f}(\xi)$ makes sense for each $\xi \in \mathbb{R}^n$.

Example 12. *i) Suppose $f \in L^1(\mathbb{R})$ is the "top hat" function, defined by:*

$$f(x) = \begin{cases} 1 & -1 < x < 1, \\ 0 & |x| \geq 1. \end{cases}$$

We calculate:

$$\hat{f}(\xi) = \int_{-1}^{1} e^{-ix\xi} dx = \left[\frac{e^{-ix\xi}}{-i\xi}\right]_{-1}^{1} = 2\frac{\sin \xi}{\xi}$$

Notice that $\hat{f}(\xi)$ is continuous (in fact smooth) on $\mathbb{R}$. We also have $\hat{f}(\xi) \to 0$ as $\xi \to \infty$.

ii) Suppose $f \in L^1(\mathbb{R})$ is defined by:

$$f(x) = \begin{cases} e^{x} & x < 0, \\ e^{-x} & x \geq 0. \end{cases}$$

Then:

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{0} e^{x(1-i\xi)}dx + \int_{0}^{\infty} e^{x(-1-i\xi)}dx \\ &= \left[\frac{e^{x(1-i\xi)}}{1-i\xi}\right]_{-\infty}^{0} + \left[\frac{e^{x(-1-i\xi)}}{-1-i\xi}\right]_{0}^{\infty} \\ &= \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2}\end{aligned}$$

Again, notice that $\hat{f}$ is smooth and decays for large ξ.

iii) Consider $g \in L^1(\mathbb{R})$ given by

$$g(x) = \frac{1}{1+x^2}.$$

We have:

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{1+x^2}dx$$

We can consider this as a limit of contour integrals:

$$\hat{g}(\xi) = \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-iz\xi}}{1+z^2}dx.$$

Where $\gamma_R = \{\Im(z) = 0,\ |\Re(z)| < R\}$. For $\xi \geq 0$, we can close the contour with a semi-circle in the lower half-plane, and we pick up a contribution from the pole at $z = -i$. The contribution from the curved part of the contour tends to zero as $R \to \infty$ by Jordan's lemma, and we find:

$$\hat{g}(\xi) = \pi e^{-\xi}, \quad \xi \geq 0.$$

For $\xi < 0$, we close the contour in the upper half-plane, picking up a contribution from the pole at $z = i$ and again discard the contribution from the curved part of the contour in the limit. We find:

$$\hat{g}(\xi) = \pi e^{\xi}, \quad \xi < 0.$$

In conclusion, we have:

$$g(\xi) = \begin{cases} \pi e^{\xi} & \xi < 0, \\ \pi e^{-\xi} & \xi \geq 0. \end{cases}$$

iv) Consider now for $x \in \mathbb{R}^n$ the Gaussian $f(x) = e^{-\frac{1}{2}|x|^2}$. We calculate:

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2 - i\xi\cdot x}dx \\ &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x-i\xi)\cdot(x-i\xi) - \frac{1}{2}|\xi|^2}dx \\ &= e^{-\frac{1}{2}|\xi|^2}\left(\int_{\mathbb{R}} e^{-\frac{1}{2}(x_1-i\xi_1)^2}dx_1\right)\cdots\left(\int_{\mathbb{R}} e^{-\frac{1}{2}(x_n-i\xi_n)^2}dx_n\right)\end{aligned}$$

By shifting a contour in the complex plane, which is justified since e^{-z^2} is entire and rapidly decaying as z approaches infinity along any line parallel to the real axis, we can show that:

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(x_1 - i\xi_1)^2} dx_1 = \int_{\mathbb{R}} e^{-\frac{1}{2}x_1^2} dx_1 = \sqrt{2\pi}.$$

We deduce that:

$$\hat{f}(\xi) = (2\pi)^{\frac{n}{2}} e^{-\frac{1}{2}|\xi|^2}$$

Notice that this is (as a function) equal to f up to a factor.

We will make some casual observations at this stage. In all of our examples, we saw that the Fourier transformed function decays towards infinity. For examples iii), iv) we see very rapid (exponential) decay of the Fourier transform, while in examples i), ii) the decay is only polynomial. In all examples the transformed function is continuous. In examples i), ii), iv) it is in fact smooth, while for iii) the Fourier transform has a discontinuous first derivative. Reflecting on this, one sees that these two features appear to be dual to one another: if f is smooth, then $\hat{f}$ has rapid decay towards infinity. If f decays rapidly near infinity, then $\hat{f}$ is smooth. This is in fact a general feature of the Fourier transform smoothness and decay are dual to one another under the transform.

We shall now make some of these observations more precise.

Lemma 3.1 (Riemann-Lebesgue Lemma)**.** *Suppose $f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^0(\mathbb{R}^n)$ with the estimate:*

$$\sup_{\xi \in \mathbb{R}^n} \left| \hat{f}(\xi) \right| \leq ||f||_{L^1(\mathbb{R}^n)} \tag{3.1}$$

and moreover $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Proof. To establish the continuity of $\hat{f}$, we use the continuity of the exponential, together with the dominated convergence theorem. Let $\{\xi_j\}_{j=1}^{\infty}$ be any sequence with $\xi_j \to \xi$ as $j \to \infty$. Recalling the definition of the integral, we have:

$$\hat{f}(\xi_j) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi_j} dx.$$

Now, clearly for $x \in \mathbb{R}^n$ we have:

$$f(x)e^{-ix\cdot\xi_j} \to f(x)e^{-ix\cdot\xi}, \qquad \text{as } j \to \infty$$

so we have pointwise convergence of the integrand. We can also estimate:

$$\left| f(x)e^{-ix\cdot\xi_j} \right| \leq |f(x)|$$

so the integrand is dominated by an integrable function, since $f \in L^1(\mathbb{R}^n)$. Applying the Dominated Convergence Theorem, we conclude:

$$\hat{f}(\xi_j) \to \hat{f}(\xi), \qquad \text{as } j \to \infty.$$

This implies that $\hat{f}(\xi)$ is continuous. We can readily estimate:

$$\sup_{\xi\in\mathbb{R}^n}\left|\hat{f}(\xi)\right| = \sup_{\xi\in\mathbb{R}^n}\left|\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx\right| \leq \sup_{\xi\in\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x)|\,dx = ||f||_{L^1(\mathbb{R}^n)}\,.$$

This establishes the first part of the Lemma. To establish the second part, we make use of an approximation argument. Chapter 1 that if $f \in L^1(\mathbb{R}^n)$, we can approximate f by an element of $C_0^\infty(\mathbb{R}^n)$. Given $\epsilon > 0$, there exists $f_\epsilon \in C_0^\infty(\mathbb{R}^n)$ with

$$||f - f_\epsilon||_{L^1(\mathbb{R}^n)} < \frac{\epsilon}{2}.$$

Now, in the integral for $\hat{f}_\epsilon$ we can integrate by parts::

$$\begin{aligned}
\hat{f}_\epsilon(\xi) &= \int_{\mathbb{R}^n} f_\epsilon(x)e^{-ix\cdot\xi}dx \\
&= \int_{\mathbb{R}^n} f_\epsilon(x)\ \mathrm{div}\left(\frac{\xi}{-i\left|\xi\right|^2}e^{-ix\cdot\xi}\right)dx \\
&= -\int_{\mathbb{R}^n}\frac{\xi}{-i\left|\xi\right|^2}\cdot Df_\epsilon(x)e^{-ix\cdot\xi}dx
\end{aligned}$$

so that for each $i = 1, \ldots, n$ we have, by the Cauchy-Schwartz inequality:

$$\begin{aligned}
\left|\hat{f}_\epsilon(\xi)\right| &= \left|\int_{\mathbb{R}^n}\frac{\xi}{i\left|\xi\right|^2}\cdot Df_\epsilon(x)e^{-ix\cdot\xi}dx\right| \\
&\leq \int_{\mathbb{R}^n}\left|\frac{\xi}{i\left|\xi\right|^2}\cdot Df_\epsilon(x)e^{-ix\cdot\xi}\right|dx \qquad (3.2)\\
&\leq \int_{\mathbb{R}^n}\frac{1}{|\xi|}\left|Df_\epsilon(x)\right|dx \\
&= \frac{1}{|\xi|}\Big|\Big|\,\left|Df_\epsilon(x)\right|\,\Big|\Big|_{L^1(\mathbb{R}^n)}
\end{aligned}$$

From this, we conclude that there exists $R > 0$ such that if $|\xi| > R$, we have $\left|\hat{f}_\epsilon(\xi)\right| < \frac{\epsilon}{2}$. For $|\xi| > R$ we calculate:

$$\begin{aligned}
\left|\hat{f}(\xi)\right| &= \left|\hat{f}(\xi) - \hat{f}_\epsilon(\xi) + \hat{f}(\xi)\right| \\
&\leq \left|\hat{f}(\xi)\right| + \left|\hat{f}(\xi) - \hat{f}_\epsilon(\xi)\right| \\
&\leq \left|\hat{f}(\xi)\right| + ||f - f_\epsilon||_{L^1(\mathbb{R}^n)} < \epsilon.
\end{aligned}$$

In the last line, we have used (3.1), together with the linearity of the Fourier transform. Since $\epsilon > 0$ was arbitrary, we have shown that $\left|\hat{f}(\xi)\right| \to 0$. □

Remark. *The argument above is another example of an approximation argument where one first proves the result on a suitably nice dense subset then extends to the full space by* *continuity. In this case, we are using the fact that the Fourier transform is a bounded (hence continuous) linear operator from $L^1(\mathbb{R}^n)$ to the Banach space of continuous functions decaying at infinity equipped with the uniform norm. The dense set is $C_0^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.*

Another tactic which we used here was to use integration by parts to exploit the rapid oscillations in the $e^{-ix\cdot\xi}$ factor when $|\xi|$ is large.

One might be tempted to infer from (3.2) that $\left|\hat{f}(\xi)\right| \leq C(1+|\xi|)^{-1}$. While this is true for each f_ϵ approximating f, in general the constant C will grow larger and larger as $\epsilon \to 0$, so we cannot quite come to this conclusion.

Exercise 3.4. For $\xi \in \mathbb{R}^n$, define $e_\xi(x) = e^{i\xi\cdot x}$. Show that $T_{e_\xi} \in \mathscr{S}'$, and that:

$$T_{e_\xi} \to 0, \qquad \text{as } |\xi| \to \infty$$

in the topology[1] of $\mathscr{S}'$.

We shall prove some important properties of the Fourier transform. Recall that $\tau_y f(x) = f(x-y)$, and introduce the character $e_y(x) = e^{iy\cdot x}$.

Lemma 3.2 (Properties of the Fourier transform). *i) Suppose $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\lambda > 0$ and $f_\lambda(y) = \lambda^{-n} f(\lambda^{-1}y)$. Then*

$$\hat{f}_\lambda(\xi) = \hat{f}(\lambda\xi) \qquad \widehat{(e_x f)}(\xi) = \tau_x \hat{f}(\xi) \qquad \widehat{\tau_x f}(\xi) = e_{-x}(\xi)\hat{f}(\xi)$$

ii) Suppose $f, g \in L^1(\mathbb{R}^n)$. Then $f \star g \in L^1(\mathbb{R}^n)$ and:

$$\widehat{f \star g}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Proof. i) Writing out the expression for $\hat{f}_\lambda(\xi)$, and changing the integration variable to $z = \lambda^{-1}x$, we see

$$\hat{f}_\lambda(\xi) = \int_{\mathbb{R}^n} f_\lambda(x) e^{-i\xi\cdot x} dx = \int_{\mathbb{R}^n} f(\lambda^{-1}x) e^{-i\xi\cdot x} \lambda^{-n} dx = \int_{\mathbb{R}^n} f(y) e^{-i\lambda\xi\cdot z} dz = \hat{f}(\lambda\xi).$$

Next, we calculate:

$$\widehat{(e_x f)}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot y} f(y) e^{-i\xi\cdot y} dy = \int_{\mathbb{R}^n} f(y) e^{-i(\xi-x)\cdot y} dy = \tau_x \hat{f}(\xi).$$

Finally, we have:

$$\widehat{\tau_x f}(\xi) = \int_{\mathbb{R}^n} f(y-x) e^{-i\xi\cdot y} dy = \int_{\mathbb{R}^n} f(z) e^{-i\xi\cdot(z+x)} dz = e^{-i\xi\cdot x} \int_{\mathbb{R}^n} f(z) e^{-i\xi\cdot z} dz = e_{-x}(\xi)\hat{f}(\xi),$$

where we have used the substitution $z = y - x$.

[1] This is defined precisely as the topology of $\mathscr{D}'(\Omega)$, *mutatis mutandis.*

ii) First we show that $f \star g \in L^1(\mathbb{R}^n)$. To see this, we first estimate:

$$|f \star g(x)| = \left| \int_{\mathbb{R}^n} f(y)g(x-y)dy \right| \leq \int_{\mathbb{R}^n} |f(y)g(x-y)|\,dy$$

Integrating and applying Fubini's theorem, we have:

$$\begin{aligned} ||f \star g||_{L^1(\mathbb{R}^n)} &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)g(x-y)|\,dy \right) dx \\ &= \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(x-y)|\,dx \right) dy \\ &= \int_{\mathbb{R}^n} |f(y)|\,||g||_{L^1(\mathbb{R}^n)}\,dy = ||f||_{L^1(\mathbb{R}^n)}\,||g||_{L^1(\mathbb{R}^n)} \end{aligned}$$

Now, we can calculate the Fourier transform:

$$\begin{aligned} \widehat{f \star g}(\xi) &= \int_{\mathbb{R}^n} f \star g(x) e^{-i\xi\cdot x} dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)g(x-y)dy \right) e^{-i\xi\cdot x} dx \\ &= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} g(x-y) e^{-i\xi\cdot x} dx \right) dy \\ &= \int_{\mathbb{R}^n} f(y) \widehat{\tau_y g}(\xi) dy \\ &= \int_{\mathbb{R}^n} f(y) \widehat{g}(\xi) e^{-i\xi\cdot y} dy = \hat{f}(\xi)\hat{g}(\xi) \end{aligned}$$

□

Exercise 3.5. Calculate the Fourier transform of the following functions $f \in L^1(\mathbb{R})$:

a) $f(x) = \dfrac{\sin x}{1+x^2}$.

b) $f(x) = \dfrac{1}{\epsilon^2 + x^2}$, for $\epsilon > 0$ a constant.

c) $f(x) = \sqrt{\dfrac{\sigma}{t}} e^{-\sigma \frac{(x-y)^2}{t}}$, where $\sigma > 0$, $t > 0$ and y are constants.

*d) $f(x) = \dfrac{1}{\cosh x}$.

We saw with the examples that there is a duality between the deacy of a function and the regularity of its Fourier transform and vice versa. To make this more precise we prove the following result, which tells us, roughly speaking, that the Fourier transform swaps coordinate functions x_j multiplying f for derivatives iD_j acting on $\hat{f}$.

Theorem 3.3. *i) Suppose $f \in C^1(\mathbb{R}^n)$ and that $f, D_j f \in L^1(\mathbb{R}^n)$ for all $j = 1, \ldots n$. Then*

$$\widehat{D_j f}(\xi) = i\xi_j \hat{f}(\xi)$$

ii) Suppose $(1 + |x|)f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^1(\mathbb{R}^n)$, and:

$$D_j \hat{f}(\xi) = -i\, \widehat{x_j f}(\xi)$$

Proof. i) We again appeal to an approximation result. For $f \in C^1(\mathbb{R}^n)$ with $f, D_i f \in L^1(\mathbb{R}^n)$, then for any $\epsilon > 0$ there exists $f_\epsilon \in C^1_0(\mathbb{R}^n)$ such that $||f - f_\epsilon||_{L^1(\mathbb{R}^n)} < \epsilon$ and $||D_j f - D_j f_\epsilon||_{L^1(\mathbb{R}^n)} < \epsilon$. Integrating by parts, we readily calculate:

$$\begin{aligned}\widehat{D_j f_\epsilon}(\xi) &= \int_{\mathbb{R}^n} D_j f_\epsilon(x) e^{-i\xi\cdot x} dx \\ &= -\int_{\mathbb{R}^n} f_\epsilon(x) D_j(e^{-i\xi\cdot x}) dx \\ &= i\xi_j \int_{\mathbb{R}^n} f_\epsilon(x) e^{-i\xi\cdot x} dx\end{aligned}$$

so that $\widehat{D_j f_\epsilon}(\xi) = i\xi_j \hat{f}_\epsilon(\xi)$. Now, we calculate:

$$\begin{aligned}\left|\widehat{D_j f}(\xi) - i\xi_j \hat{f}(\xi)\right| &= \left|\widehat{D_j f}(\xi) - \widehat{D_j f_\epsilon}(\xi) + i\xi_j \hat{f}_\epsilon(\xi) - i\xi_j \hat{f}(\xi)\right| \\ &\leq ||D_j f - D_j f_\epsilon||_{L^1(\mathbb{R}^n)} + |\xi|\, ||f - f_\epsilon||_{L^1(\mathbb{R}^n)} \\ &\leq \epsilon(1 + |\xi|)\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we must have that $\left|\widehat{D_j f}(\xi) - i\xi_j \hat{f}(\xi)\right| = 0$, and the result follows.

ii) From the condition on f it is clear that $x_j f \in L^1(\mathbb{R}^n)$, so $-i\, \widehat{x_j f}$ is continuous. It suffices to prove then that:

$$\Delta_j^{h_k} \hat{f}(\xi) \to -i\, \widehat{x_j f}(\xi), \qquad \text{as } k \to \infty$$

for any sequence $\{h_k\}_{k=1}^\infty \subset \mathbb{R}$ with $h_k \to 0$. We calculate:

$$\Delta_j^{h_k} \hat{f}(\xi) = \frac{1}{h_k}\left(\hat{f}(\xi + h_k e_j) - \hat{f}(\xi)\right) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \left(\frac{e^{-ix_j h_k} - 1}{h_k}\right) dx.$$

Now for $x \in \mathbb{R}^n$ we have:

$$f(x) e^{-ix\cdot\xi} \left(\frac{e^{-ix_j h_k} - 1}{h_k}\right) \to -ix_j f(x) e^{-ix\cdot\xi}$$

as $k \to \infty$. Noting that $\left|e^{i\theta} - 1\right| = 2\left|\sin\frac{\theta}{2}\right| \leq \theta$ for any $\theta \in \mathbb{R}$, we have that:

$$\left|f(x) e^{-ix\cdot\xi} \left(\frac{e^{-ix_j h_k} - 1}{h_k}\right)\right| \leq |x_j f(x)|$$

where the right hand side is integrable. By the Dominated Convergence Theorem, we have:

$$\lim_{k\to\infty} \Delta_j^{h_k} \hat{f}(\xi) = \int_{\mathbb{R}^n} -ix_j f(x) e^{-ix\cdot\xi} dx = -i\,\widehat{x_j f}(\xi).$$

We deduce that $\hat{f} \in C^1(\mathbb{R}^n)$.

□

Exercise 3.6. Suppose $f \in C^1(\mathbb{R}^n)$ and that $f, D_j f \in L^1(\mathbb{R}^n)$. Fix $\epsilon > 0$. Show that there exists $f_\epsilon \in C^1_0(\mathbb{R}^n)$ such that

$$||f - f_\epsilon||_{L^1(\mathbb{R}^n)} + ||D_j f - D_j f_\epsilon||_{L^1(\mathbb{R}^n)} < \frac{\epsilon}{2}.$$

[Hint: First construct, for large R, a smooth cut-off function $\chi_R(x)$ with $\chi_R(x) = 1$ for $|x| < R$, $\chi_R(x) = 0$ for $|x| > 2R$ and $|D\chi_R(x)| < C$, where C is independent of R.]

Corollary 3.4. *i) Suppose $f \in C^k(\mathbb{R}^n)$ and $D^\alpha f \in L^1(\mathbb{R}^n)$ for $|\alpha| \leq k$. Then there is some constant $C_k > 0$ depending only on k such that:*

$$\sup_{\xi\in\mathbb{R}^n} \left|(1+|\xi|)^k \hat{f}(\xi)\right| \leq C_k \sum_{|\alpha|\leq k} ||D^\alpha f||_{L^1(\mathbb{R}^n)}$$

ii) Suppose $(1+|x|)^k f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^k(\mathbb{R}^n)$ and for any $|\alpha| \leq k$ we have:

$$\sup_{\xi\in\mathbb{R}^n} \left|D^\alpha \hat{f}(\xi)\right| \leq \left|\left|(1+|x|)^k f\right|\right|_{L^1(\mathbb{R}^n)}$$

iii) The Fourier transform is a continuous linear map from $\mathscr{S}$ into $\mathscr{S}$:

$$\mathcal{F} : \mathscr{S} \to \mathscr{S}.$$

Proof. i) First we note the algebraic fact that for any k there is some constant C_k such that[2]:

$$(1+|\xi|)^k \leq C_k \sum_{|\alpha|\leq k} |\xi^\alpha|$$

holds for any $\xi \in \mathbb{R}^n$. Repeatedly applying the part $i)$ of Theorem 3.3 we know that:

$$i^{|\alpha|}\xi^\alpha \hat{f}(\xi) = \widehat{D^\alpha f}(\xi).$$

We therefore have:

$$(1+|\xi|)^k \left|\hat{f}(\xi)\right| \leq C_k \sum_{|\alpha|\leq k} \left|i^{|\alpha|}\xi^\alpha \hat{f}(\xi)\right| = C_k \sum_{|\alpha|\leq k} \left|\widehat{D^\alpha f}(\xi)\right|$$

taking the supremum over $\xi \in \mathbb{R}^n$ and applying the estimate (3.1) we conclude

$$\sup_{\xi\in\mathbb{R}^n} \left|(1+|\xi|)^k \hat{f}(\xi)\right| \leq C_k \sum_{|\alpha|\leq k} ||D^\alpha f||_{L^1(\mathbb{R}^n)}.$$

[2] recall that $\xi^\alpha := \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$

ii) By iterating part *ii)* of Theorem 3.3 we have that for $|\alpha| \leq k$:

$$D^\alpha \hat{f}(\xi) = (-i)^{|\alpha|}\widehat{x^\alpha f}(\xi).$$

Taking the supremum of the absolute value over $\xi \in \mathbb{R}^n$ and applying the estimate (3.1) we have:

$$\sup_{\xi\in\mathbb{R}^n} \left|D^\alpha \hat{f}(\xi)\right| \leq ||x^\alpha f||_{L^1(\mathbb{R}^n)} \leq \left|\left|(1+|x|)^k f\right|\right|_{L^1(\mathbb{R}^n)}$$

iii) Note that if:

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^N |f(x)| < K,$$

we have:

$$||f||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|\, dx \leq K \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^N} dx < \infty$$

provided $N > n$. Thus in particular if $f \in \mathscr{S}$ then there exists some constant C_n such that:

$$\left|\left|(1+|x|)^M D^\alpha f\right|\right|_{L^1(\mathbb{R}^n)} \leq C_n \sup_{x\in\mathbb{R}^n} (1+|x|)^{M+n+1} |D^\alpha f(x)|$$

for all $M \in \mathbb{N}$ and all multi-indices α. Applying the previous two parts we conclude that $\hat{f} \in C^\infty(\mathbb{R}^n)$ and:

$$\sup_{\xi\in\mathbb{R}^n, |\beta|\leq M} (1+|\xi|)^N \left|D^\beta \hat{f}(\xi)\right| \leq C_{N,M,n} \sup_{x\in\mathbb{R}^n, |\alpha|\leq N} (1+|x|)^{M+n+1} |D^\alpha f(x)|$$

For some constant $C_{N,M,n}$ depending only on N, M, n. Thus $\hat{f} \in \mathscr{S}$. Moreover, if $\{f_j\}_{j=1}^\infty \subset \mathscr{S}$ is a sequence with $f_j \to 0$ in $\mathscr{S}$, then $\hat{f}_j \to 0$ in $\mathscr{S}$, so that $\mathcal{F}$ is continuous.

□

Notice that while the Fourier transform maps $\mathscr{S}$ to itself, the same is not true of $\mathscr{D}(\mathbb{R}^n)$. Suppose $f \in C_0^\infty(\mathbb{R}^n)$, then provided $\operatorname{supp} f \subset K$ for K a compact set we have:

$$\hat{f}(\xi) = \int_K f(x)e^{-ix\cdot\xi}dx$$

By repeatedly differentiating, it is possible to show that $\hat{f}$ is in fact *real analytic*, and hence $\hat{f}$ cannot vanish on any open set without vanishing everywhere. In particular, $\hat{f}$ cannot vanish outside a compact set.

Exercise 3.7. Suppose $f \in L^1(\mathbb{R}^n)$, with $\operatorname{supp} f \subset B_R(0)$ for some $R > 0$.

a) Show that $\hat{f} \in C^\infty(\mathbb{R}^n)$ and for any multi-index:

$$\sup_{\xi\in\mathbb{R}^n} \left|D^\alpha \hat{f}(\xi)\right| \leq R^{|\alpha|} ||f||_{L^1(\mathbb{R}^n)}$$

b) Show that $\hat{f}$ is real analytic, with an infinite radius of convergence, i.e.:

$$\hat{f}(\xi) = \sum_{\alpha} D^\alpha \hat{f}(0) \frac{\xi^\alpha}{\alpha!}$$

holds for all $\xi \in \mathbb{R}^n$.

c) Show that if $\hat{f}(\xi)$ vanishes on an open set, it must vanish everywhere. *[Hint: use part i) of Lemma 3.2]*

You may assume the following form of Taylor's theorem. Suppose $g \in C^{k+1}(\overline{B_r(0)})$. Then for $x \in B_r(0)$:

$$g(x) = \sum_{|\alpha| \leq k} D^\alpha \hat{f}(0) \frac{\xi^\alpha}{\alpha!} + \sum_{\beta = k+1} R_\beta(x) x^\beta$$

where the remainder $R_\beta(x)$ satisfies the following estimate in $B_r(0)$:

$$|R_\beta(x)| \leq \frac{1}{\beta!} \max_{|\alpha| = |\beta|} \max_{y \in \overline{B_r(0)}} |D^\alpha g(y)|\,.$$

See §A.1 of the notes for notation.

To complete this section, we are going to establish the invertibility of the Fourier transform, under some reasonable assumptions on f and $\hat{f}$. In particular, this will permit us to show that $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is in fact a bijection.

Theorem 3.5 (Fourier inversion theorem)**.** *Suppose $f \in L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ and assume $\hat{f} \in L^1(\mathbb{R}^n)$. Then:*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi. \tag{3.3}$$

Proof. We shall establish the result by looking at the limit $\epsilon \to 0$ of

$$I_\epsilon(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\frac{1}{2}\epsilon^2 |\xi|^2} e^{ix\cdot\xi} d\xi.$$

in two different ways. Firstly note that for $\xi \in \mathbb{R}^n$ we have:

$$\hat{f}(\xi) e^{-\frac{1}{2}\epsilon^2 |\xi|^2} e^{ix\cdot\xi} \to \hat{f}(\xi) e^{ix\cdot\xi}.$$

Moreover, we can estimate

$$\left| \hat{f}(\xi) e^{-\frac{1}{2}\epsilon^2 |\xi|^2} e^{ix\cdot\xi} \right| \leq \left| \hat{f}(\xi) \right|$$

so that the integrand is dominated by an integrable function. Thus by the Dominated Convergence Theorem we have:

$$I_\epsilon(x) \to \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi, \qquad \text{as } \epsilon \to 0.$$

On the other hand, we have, using Fubini's thorem:

$$\begin{aligned} I_\epsilon(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(y)e^{-i\xi\cdot y}dy\right)e^{-\frac{1}{2}\epsilon^2|\xi|^2}e^{ix\cdot\xi}d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(y)\left(\int_{\mathbb{R}^n} e^{-\frac{1}{2}\epsilon^2|\xi|^2}e^{-i\xi\cdot(y-x)}d\xi\right)dy \\ &= \int_{\mathbb{R}^n} f(y)\frac{1}{\epsilon^n(2\pi)^{\frac{n}{2}}}e^{-\frac{|y-x|^2}{2\epsilon^2}}dy \\ &= f\star\psi_\epsilon(x) \end{aligned}$$

where $\psi_\epsilon(x) = \epsilon^{-n}\psi(\epsilon^{-1}x)$ for

$$\psi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{1}{2}|x|^2}.$$

Note that $\psi \in L^1(\mathbb{R}^n)$, $\psi(x) \geq 0$ and

$$\int_{\mathbb{R}^n}\psi(x)dx = 1$$

so by Theorem 1.13, c), we have that:

$$f\star\psi_\epsilon(x) \to f(x),$$

for each $x \in \mathbb{R}^n$. Equating the two expressions for the limit of $I_\epsilon(x)$ we are done. □

We can summarise the inversion formula quite neatly by noting that:

$$\mathcal{F}^2 f = (2\pi)^n\check{f}.$$

An immediate corollary of the above result is that $\mathcal{F}:\mathscr{S}\to\mathscr{S}$ is a bijection, and that $\mathcal{F}^{-1}:\mathscr{S}\to\mathscr{S}$ is continuous.

Exercise 4.1. Consider the following ODE problem. Given $f:\mathbb{R}\to\mathbb{C}$, find ϕ such that:

$$-\phi'' + \phi = f. \tag{3.4}$$

a) Show that if $f\in\mathscr{S}$, there is a unique $\phi\in\mathscr{S}$ solving (3.4), and give an expression for $\hat{\phi}$.

b) Show that

$$\phi(x) = \int_{\mathbb{R}} f(y)G(x-y)dy$$

where

$$G(x) = \begin{cases} \frac{1}{2}e^{x} & x<0, \\ \frac{1}{2}e^{-x} & x\geq 0. \end{cases}$$

Exercise 4.2. Suppose $f\in L^1(\mathbb{R}^3)$ is a radial function, i.e. $f(Rx) = f(x)$, whenever $R\in SO(3)$ is a rotation.

a) Show that $\hat{f}$ is radial.

b) Suppose that $\xi = (0, 0, \zeta)$. By writing the Fourier integral in polar coordinates, show that

$$\hat{f}(\xi) = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(r) e^{-i\zeta r \cos\theta} r^2 \sin\theta d\theta dr d\phi.$$

c) Making the substitution $s = \cos\theta$, and using the fact that $\hat{f}$ is radial, deduce:

$$\hat{f}(\xi) = 4\pi \int_0^{\infty} f(r) \frac{\sin r\,|\xi|}{r\,|\xi|} r^2 dr$$

for any $\xi \in \mathbb{R}^n$.

3.2 The Fourier transform on $L^2(\mathbb{R}^n)$

Having defined the Fourier transform acting on functions in $L^1(\mathbb{R}^n)$, we are going to extend it to act on more general functions (and eventually distributions). Firstly, we shall see how the Fourier transform extends very nicely to act on functions in $L^2(\mathbb{R}^n)$. This is a particularly nice function space because is is a *Hilbert space*. Let us recall a few facts about $L^2(\mathbb{R}^n)$. Firstly it is equipped with an inner product:

$$(f, g) = \int_{\mathbb{R}^n} \overline{f}(x) g(x) dx,$$

which induces the norm via:

$$||f||_{L^2(\mathbb{R}^n)} = (f, f)^{\frac{1}{2}}$$

and moreover it is complete, which means that all Cauchy sequences converge in $L^2(\mathbb{R}^n)$.

Roughly you should think of functions in $L^2(\mathbb{R}^n)$ as being locally better behaved than those in $L^1(\mathbb{R}^n)$, but with less control on their growth towards infinity. In particular, $|x|^{-p} \in L^1(B_1(0))$ if and only if $p < n$, while $|x|^{-p} \in L^2(B_1(0))$ if and only if $p < n/2$, so locally L^1 functions may be more singular than locally L^2. On the other hand, $|x|^{-p} \in L^1(\mathbb{R}^n \setminus B_1(0))$ if and only if $p > n$, while $|x|^{-p} \in L^2(\mathbb{R}^n \setminus B_1(0))$ if and only if $p > n/2$, so that functions in L^2 are permitted to decay more slowly towards infinity than those in L^1.

We shall first establish that the Fourier transform maps $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, and moreover show that the L^2 inner product is preserved by the Fourier transform (up to multiplication by a constant).

Theorem 3.6 (Parseval's Formula)**.** *Suppose* $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. *Then* $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$ *and moreover:*

$$(f, g) = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g}).$$

Proof. We will again use a density argument to prove this result. First suppose that $f, g \in \mathscr{S}$. Then using the Fourier Inversion Theorem (Theorem 3.5) and Fubini's theorem we can calculate:

$$\begin{aligned}
(f,g) &= \int_{\mathbb{R}^n} \overline{f}(x) g(x) dx \\
&= \int_{\mathbb{R}^n} \overline{f}(x) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi) e^{ix\cdot\xi} d\xi \right) dx \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \overline{f}(x) e^{ix\cdot\xi} dx \right) \hat{g}(\xi) d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\left(\int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx \right)} \hat{g}(\xi) d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\hat{f}}(\xi) \hat{g}(\xi) d\xi = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})
\end{aligned}$$

Now suppose that $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By Theorem 1.13 part *b*), there exists a sequence $\{f_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^n) \subset \mathscr{S}$ such that:

$$||f_j - f||_{L^1(\mathbb{R}^n)} + ||f_j - f||_{L^2(\mathbb{R}^n)} < \frac{1}{j}$$

and similarly for g. We know that:

$$\sup_{\xi \in \mathbb{R}^n} \left| \hat{f}_j(\xi) - \hat{f}(\xi) \right| \leq ||f_j - f||_{L^1(\mathbb{R}^n)} < \frac{1}{j}$$

so that $\hat{f}_j \to f$ uniformly on $\mathbb{R}^n$. We also have by the calculation above:

$$\left|\left| \hat{f}_j - \hat{f}_k \right|\right|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} ||f_j - f_k||_{L^2(\mathbb{R}^n)} .$$

Now since $f_j \to f$ in $L^2(\mathbb{R}^n)$, we have that $\{f_j\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. Thus $\hat{f}_j$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. By the completeness of $L^2(\mathbb{R}^n)$, we have that $\hat{f}_j$ converges in $L^2(\mathbb{R}^n)$ and hence $\hat{f} \in L^2(\mathbb{R}^n)$. Furthermore, we know that

$$(f_j, g_j) = \frac{1}{(2\pi)^n} (\hat{f}_j, \hat{g}_j)$$

since each of the sequences $\{f_j\}, \{g_j\}, \{\hat{f}_j\}, \{\hat{g}_j\}$ converge in $L^2(\mathbb{R}^n)$, we can take the limit[3] $j \to \infty$ to conclude:

$$(f,g) = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})$$ □

Thus we have shown that the Fourier transform $\mathcal{F}$ maps $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Moreover, we have that it is a *bounded* as an operator from $L^2(\mathbb{R}^n)$ to itself, since

$$\left|\left| \hat{f} \right|\right|_{L^2(\mathbb{R}^n)} \leq (2\pi)^{\frac{n}{2}} ||f||_{L^2(\mathbb{R}^n)} .$$

[3] You should check that you understand why this is valid.

This means that $\mathcal{F}$ is a bounded linear map defined on a dense subset of a Banach space. A general result tells us that the map extends uniquely to a bounded linear map on the entire space. Rather than invoke an abstract result, we can show this directly.

Corollary 3.7. *There is a unique continuous linear operator* $\overline{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *such that:*

$$\overline{\mathcal{F}}[f] = \mathcal{F}[f], \qquad \text{for all } \ f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{3.5}$$

We say that $\overline{\mathcal{F}}$ *is the extension of the Fourier transform to* $L^2(\mathbb{R}^n)$*. It is sometimes known as the* Fourier-Plancherel *transform.*

Proof. For any $f \in L^2(\mathbb{R}^n)$, we can take a sequence $\{f_j\}_{j=1}^\infty \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $f_j \to f$ in $L^2(\mathbb{R}^n)$ (for example by approximating f with smooth functions of compact support). By Theorem 3.6 we have that:

$$\left|\left|\hat{f}_j - \hat{f}_k\right|\right|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} \left|\left|f_j - f_k\right|\right|_{L^2(\mathbb{R}^n)}. \tag{3.6}$$

Now, since f_j converges in $L^2(\mathbb{R}^n)$, it is in particular a Cauchy sequence in $L^2(\mathbb{R}^n)$. Equation (3.6) shows that $\hat{f}_j$ is also a Cauchy sequence in $L^2(\mathbb{R}^n)$, hence has a limit, say $F \in L^2(\mathbb{R}^n)$ by the completeness of $L^2(\mathbb{R}^n)$. Suppose $\{f'_j\}_{j=1}^\infty \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is another sequence with $f'_j \to f$, and suppose $\hat{f}_j \to F'$. Then we have:

$$\left|\left|F - F'\right|\right|_{L^2(\mathbb{R}^n)} = \lim_{j\to\infty} \left|\left|\hat{f}_j - \hat{f}'_j\right|\right|_{L^2(\mathbb{R}^n)} = \lim_{j\to\infty} (2\pi)^{\frac{n}{2}} \left|\left|f_j - f'_j\right|\right|_{L^2(\mathbb{R}^n)} = 0$$

since both f_j and f'_j tend to f. Thus F depends only f, and not on the sequence f_j which we chose to approximate f.

We define $\overline{\mathcal{F}}[f] = F$, i.e.:

$$\overline{\mathcal{F}}[f] = \lim_{j\to\infty} \mathcal{F}[f_j], \quad \text{where } \{f_j\}_{j=1}^\infty \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad f_j \to f \text{ in } L^2(\mathbb{R}^n),$$

and the limit is to be understood to be in $L^2(\mathbb{R}^n)$. This certainly satisfies (3.5), since we can take our approximating sequence to be the constant sequence $f_j = f$ for all j when $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. $\overline{\mathcal{F}}$ is clearly linear and moreover, we have that

$$\begin{aligned}
\left|\left|\overline{\mathcal{F}}[f]\right|\right|_{L^2(\mathbb{R}^n)} &= \left|\left|\lim_{j\to\infty} \mathcal{F}[f_j]\right|\right|_{L^2(\mathbb{R}^n)} \\
&= \lim_{j\to\infty} \left|\left|\mathcal{F}[f_j]\right|\right|_{L^2(\mathbb{R}^n)} \\
&= \lim_{j\to\infty} (2\pi)^{\frac{n}{2}} \left|\left|f_j\right|\right|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} \left|\left|f\right|\right|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

so $\overline{\mathcal{F}}$ is bounded and hence continuous[4]. It remains to show that $\overline{\mathcal{F}}$ is unique. Suppose that $\overline{\mathcal{F}}'$ is another continuous linear operator satisfying (3.5). For any $f \in L^2(\mathbb{R}^n)$, take

[4]If $\{f_j\}_{j=1}^\infty \subset L^2(\mathbb{R}^n)$ is a sequence with $f_j \to f$ in $L^2(\mathbb{R}^n)$, then

$$\left|\left|\overline{\mathcal{F}}[f_j] - \overline{\mathcal{F}}[f]\right|\right|_{L^2(\mathbb{R}^n)} = \left|\left|\overline{\mathcal{F}}[f_j - f]\right|\right|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} \left|\left|f_j - f\right|\right|_{L^2(\mathbb{R}^n)} \to 0$$

so $\overline{\mathcal{F}}[f_j] \to \overline{\mathcal{F}}[f]$ in $L^2(\mathbb{R}^n)$.

a sequence $\{f_j\}_{j=1}^{\infty} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $f_j \to f$ in $L^2(\mathbb{R}^n)$. We have:

$$\overline{\mathcal{F}}'[f] = \lim_{j\to\infty} \overline{\mathcal{F}}'[f_j] = \lim_{j\to\infty} \overline{\mathcal{F}}[f_j] = \overline{\mathcal{F}}[f]$$

so that $\overline{\mathcal{F}}' = \overline{\mathcal{F}}$. □

Exercise 4.3. (*) Suppose that $f, g \in L^2(\mathbb{R}^n)$, and denote the Fourier-Plancherel transform by $\overline{\mathcal{F}}$. You may assume any results already established for the Fourier transform.

a) Show that

$$(f, g) = \frac{1}{(2\pi)^n} \left(\overline{\mathcal{F}}[g], \overline{\mathcal{F}}[g]\right).$$

b) Recall that $\check{f}(y) = f(-y)$. Show that:

$$\overline{\mathcal{F}}\left[\overline{\mathcal{F}}[f]\right] = \check{f}.$$

Hence, or otherwise, deduce that $\overline{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bijection, and that $\overline{\mathcal{F}}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear map.

c) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \lim_{R\to\infty} \int_{B_R(0)} f(x) e^{-ix\cdot\xi} dx$$

with convergence in the sense of $L^2(\mathbb{R}^n)$.

d) Suppose that $f \in C^1(\mathbb{R}^n)$ and $f, D_j f \in L^2(\mathbb{R}^n)$. Show that $\xi_j \overline{\mathcal{F}}[f](\xi) \in L^2(\mathbb{R}^n)$ and:

$$\overline{\mathcal{F}}[D_j f](\xi) = i\xi_j \overline{\mathcal{F}}[f](\xi)$$

e) For $x \in \mathbb{R}$ let:

$$f(x) = \frac{\sin x}{x}$$

i) Show that $f \in L^2(\mathbb{R})$.

ii) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \begin{cases} \pi & -1 < \xi < 1, \\ 0 & |\xi| \geq 1. \end{cases}$$

f) i) Show that for all $x \in \mathbb{R}^n$:

$$|f \star g(x)| \leq ||f||_{L^2(\mathbb{R}^n)} ||g||_{L^2(\mathbb{R}^n)}.$$

ii) Show that $f \star g \in C^0(\mathbb{R}^n)$ and:

$$f \star g = \mathcal{F}^{-1}\left[\overline{\mathcal{F}}[f] \cdot \overline{\mathcal{F}}[g]\right]$$

where:

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi\cdot x} d\xi.$$

[Hint for parts a), b), d), f): approximate by Schwartz functions]

Exercise 4.4. Work in $\mathbb{R}^3$. For $k > 0$, define the function:

$$G(x) = \frac{e^{-k|x|}}{4\pi |x|}$$

a) Show that $G \in L^1(\mathbb{R}^3)$.

b) Show that:

$$\hat{G}(\xi) = \frac{1}{|\xi|^2 + k^2}$$

[Hint: use Exercise 4.2, part c)]

Exercise 4.5. Consider the inhomogeneous Helmholtz equation on $\mathbb{R}^3$:

$$-\Delta\phi + k^2\phi = f \tag{3.7}$$

where $f \in \mathscr{S}$. Show that there exists a unique $\phi \in \mathscr{S}$ satisfying (3.7) given by:

$$\phi(x) = \int_{\mathbb{R}^3} f(y)G(x-y)dy,$$

where

$$G(x) = \frac{e^{-k|x|}}{4\pi |x|}.$$

[Hint: first derive an equation satisfied by $\hat{\phi}$]

Usually one does not labour the distinction between the Fourier transform acting on $L^1(\mathbb{R}^n)$ and the Fourier-Plancherel transform acting on $L^2(\mathbb{R}^n)$. From now on we shall use the same notation for both, so that for $f \in L^2(\mathbb{R}^n)$ we write $\overline{\mathcal{F}}[f] = \mathcal{F}[f] = \hat{f}$. Since the two transforms agree wherever both are defined, there is no ambiguity in this. The majority of the results that we have already established for the Fourier transform extend to the Fourier-Plancherel transform in a straightforward way, see Exercise 4.3.

3.3 The Fourier transform on $\mathscr{S}'$

We are now going to extend the Fourier transform in a slightly different way, such that it acts on distributions. Suppose that $f \in L^1(\mathbb{R}^n)$, and $\phi \in \mathscr{S}$. Then since $\hat{f} \in C^0(\mathbb{R}^n)$ and $\hat{f}$ decays towards infinity, we have that $T_{\hat{f}} \in \mathscr{S}'$. By Fubini we have:

$$\begin{aligned} T_{\hat{f}} = \int_{\mathbb{R}^n} \hat{f}(x)\phi(x)dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)e^{-ix\cdot y}dy \right) \phi(x)dx \\ &= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} \phi(x)e^{-ix\cdot y}dx \right) dy \\ &= \int_{\mathbb{R}^n} f(x)\hat{\phi}(x)dx. \end{aligned}$$

Thus for $f \in L^1(\mathbb{R}^n)$ we have that:

$$T_{\hat{f}}[\phi] = T_f[\hat{\phi}], \qquad \text{for all } \phi \in \mathscr{S}.$$

Motivated by this, we define:

Definition 3.1. *For a distribution $u \in \mathscr{S}'$, we define the Fourier transform of u, written $\hat{u} \in \mathscr{S}'$ to be the distribution satisfying:*

$$\hat{u}[\phi] = u[\hat{\phi}], \qquad \textit{for all } \phi \in \mathscr{S}.$$

Notice that the definition makes sense because the Fourier transform maps $\mathscr{S}$ to $\mathscr{S}$ continuously. If we tried to use the above definition but with $\phi \in \mathscr{D}(\mathbb{R}^n)$ and $u \in \mathscr{D}'(\mathbb{R}^n)$, we would run into difficulties because $\hat{\phi} \notin \mathscr{D}(\mathbb{R}^n)$.

Example 13. *a) For $\xi \in \mathbb{R}^n$ we have:*

$$\widehat{\delta_\xi} = T_{e_{-\xi}}$$

To see this, we use the definition. For $\phi \in \mathscr{S}$:

$$\widehat{\delta_\xi}[\phi] = \delta_\xi[\hat{\phi}] = \hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(x)dx = T_{e_{-\xi}}[\phi]$$

Since ϕ was arbitrary, the distributions are equal.

b) For $x \in \mathbb{R}^n$ we have:

$$\widehat{T_{e_x}} = (2\pi)^n \delta_x.$$

To see this, we note for $\phi \in \mathscr{S}$:

$$\widehat{T_{e_x}}[\phi] = T_{e_x}[\hat{\phi}] = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\phi}(\xi)d\xi = (2\pi)^n\phi(x) = (2\pi)^n\delta_x[\phi].$$

Again, as ϕ is arbitrary the distributions are equal. Note that a particular case is $\widehat{T_1} = (2\pi)^n\delta_0$.

c) For α a multi-index, denote by X^α the map

$$X^\alpha : x \mapsto x^\alpha.$$

Then we have:

$$\widehat{T_{X^\alpha}} = (2\pi)^n i^{|\alpha|} D^\alpha \delta_0$$

For $\phi \in \mathscr{S}$:

$$\begin{aligned}
\widehat{T_{X^\alpha}}[\phi] = T_{X^\alpha}[\hat{\phi}] &= \int_{\mathbb{R}^n} \xi^\alpha \hat{\phi}(\xi)d\xi \\
&= (-i)^{|\alpha|}\int_{\mathbb{R}^n} \widehat{D^\alpha\phi}(\xi)d\xi \\
&= (2\pi)^n(-i)^{|\alpha|}D^\alpha\phi(0) = (2\pi)^n i^{|\alpha|} \times (-1)^{|\alpha|}\delta_0\left[D^\alpha\phi\right] \\
&= (2\pi)^n i^{|\alpha|}D^\alpha\delta_0[\phi]
\end{aligned}$$

Most of the properties of the Fourier transform defined on $\mathscr{S}$ are inherited by the transform defined on $\mathscr{S}'$. We first need to define a couple of operations on $\mathscr{S}'$. Recall that if $\phi \in \mathscr{S}$, then $\tau_x\phi \in \mathscr{S}$ is the translate of ϕ, given by $\tau_x\phi(y) = \phi(y-x)$, and $\check{\phi} \in \mathscr{S}$ is given by $\check{\phi}(y) = \phi(-y)$. For $u \in \mathscr{S}$, we define:

$$\tau_x u[\phi] = u[\tau_{-x}\phi], \qquad \check{u}[\phi] = u[\check{\phi}]$$

Notice also that if $f \in C^\infty(\mathbb{R}^n)$ is a function of tempered growth, i.e., if for each α and there exists a constant C_α and integer N_α such that:

$$|D^\alpha f(x)| \leq C_\alpha(1+|x|)^{N_\alpha}, \qquad \forall x \in \mathbb{R}^n.$$

then $\phi f \in \mathscr{S}$ when $\phi \in \mathscr{S}$ and we can define $fu \in \mathscr{S}'$ by

$$fu[\phi] = u[f\phi]$$

Exercise 4.6. Verify that if $f \in L^1_{loc.}$ is such that $T_f \in \mathscr{S}'$, then:

$$\tau_x T_f = T_{\tau_x f}, \quad \text{and} \quad \check{T}_f = T_{\check{f}}$$

Lemma 3.8. *Suppose $u \in \mathscr{S}'$ is a tempered distribution. Then:*

$$\widehat{e_x u} = \tau_x \hat{u}, \qquad \widehat{\tau_x u} = e_{-x}\hat{u}, \qquad \widehat{D^\alpha u} = i^{|\alpha|}X^\alpha \hat{u} \qquad D^\alpha \hat{u} = (-i)^{|\alpha|}\widehat{X^\alpha u}$$

Moreover:

$$\hat{\hat{u}} = (2\pi)^n \check{u},$$

so that the Fourier transform on $\mathscr{S}'$ is invertible.

Proof. These are all calculations using the corresponding results for $\mathscr{S}$. Take $\phi \in \mathscr{S}$. We have:

$$\widehat{e_x u}[\phi] = e_x u[\hat{\phi}] = u[e_x\hat{\phi}] = u\left[\widehat{\tau_{-x}\phi}\right] = \hat{u}[\tau_{-x}\phi] = \tau_x u[\phi].$$

Since ϕ was arbitrary, we have $\widehat{e_x u} = \tau_x \hat{u}$. Similarly, we calculate:

$$\widehat{\tau_x u}[\phi] = \tau_x u[\hat{\phi}] = u[\tau_{-x}\hat{\phi}] = u\left[\widehat{e_{-x}\phi}\right] = \hat{u}[e_{-x}\phi] = e_{-x}u[\phi].$$

Next we have

$$\begin{aligned}
\widehat{D^\alpha u}[\phi] &= D^\alpha u\left[\hat{\phi}\right] = (-1)^{|\alpha|}u\left[D^\alpha\hat{\phi}\right] \\
&= (-1)^{|\alpha|}u\left[(-i)^{|\alpha|}\widehat{X^\alpha\phi}\right] = i^{|\alpha|}u\left[\widehat{X^\alpha\phi}\right] \\
&= i^{|\alpha|}\hat{u}\left[X^\alpha\phi\right] = \left(i^{|\alpha|}X^\alpha\hat{u}\right)[\phi]
\end{aligned}$$

similarly:

$$\begin{aligned}
D^\alpha\hat{u}[\phi] &= (-1)^{|\alpha|}\hat{u}[D^\alpha\phi] = (-1)^{|\alpha|}u[\widehat{D^\alpha\phi}] \\
&= (-1)^{|\alpha|}u[i^{|\alpha|}X^\alpha\hat{\phi}] = (-i)^{|\alpha|}X^\alpha u[\hat{\phi}] \\
&= \left((-i)^{|\alpha|}\widehat{X^\alpha u}\right)[\phi].
\end{aligned}$$

Finally, we have

$$\hat{u}[\phi] = \hat{u}[\hat{\phi}] = u[\hat{\hat{\phi}}] = u[(2\pi)^n \check{\phi}] = (2\pi)^n \check{u}[\phi]$$

Since $\check{\check{u}} = u$, we have that the Fourier transform is invertible. □

Importantly, the Fourier transform is also a continuous linear map $\mathscr{S}' \to \mathscr{S}'$.

Lemma 3.9. *The map:*

$$\begin{array}{cccc} \mathcal{F} & : & \mathscr{S}' & \to & \mathscr{S}' \\ & & u & \mapsto & \hat{u} \end{array}$$

is a linear homeomorphism.

Proof. We already have that $\mathcal{F}$ is linear. From the definition of the weak-$\star$ topology, a sequence $\{u_j\}_{j=1}^{\infty} \subset \mathscr{S}'$ converges to u if

$$u_j[\phi] \to u[\phi]$$

for all $\phi \in \mathscr{S}$. Suppose that we have such a convergent sequence in $\mathscr{S}'$. We calculate:

$$\hat{u}_j[\phi] = u_j[\hat{\phi}] \to u[\hat{\phi}] = \hat{u}[\phi].$$

Thus if $u_j \to u$ we have $\mathcal{F}(u_j) \to \mathcal{F}(u)$. Thus $\mathcal{F}$ is continuous. Since $\mathcal{F}^4 = (2\pi)^{2n}\iota$, we have that $\mathcal{F}$ is invertible and the inverse is also continuous. □

Remark. *Strictly, we have only established that $\mathcal{F}$ is sequentially continuous with respect to the weak-$\star$ topology induced on $\mathscr{S}'$ by $\mathscr{S}$. Establishing genuine continuity is not difficult, but would require a full description of the weak-$\star$ topology, which takes us a bit beyond the scope of the course.*

Exercise 4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be the sign function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and define $f_R(x) = f(x)\mathbb{1}_{[-R,R]}(x)$.

a) Sketch $f_R(x)$.

b) Show that:

$$T_{f_R} \to T_f \text{ in } \mathscr{S}' \text{ as } R \to \infty.$$

c) Show that:

$$\hat{f}_R(\xi) = 2i\frac{\cos R\xi - 1}{\xi}$$

d) For $\phi \in \mathscr{S}$, show that:

$$T_{\hat{f}_R}[\phi] = -2i\int_0^\infty \frac{\phi(x) - \phi(-x)}{x}dx + 2i\int_0^\infty \left(\frac{\phi(x) - \phi(-x)}{x}\right)\cos Rxdx$$

e) By applying the Riemann-Lebesgue Lemma, or otherwise, show that for any $\psi \in \mathscr{S}$:

$$\int_0^\infty \psi(x) \cos Rx dx \to 0$$

as $R \to \infty$.

f) Deduce that

$$\widehat{T_f} = -2i P.V. \left(\frac{1}{x}\right)$$

g) Write down $\widehat{T_H}$, where H is the Heaviside function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

3.3.1 Convolutions

We have generalised almost all of the properties of the Fourier transform to distributions. The final result that we shall establish concerns convolutions. Recall that if $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$ then $u \star \phi \in C^\infty(\mathbb{R}^n)$ is given by:

$$u \star \phi(x) = u\left[\tau_x \check{\phi}\right].$$

Notice that this definition continues to make sense for each x, provided $u \in \mathscr{S}'$ and $\phi \in \mathscr{S}$, although it is no longer clear that the resulting function is smooth. We have the following results concerning this convolution.

Theorem 3.10. *Suppose $u \in \mathscr{S}'$ and $\phi \in \mathscr{S}$ are given. Then the function:*

$$u \star \phi : \mathbb{R}^n \to \mathbb{C}$$

has the following properties

a) $u \star \phi \in C^\infty(\mathbb{R}^n)$ with

$$D^\alpha(u \star \phi) = D^\alpha u \star \phi = u \star D^\alpha \phi.$$

b) There exist constants $N \in \mathbb{N}$, $K > 0$ depending on u and ϕ such that:

$$|u \star \phi(x)| \leq K(1 + |x|)^N.$$

c) $T_{u\star\phi} \in \mathscr{S}'$ and moreover:

$$\widehat{T_{u\star\phi}} = \hat{\phi}\hat{u}.$$

d) For any $\psi \in \mathscr{S}$, we have:

$$(u \star \phi) \star \psi = u \star (\phi \star \psi)$$

e) We have:

$$T_{\hat{u}\star\hat{\phi}} = (2\pi)^n \widehat{\phi u}$$

Proof. a) The smoothness of $u \star \phi$ is proven exactly as in Lemma 2.6, *ii*). The only modification to the argument required is to note that for $\phi \in \mathscr{S}$, we have

$$\Delta_i^h \phi \to D_i \phi \quad \text{in} \quad \mathscr{S}, \quad \text{as} \quad h \to 0.$$

b) First, we note the following simple inequality which holds for all $x, y \in \mathbb{R}^n$:

$$1 + |x + y| \leq 1 + |x| + |y| \leq (1 + |x|)(1 + |y|).$$

Next, recall from Lemma 2.11 that there exist $N, k \in \mathbb{N}$ and $C > 0$ such that:

$$|u[\psi]| \leq C \sup_{y \in \mathbb{R}^n; |\alpha| \leq k} \left|(1 + |y|)^N D^\alpha \psi(y)\right|, \qquad \text{for all } \psi \in \mathscr{S}.$$

Applying this inequality with $\psi = \tau_x \check{\phi}$, we calculate:

$$\begin{aligned} |u \star \phi(x)| = \left|u\left[\tau_x \check{\phi}\right]\right| &\leq C \sup_{y \in \mathbb{R}^n; |\alpha| \leq k} \left|(1 + |y|)^N D^\alpha \phi(y - x)\right| \\ &= C \sup_{y \in \mathbb{R}^n; |\alpha| \leq k} \left|(1 + |z + x|)^N D^\alpha \phi(z)\right| \\ &\leq \left[C \sup_{y \in \mathbb{R}^n; |\alpha| \leq k} \left|(1 + |z|)^N D^\alpha \phi(z)\right|\right] (1 + |x|)^N \end{aligned}$$

which gives the result on setting:

$$K = C \sup_{y \in \mathbb{R}^n; |\alpha| \leq k} \left|(1 + |z|)^N D^\alpha \phi(z)\right|.$$

c) Combining the above two results, we have that $T_{u \star \phi} \in \mathscr{S}'$, since $u \star \phi \in L^1_{loc.}(\mathbb{R}^n)$ and $u \star \phi$ grows at most polynomially. It therefore makes sense to consider the Fourier transform. Suppose that $\psi \in \mathscr{D}(\mathbb{R}^n)$. We calculate:

$$\begin{aligned} \widehat{T_{u\star\phi}}[\hat{\psi}] = T_{u\star\phi}\left[\hat{\hat{\psi}}\right] &= (2\pi)^n T_{u\star\phi}\left[\check{\psi}\right] && \text{Fourier Inversion Thm} \\ &= (2\pi)^n \int_{\mathbb{R}^n} u \star \phi(x) \psi(-x) dx && \text{Defn. of } T_f \\ &= (2\pi)^n \int_{\mathbb{R}^n} u\left[\tau_x \check{\phi}\right] \psi(-x) dx && \text{Defn. of } u \star \phi \\ &= (2\pi)^n \int_{\mathbb{R}^n} u\left[\psi(-x) \tau_x \check{\phi}\right] dx && \text{Linearity of } u \\ &= (2\pi)^n u\left[\int_{\mathbb{R}^n} \psi(-x) \tau_x \check{\phi} dx\right] && (!!) \\ &= (2\pi)^n u\left[(\phi \check{\star} \psi)\right] && \text{Defn. of } \phi \star \psi \\ &= u\left[\widehat{\widehat{\phi \star \psi}}\right] = \hat{u}\left[\widehat{\phi \star \psi}\right] && \text{Fourier Inversion Thm} \\ &= \hat{u}\left[\hat{\phi}\hat{\psi}\right] = \left(\hat{\phi}\hat{u}\right)[\hat{\psi}] && \text{F.T. of convolution} \end{aligned}$$

Most of the manipulations here are relatively straightforward. We have used Theorems 3.2, 3.5 in addition to various definitions. The step marked (!!), in which we interchanged an integration and an application of u requires some justification. Crudely this is true because we can replace the integration with an appropriately convergent Riemann sum and use the linearity and continuity of u. We shall justify this step in Lemma 3.11. The conclusion of this calculation is that:

$$\widehat{T_{u\star\phi}}[\hat{\psi}] = \left(\hat{\phi}\hat{u}\right)[\hat{\psi}]$$

This holds for all $\psi \in \mathscr{D}(\mathbb{R}^n)$. Now, since $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}$ and $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is a homeomorphism, we have that:

$$\mathcal{F}\left[\mathscr{D}(\mathbb{R}^n)\right] = \left\{\hat{\psi} : \psi \in \mathscr{D}(\mathbb{R}^n)\right\}$$

is dense in $\mathscr{S}$. Thus, by approximation,

$$\widehat{T_{u\star\phi}}[\chi] = \left(\hat{\phi}\hat{u}\right)[\chi]$$

holds for any $\chi \in \mathscr{S}$ and we're done.

d) Note that in the process of proving the previous part, we established that for any $\psi \in \mathscr{D}(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} u \star \phi(x)\psi(-x)dx = u\left[(\phi \check{\star} \psi)\right]$$

which is equivalent to:

$$(u \star \phi) \star \psi(0) = u \star (\phi \star \psi)(0). \tag{3.8}$$

Now, note that:

$$u \star \tau_y\phi = \tau_y (u \star \phi), \qquad \phi \star \tau_y\psi = \tau_y (\phi \star \psi)$$

as can bee easily seen from the definitions. Applying (3.8) with ψ replaced by $\tau_y\psi$, we conclude that:

$$(u \star \phi) \star \psi(y) = u \star (\phi \star \psi)(y).$$

Since this holds for any $\psi \in \mathscr{D}(\mathbb{R}^n)$ and $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}$, we're done.

e) This result follows by applying part *c*) to $\hat{u} \star \hat{\phi}$ and repeatedly making use of the Fourier inversion theorem. We calculate:

$$\begin{aligned}\widehat{T_{\hat{u}\star\hat{\phi}}} &= \hat{\hat{\phi}}\hat{\hat{u}} = (2\pi)^{2n}\check{\phi}\check{u} \\ &= (2\pi)^{2n}(\check{\phi u}) = (2\pi)^n\widehat{\widehat{(\phi u)}}\end{aligned}$$

Since the Fourier transform is a bijection on $\mathscr{S}'$, the result follows.

□

In order to complete the proof of the above result, we need to justify the step marked (!!) in which a tempered distribution and an integration were interchanged. We will first prove a result concerning the convergence of Riemann sums, which will enable us to establish that the (!!) step was justified. Let us suppose that $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ are open and that $f \in C^0(\Omega \times \Omega')$ is uniformly continuous. We will also assume that there is some $R > 0$ such that:

$$\operatorname{supp} f(\cdot, y) \subset [-R, R)^n \subset \Omega$$

for each $y \in \Omega'$.

Next, we define a dyadic family of partitions of $[-R, R)^n$ into cubes as follows:

$$\Pi_k = \left\{ \left[-\frac{R}{2^k} i_1, \frac{R}{2^k}(i_1 + 1) \right) \times \cdots \times \left[-\frac{R}{2^k} i_n, \frac{R}{2^k}(i_n + 1) \right) : i_l \in [-2^k, 2^k - 1] \cap \mathbb{Z} \right\}$$

where $k = 0, 1, \ldots$. The $(k+1)^{st}$ partition is obtained by chopping each cube in the k^{th} partition into cubes with half the side length. Clearly for each fixed k:

$$\bigcup \Pi_k = [-R, R)^n$$

For $\pi \in \Pi_k$, we define x_π to be the point at the centre of the cube π. We define the k^{th} Riemann sum with respect to this partition by:

$$S_k(y) = \sum_{\pi \in \Pi_k} f(x_\pi, y) \left| \pi \right| .$$

Lemma 3.11. *With the definitions as above,* (U)

$$S_k(y) \to \int_\Omega f(x, y) dx$$

uniformly in $y \in \Omega'$.

Proof. First note that $x \mapsto f(x, y)$ is continuous and of compact support, hence Riemann integrable on Ω. Thus for each fixed y we have:

$$S_k(y) \to \int_\Omega f(x, y) dx$$

Next consider $k' \geq k$. We have that $\Pi_{k'}$ is a refinement of Π_k, i.e. if $\pi' \in \Pi_{k'}$, then there is a unique $\pi \in \Pi_k$ with $\pi' \subset \pi$. We calculate:

$$\begin{aligned} S_k(y) - S_{k'}(y) &= \sum_{\pi \in \Pi_k} f(x_\pi, y) \left| \pi \right| - \sum_{\pi \in \Pi_k} \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} f(x_{\pi'}, y) \left| \pi' \right| \\ &= \sum_{\pi \in \Pi_k} \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} \left(f(x_\pi, y) - f(x_{\pi'}, y) \right) \left| \pi' \right| \end{aligned}$$

here we have used that:

$$|\pi| = \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} |\pi'|$$

Now, since f is uniformly continuous, we know that for any $\epsilon > 0$ there exists a δ, independent of y, such that

$$|f(x, y) - f(x', y)| < \epsilon$$

for all $|x - x'| < \delta$. Notice that for $\pi' \subset \pi$ we have:

$$|x'_\pi - x_\pi| \leq \frac{R}{2^{k+1}}\sqrt{n}.$$

Thus given $\epsilon > 0$, there exists K such that for all $k \geq K$:

$$|f(x_\pi, y) - f(x_{\pi'}, y)| < \frac{\epsilon}{(2R)^n}.$$

Now suppose $k' \geq k \geq K$. We estimate:

$$\begin{aligned} |S_k(y) - S_{k'}(y)| &\leq \sum_{\pi \in \Pi_k} \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} |f(x_\pi, y) - f(x_{\pi'}, y)| \, |\pi'| \\ &\leq \frac{\epsilon}{(2R)^n} \sum_{\pi \in \Pi_k} \sum_{\substack{\pi' \in \Pi_{k'} \\ \pi' \subset \pi}} |\pi'| = \epsilon, \end{aligned}$$

since the sum over the partition simply gives us back the volume of the large cube. Sending k' to infinity, we have the result we require. □

This result allows us to establish the result we require:

Corollary 3.12. *Suppose $u \in \mathscr{S}'$, $\phi \in \mathscr{S}$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$. Then:*

$$u\left[\int_{\mathbb{R}^n} \psi(-x)\tau_x\check{\phi}dx\right] = \int_{\mathbb{R}^n} u\left[\psi(-x)\tau_x\check{\phi}\right]dx$$

Proof. Fix Ω, $R > 0$ such that $\operatorname{supp}\check{\psi} \subset [-R, R)^n \subset \Omega$. Define the map:

$$\begin{aligned} f \quad : \quad \Omega \times \mathbb{R}^n &\to \mathbb{C} \\ (x, y) &\mapsto \psi(-x)\phi(y - x) \end{aligned}$$

Notice that $(1+|y|)^N D_y^\alpha f$ is uniformly continuous on $\Omega \times \mathbb{R}^n$ for any α, N. Thus applying Lemma 3.11 we deduce that:

$$S_k \to \int_{\mathbb{R}^n} \psi(-x)\tau_x\check{\phi}dx, \qquad \text{in } \mathscr{S}.$$

By the continuity of u, we deduce that:

$$u\left[\int_{\mathbb{R}^n} \psi(-x)\tau_x\check{\phi}dx\right] = u\left[\lim_{k\to\infty} S_k\right] = \lim_{k\to\infty} u\left[S_k\right]$$

By the linearity of u, we calculate:

$$u\left[S_k\right] = u\left[\sum_{\pi\in\Pi_k} f(x_\pi, \cdot)\,|\pi|\right] = \sum_{\pi\in\Pi_k} u\left[f(x_\pi,\cdot)\right]|\pi| = \sum_{\pi\in\Pi_k} u\left[\psi(-x_\pi)\tau_{x_\pi}\check{\phi}\right]|\pi|$$

But $x \mapsto u\left[\psi(-x)\tau_x\check{\phi}\right]$ is smooth, hence Riemann integrable, and we have that

$$\lim_{k\to\infty}\sum_{\pi\in\Pi_k} u\left[\psi(-x_\pi)\tau_{x_\pi}\check{\phi}\right]|\pi| = \int_{\mathbb{R}^n} u\left[\psi(-x)\tau_x\check{\phi}\right]dx.$$

□

3.4 The Fourier–Laplace transform on $\mathscr{E}'(\mathbb{R}^n)$

Recall that $\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ is a continuously embedded subspace, consisting of the distributions of *compact support*. These distributions are precisely those which extend to continuous linear maps from $\mathscr{E}(\mathbb{R}^n)$ to $\mathbb{C}$ (see Theorem 2.10). For these distributions, we can express the Fourier transform in a very clean fashion.

Theorem 3.13. *Suppose that $u \in \mathscr{E}'(\mathbb{R}^n)$. Then $\hat{u} = T_{\hat{v}}$ for some $\hat{v} \in C^\infty(\mathbb{R}^n)$ with:*

$$\hat{v}(\xi) = u[e_{-\xi}].$$

Proof. Suppose that $\operatorname{supp} u \subset B_R(0)$. Pick $\psi \in \mathscr{D}(\mathbb{R}^n)$ with $\psi = 1$ on $B_{R+1}(0)$, so that $\psi u = u$. We calculate:

$$\hat{u} = \widehat{\psi u} = \frac{1}{(2\pi)^n} T_{\hat{u}\star\hat{\psi}}.$$

By Theorem 3.10 e). Thus we have $\hat{u} = T_{\hat{v}}$ with $\hat{v} = (2\pi)^{-n}\hat{u}\star\hat{\psi} \in C^\infty(\mathbb{R}^n)$, by Theorem 3.10 a).

Now let $\phi \in \mathscr{S}$ be such that $\hat{\phi} = \psi$. We calculate:

$$\begin{aligned}\hat{v}(\xi) &= \frac{1}{(2\pi)^n}\hat{u}\star\hat{\psi}(\xi) = \hat{u}\star\check{\phi}(\xi)\\ &= \hat{u}\left[\tau_\xi\phi\right] = u\left[\widehat{\tau_\xi\phi}\right] = u[e_{-\xi}\psi] = (\psi u)[e_{-\xi}]\\ &= u[e_{-\xi}].\end{aligned}$$

□

In practice, one does not distinguish between the distribution $\hat{u}$ and the function $\hat{v}$ and one uses the same letter to denote both. Notice that for $u \in \mathscr{E}'(\mathbb{R}^n)$, the expression $u[e_{-z}]$ makes sense for $z \in \mathbb{C}^n$. Moreover, this function is in fact *holomorphic* on $\mathbb{C}^n$. The analytic extension of a Fourier transform from $\mathbb{R}^n$ to $\mathbb{C}^n$ (or a subset thereof) is sometimes called the *Fourier-Laplace* transform.

3.5 Periodic distributions and Poisson's summation formula

Recall that the translate of a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ is defined by:

$$\tau_z u[\phi] = u\left[\tau_{-z}\phi\right], \qquad \text{for all } \phi \in \mathscr{D}(\mathbb{R}^n),$$

Bearing this in mind, we can make a very natural definition of what it means to be periodic:

Definition 3.2. *We say that a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ is* periodic *if for each $g \in \mathbb{Z}^n$ we have:*

$$\tau_g u = u.$$

Example 14. *a) The distribution $u = T_{e_{2\pi g}}$ is periodic for any $g \in \mathbb{Z}^n$. Suppose $g' \in \mathbb{Z}^n$. Then:*

$$\begin{aligned}\tau_{g'} T_{e_{2\pi g}}[\phi] &= T_{e_{2\pi g}}\left[\tau_{-g'}\phi\right] = \int_{\mathbb{R}^n} e^{2\pi i g\cdot y}\phi(y+g')dy \\ &= \int_{\mathbb{R}^n} e^{2\pi i g\cdot(z-g')}\phi(z)dz = e^{-2\pi i g\cdot g'}\int_{\mathbb{R}^n} e^{2\pi i g\cdot z}\phi(z)dz \\ &= T_{e_{2\pi g}}[\phi]\end{aligned}$$

b) Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$. Then

$$u = \sum_{g\in\mathbb{Z}^n} \tau_g v$$

is periodic. It is straightforward to show that u defines a distribution (see Exercise below). To check periodicity, we have, for $g \in \mathbb{Z}^n$:

$$\tau_{g'}u[\phi] = u[\tau_{-g'}\phi] = \sum_{g\in\mathbb{Z}^n} \tau_g v\left[\tau_{-g'}\phi\right] = \sum_{g\in\mathbb{Z}^n} v\left[\tau_{-g-g'}\phi\right] = \sum_{g\in\mathbb{Z}^n} \tau_{g+g'} v\left[\phi\right] = u[\phi],$$

where we shift the dummy variable in the sum for the last step.

Exercise 5.1. Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$ and let:

$$u = \sum_{g\in\mathbb{Z}^n} \tau_g v.$$

Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ with supp $\subset K$ for some compact $K \subset \mathbb{R}^n$ then

$$u[\phi] = \sum_{g\in A} \tau_g v[\phi],$$

for some finite set $A \subset \mathbb{Z}^n$ which depends only on K. Deduce that u defines a distribution.

When dealing with a periodic function $f \in C^\infty(\mathbb{R}^n)$, many quantities of interest are averages over a fundamental cell of the periodic lattice:

$$q = \left\{ x \in \mathbb{R}^n : -\frac{1}{2} \leq x_i < \frac{1}{2}, i = 1, \ldots, n \right\}$$

For example:

$$M(f) = \int_q f(x) dx$$

is the mean value that f attains. We'd like to extend this notion to makes sense for periodic functions, but we're presented with a difficulty. The obvious definition would be to set:

$$M(u) \stackrel{!}{=} u[\mathbb{1}_q]$$

but of course $\mathbb{1}_q \notin \mathscr{D}(\mathbb{R}^n)$ so we're not able to do this. Instead we will 'smear out' the function $\mathbb{1}_q$. To do this, notice that a crucial property of $\mathbb{1}_q$ is the following identity:

$$\sum_{g \in \mathbb{Z}^n} \tau_g \mathbb{1}_q = 1,$$

which tells us that $\mathbb{1}_q$ generates a partition of unity.

We shall construct a smooth 'partition of unity', which will allow us to localise various objects, and thus render them easier to deal with, and will enable us to define the mean of a periodic distribution. This is a slightly technical result, but the basic idea is important and crops up in many areas of analysis.

Lemma 3.14. *Let*

$$Q = \{x \in \mathbb{R}^n : |x_i| < 1, i = 1, \ldots, n\}$$

be the cube of side length 2 *centred at the origin. There exists a function* $\psi \in C^\infty(\mathbb{R}^n)$ *with* $\psi \geq 0$ *and* $\operatorname{supp} \psi \subset Q$ *such that:*

$$\sum_{g \in \mathbb{Z}^n} \tau_g \psi = 1.$$

Suppose that $u \in \mathscr{D}'(\mathbb{R}^n)$ *is periodic, and* ψ, ψ' *are both as above. Then:*

$$u[\psi] = u[\psi']$$

We then define:

$$M(u) := u[\psi]$$

Proof. Note

$$\overline{q} = \left\{ x \in \mathbb{R}^n : |x_i| \leq \frac{1}{2}, i = 1, \ldots, n \right\}.$$

By Lemma 1.14, there exists a function $\psi_0 \in C_0^\infty(Q)$, with $\psi_0(x) = 1$ for $x \in \overline{q}$ and $\psi_0 \geq 0$. Consider:

$$S(x) := \sum_{g \in \mathbb{Z}^n} \psi_0(x - g).$$

For any bounded open set Ω, we have that

$$A = \{g \in \mathbb{Z}^n : (\Omega - g) \cap Q \neq \emptyset\}$$

is finite. For $x \in \Omega$, we have:

$$S(x) = \sum_{g \in A} \psi_0(x - g),$$

so $S(x)$ is smooth. Moreover, for each $x \in \mathbb{R}^n$, there is at least one $g \in \mathbb{Z}^n$ with $x - g \in \overline{q}$. Thus $S(x) \geq 1$. We can thus take:

$$\psi(x) = \frac{\psi_0(x)}{S(x)}.$$

This is smooth, positive, supported in Q and moreover:

$$\sum_{g \in \mathbb{Z}^n} \tau_g \psi(x) = \frac{1}{S(x)} \sum_{g \in \mathbb{Z}^n} \psi_0(x - g) = 1.$$

Now suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic and ψ, ψ' are both partitions of unity as above. We calculate:

$$\begin{aligned} u[\psi] &= u\left[\psi \sum_{g \in \mathbb{Z}^n} \tau_g \psi'\right] = \sum_{g \in \mathbb{Z}^n} u\left[\psi \tau_g \psi'\right] \\ &= \sum_{g \in \mathbb{Z}^n} \tau_{-g} u\left[\tau_{-g} \psi \psi'\right] = u\left[\psi' \sum_{g \in \mathbb{Z}^n} \tau_{-g} \psi\right] = u[\psi'] \end{aligned}$$ □

We thus have an acceptable definition of the mean of a periodic distribution. Notice that if $u = T_f$ for some locally integrable periodic function f, then by choosing a sequence of ψ_j's such that $\psi_j \to \mathbb{1}_{\overline{q}}$ in $L^1(\mathbb{R}^n)$, we can show that:

$$M(T_f) = \int_{\overline{q}} f(x) dx,$$

justifying calling M the mean of the distribution.

To see why this technical lemma is useful, let us apply it to show that a periodic distribution is necessarily tempered, and in fact the periodic distributions we found by translating a compact distribution are indeed all of the periodic distributions.

Lemma 3.15. *Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$ is a compact distribution. Then:*

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v \tag{3.9}$$

converges in $\mathscr{S}'$. Conversely, suppose that $u \in \mathscr{D}'(\mathbb{R}^n)$ is a periodic distribution. Then there exists $v \in \mathscr{E}'(\mathbb{R}^n)$ such that (3.9) holds and thus u extends uniquely to a tempered distribution $u \in \mathscr{S}'$.

Proof. Let $K = \operatorname{supp} v$. Since $v \in \mathscr{E}'(\mathbb{R}^n)$, by Lemma 2.9 there exists $C > 0, N$ such that:

$$|v[\phi]| \leq C \sup_{x \in K; |\alpha| \leq N} |D^\alpha \phi(x)|, \qquad \text{for all } \phi \in \mathscr{E}(\mathbb{R}^n).$$

Now suppose $\phi \in \mathscr{S} \subset \mathscr{E}(\mathbb{R}^n)$. We have:

$$|\tau_g v\,[\phi]| = |v\,[\tau_{-g}\phi]| \leq C \sup_{x \in K; |\alpha| \leq N} |D^\alpha \phi(x+g)|\,.$$

Since K is bounded, we have that $K \subset B_R(0)$ for some $R > 0$. We calculate:

$$1 + |g| = 1 + |x + g - x| \leq 1 + R + |x + g| \leq (1 + R)(1 + |x + g|)$$

for all $x \in K$, so that:

$$1 \leq (1 + R)\frac{1 + |x + g|}{1 + |g|}.$$

We conclude that for any $M \geq 1$:

$$\begin{aligned} |\tau_g v\,[\phi]| &\leq \frac{C(1+R)^M}{(1+|g|)^M} \sup_{x \in K; |\alpha| \leq N} (1 + |x+g|)^M |D^\alpha \phi(x+g)| \\ &\leq \frac{C(1+R)^M}{(1+|g|)^M} \sup_{y \in \mathbb{R}^n; |\alpha| \leq N} (1 + |y|)^M |D^\alpha \phi(y)|\,. \end{aligned}$$

Since $\phi \in \mathscr{S}$, in particular we have:

$$|\tau_g v\,[\phi]| \leq \frac{C'}{(1+|g|)^{n+1}}$$

where C' depends on v, ϕ. Now, since:

$$\sum_{g \in \mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty,$$

(see Exercise below) we deduce that for each $\phi \in \mathscr{S}$ the sum:

$$\sum_{g \in \mathbb{Z}^n} \tau_g v\,[\phi]$$

converges. This is precisely the statement that the sum in (3.9) converges in $\mathscr{S}'$.

Now suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic, and take ψ as in Lemma 3.14. Suppose $\phi \in \mathscr{D}(\mathbb{R}^n)$ is arbitrary. We have:

$$u[\phi] = \left(\sum_{g \in \mathbb{Z}^n} \tau_g \psi\right) u[\phi] = \sum_{g \in \mathbb{Z}^n} u\,[\tau_g \psi \phi]\,. \tag{3.10}$$

Now, since u is periodic,:

$$u\,[\tau_g \psi \phi] = \tau_g u\,[\tau_g \psi \phi] = u\,[\psi \tau_{-g} \phi] = (\psi u)\,[\tau_{-g}\phi] = \tau_g(\psi u)[\phi]$$

Now ψu has compact support, so by Theorem 2.10 extends uniquely to $v \in \mathscr{E}'(\mathbb{R}^n)$. Thus we have:

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v$$

which by the first part of the proof converges in $\mathscr{S}'$, thus $u \in \mathscr{S}'$. □

Exercise 5.2. Recall that for $x \in \mathbb{R}^n$:

$$||x||_1 := \sum_{i=1}^{n} |x_i| \, .$$

For $k \in \mathbb{N}$ set:

$$Q_k = \left\{ g \in \mathbb{Z}^n : k - \frac{1}{2} \leq ||g||_1 < k + \frac{1}{2} \right\}$$

a) Show that:

$$\# Q_k = (2k+1)^n - (2k-1)^n$$

so that $\# Q_k \leq c(1+k)^{n-1}$ for some $c > 0$.

b) By applying the Cauchy-Schwartz identity to estimate $a \cdot b$ for $a = (1, \ldots, 1)$ and $b = (|g_1|, \ldots, |g_n|)$, deduce that:

$$||g||_1 \leq \sqrt{n}\, |g|$$

c) Show that there exists a constant $C > 0$, depending only on n such that:

$$\sum_{g \in \mathbb{Z}^n ; ||g||_1 \leq K} \frac{1}{(1+|g|)^{n+1}} \leq 1 + C \sum_{k=1}^{K} \frac{1}{k^2}$$

holds for all k. Deduce that:

$$\sum_{g \in \mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty.$$

We have shown that a periodic distribution is necessarily tempered. It is therefore reasonable to ask what one can say about the Fourier transform of a periodic distribution. In fact, it will turn out that the Fourier transform of a periodic distribution has a very simple form: it consists of a sum over δ-distributions with support on the points of an integer lattice.

Before we establish this, we will first need a technical result regarding distributions.

Lemma 3.16. *Suppose that $u \in \mathscr{S}$ satisfies:*

$$\left(e_{-g'} - 1 \right) u = 0 \tag{3.11}$$

for all $g' \in \mathbb{Z}^n$. Then:

$$u = \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g},$$

where $c_g \in \mathbb{C}$ *satisfy the bound*

$$|c_g| \leq K(1+|g|)^N$$

for some $K > 0$*,* $N \in \mathbb{Z}$*, and the sum converges in* $\mathscr{S}'$*.*

Proof. First, we claim that $\operatorname{supp} u \subset \Lambda$, with

$$\Lambda = \{2\pi g : g \in \mathbb{Z}^n\}\,.$$

Suppose $\phi \in \mathscr{D}(R^n)$ with $\operatorname{supp}\phi \subset \mathbb{R}^n \backslash \Lambda$. Then for each $g' \in \mathbb{Z}^n$, we have $\left(e_{-g'} - 1\right)^{-1}\phi \in \mathscr{S}$, since ϕ vanishes near any zeros of $e_{-g'} - 1$. Applying the condition (3.11), we deduce:

$$0 = \left(e_{-g'} - 1\right) u \left[\left(e_{-g'} - 1\right)^{-1}\phi\right] = u[\phi]$$

so u vanishes. Thus $\operatorname{supp} u \subset \Lambda$.

Now, let us take ψ as in Lemma 3.14, and define $\tilde{\psi}(x) = \psi\left(\frac{x}{2\pi}\right)$. It's straightforward to check that:

$$\sum_{g\in\mathbb{Z}^n} \tau_{2\pi g}\tilde{\psi} = 1, \qquad \operatorname{supp}\tilde{\psi} \subset \{x \in \mathbb{R}^n : |x_i| < 2\pi\}.$$

For $g \in \mathbb{Z}^n$, let us consider $v_g = (\tau_{2\pi g}\tilde{\psi})u$. This distribution is supported at $2\pi g$, and by multiplying (3.11) by $\tau_{2\pi g}\tilde{\psi}$ we have:

$$\left(e_{-g'} - 1\right) v_g = 0$$

In particular, we have, taking $g' = l_j$ for $j = 1, \ldots n$, where $\{l_j\}$ is the canonical basis for $\mathbb{R}^n$:

$$\left(e^{-i(x_j - 2\pi g_j)} - 1\right) v_g = 0.$$

Now,

$$\left(e^{-i(x_j - 2\pi g_j)} - 1\right) = (x_j - 2\pi g_j)\kappa(x_j)$$

where $\kappa(x_j)$ is non-zero on a neighbourhood of g_j. Thus we conclude that:

$$(x_j - 2\pi g_j)v_g = 0, \qquad j = 1, \ldots n.$$

Now suppose $\phi \in \mathscr{S}$. We can write:

$$\phi(x) = \phi(2\pi g) + \sum_{j=1}^{n}(x_j - 2\pi g_j)\phi_j(x)$$

where $\phi_j(x) \in C^\infty(\mathbb{R}^n)$. Since v_g has compact support, it extends to smoothly to act on $\mathscr{E}(\mathbb{R}^n)$ and we calculate:

$$v_g[\phi] = v_g[\phi(2\pi g)] + \sum_{j=1}^{n}(x_j - 2\pi g_j)v_g[\phi_j] = v_g[\phi(2\pi g)]$$

Returning to the definition of v_g, we have:

$$(\tau_{2\pi g}\tilde{\psi})u[\phi] = (\tau_{2\pi g}\tilde{\psi})u[\phi(2\pi g)] = u[\tau_{2\pi g}\tilde{\psi}]\delta_{2\pi g}[\phi]$$

so that

$$(\tau_{2\pi g}\tilde{\psi})u = u[\tau_{2\pi g}\tilde{\psi}]\delta_{2\pi g}$$

Summing over $g \in \mathbb{Z}^n$, we recover:

$$\sum_{g\in\mathbb{Z}^n}(\tau_{2\pi g}\tilde{\psi})u = \left(\sum_{g\in\mathbb{Z}^n}(\tau_{2\pi g}\tilde{\psi})\right)u = u = \sum_{g\in\mathbb{Z}^n} c_g\delta_{2\pi g}$$

Where

$$c_g = u[\tau_{2\pi g}\tilde{\psi}].$$

To establish the estimate for c_g, we recall from Lemma 2.11 that there exist $N, k \in \mathbb{N}$ and $C > 0$ such that:

$$|u[\phi]| \leq C \sup_{x\in\mathbb{R}^n;|\alpha|\leq k} \left|(1+|x|)^N D^\alpha \phi(x)\right|, \qquad \text{for all } \phi \in \mathscr{S}.$$

Applying this to $\tau_{2\pi g}\tilde{\psi}$, we have:

$$\begin{aligned}
|c_g| &\leq C \sup_{x\in\mathbb{R}^n;|\alpha|\leq k} \left|(1+|x|)^N D^\alpha \tilde{\psi}(x-2\pi g)\right| \\
&\leq C \sup_{x\in\mathbb{R}^n;|\alpha|\leq k} \left|(1+|x+2\pi g|)^N D^\alpha \tilde{\psi}(x)\right| \\
&\leq C' \sup_{x\in\mathbb{R}^n;|\alpha|\leq k} \left|(1+|x|)^N D^\alpha \tilde{\psi}(x)\right| \times (1+|g|)^N \\
&\leq K(1+|g|)^N
\end{aligned}$$

With this bound, it is a straightforward exercise to verify that the sum converges in $\mathscr{S}'$. □

Exercise 5.3. Show that if c_g satisfy:

$$|c_g| \leq K(1+|g|)^N$$

for some $K > 0$ and $N \in \mathbb{N}$, then:

$$\sum_{g\in\mathbb{Z}^n} c_g\delta_{2\pi g}$$

converges in $\mathscr{S}'$.

This now enables us to establish a result regarding the Fourier series of a periodic distribution.

Theorem 3.17. *Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is a periodic distribution. Then there exist constants $c_g \in \mathbb{C}$ such that:*

$$u = \sum_{g \in \mathbb{Z}^n} c_g T_{e_{2\pi g}}.$$

with c_g are given by:

$$c_g = M(e_{-2\pi g} u).$$

and satisfy the bound:

$$|c_g| \leq K(1 + |g|)^N \tag{3.12}$$

for some $K > 0$, $N \in \mathbb{Z}$.

Proof. Since u is periodic, it is tempered by Lemma 3.15. Thus we may take the Fourier transform. Noting that:

$$\tau_{g'} u = u$$

for all $g' \in \mathbb{Z}^n$, we have that

$$e_{-g'} \hat{u} = \hat{u} \qquad \Longrightarrow \qquad (e_{-g'} - 1)\hat{u} = 0.$$

By Lemma 3.16, we deduce that:

$$\hat{u} = (2\pi)^n \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g},$$

for some c_g satisfying (3.12), where the sum converges in $\mathscr{S}'$. We can apply the inverse Fourier transform, making use of the fact that it is continuous on $\mathscr{S}'$ to deduce:

$$u = \sum_{g \in \mathbb{Z}^n} c_g T_{e_{2\pi g}},$$

with convergence again in $\mathscr{S}'$. To establish the formula for c_g, we make use of the comments after Lemma 3.14 to note that:

$$M(e_{-2\pi g} T_{e_{2\pi g'}}) = \int_{\overline{q}} e^{2\pi i (g - g') \cdot x} dx = \delta_{gg'}$$

Since $u \mapsto M(e_{-2\pi g} u)$ is a continuous map from $\mathscr{S}'$ to $\mathbb{C}$, we deduce that:

$$M(e_{-2\pi g} u) = \sum_{g' \in \mathbb{Z}^n} c_g M(e_{-2\pi g} T_{e_{2\pi g'}}) = c_{g'}.$$

□

Remark. *Usually one writes the Fourier series for u as:*

$$u = \sum_{g \in \mathbb{Z}^n} c_g e_{2\pi g},$$

ignoring the distinction between the function $e_{2\pi g}$ and the distribution it defines.

As a simple example, let us consider the distribution:

$$u = \sum_{g\in\mathbb{Z}^n} \delta_g.$$

By Lemma 3.15, this defines a periodic distribution, since $\delta_g = \tau_g\delta_0$ and $\delta_0 \in \mathscr{E}'(\mathbb{R}^n)$. Notice also that if ψ satisfies the conditions of Lemma 3.14, then since $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x_j| < 1\}$, we have that $\tau_g\psi(0) = 0$ for $g \in \mathbb{Z}^n$ with $g \neq 0$. Thus, since $\sum_{g\in\mathbb{Z}^n} \tau_g\psi = 1$, we must have $\psi(0) = 1$. We can then calculate:

$$c_g = M(e_{-2\pi g}u) = u[\psi e_{-2\pi g}] = \psi(0)e^{-2\pi i g\cdot 0} = 1.$$

Thus we have established *Poisson's formula*:

$$\sum_{g\in\mathbb{Z}^n} \delta_g = \sum_{g\in\mathbb{Z}^n} T_{e_{2\pi g}},$$

where we understand both sums to converge in $\mathscr{S}'$. This is sometimes written, with an abuse of notation:

$$\sum_{g\in\mathbb{Z}^n} \delta(x-g) = \sum_{g\in\mathbb{Z}^n} e^{2\pi g\cdot xi}$$

You may want to refer back to (7), which we heuristically justified in the introduction to the course.

We can specialise various results concerning Fourier transforms to the case of Fourier series.

Corollary 3.18. *i) Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic and may be written as:*

$$u = \sum_{g\in\mathbb{Z}^n} c_g T_{e_{2\pi g}}.$$

Then $D_j u \in \mathscr{D}'(R^n)$ is periodic and has Fourier series:

$$D_j u = \sum_{g\in\mathbb{Z}^n} (2\pi i g_j c_g) T_{e_{2\pi g}}.$$

ii) Suppose $f \in L^1_{loc.}(\mathbb{R}^n)$, then:

$$|c_g| \leq ||f||_{L^1(q)},$$

and moreover, $c_g \to 0$ as $|g| \to \infty$.

iii) Suppose $f \in C^{n+1}(\mathbb{R}^n)$ is periodic. Then:

$$f(x) = \sum_{g\in\mathbb{Z}^n} c_g e^{2\pi i g\cdot x}$$

with the sum converging uniformly.

iv) *Suppose $f, h \in L^2_{loc.}(\mathbb{R}^n)$ are periodic with Fourier coefficients f_g, h_g respectively. Then:*

$$\int_q \overline{f}(x)h(x)dx = \sum_{g\in\mathbb{Z}^n} \overline{f}_g h_g.$$

This is the Fourier series version of Parseval's formula. Moreover,

$$f(x) = \sum_{g\in\mathbb{Z}^n} c_g e^{2\pi i g\cdot x}$$

holds, with the sum converging in $L^2(q)$.

Proof. i) Since the Fourier series for u converges in $\mathscr{S}'$, we may differentiate term by term (as differentiation is a continuous operation from $\mathscr{S}'$ to itself). Since

$$D_j T_{e_{2\pi g}} = (2\pi i g_j) T_{e_{2\pi g}},$$

the result follows.

ii) Note that if $f \in L^1_{loc.}(\mathbb{R}^n)$, then:

$$|c_g| = \left| \int_q e^{-2\pi i g\cdot x} f(x)dx \right| \leq \int_q |f(x)|\, dx = ||f||_{L^1(q)}\,.$$

Now, given $\epsilon > 0$, we can approximate[5] f by a smooth periodic function f_ϵ, with Fourier coefficients c'_g, such that

$$||f - f_\epsilon||_{L^1(q)} < \frac{\epsilon}{2}.$$

Since $D_j D_j f_\epsilon \in L^1_{loc.}(\mathbb{R}^n)$, we have that $|g|^2 \left|c'_g\right| < C$, for each $j = 1, \dots, n$ so there exists $R > 0$ such that $\left|c'_g\right| < \frac{\epsilon}{2}$ for $|g| > R$. We have:

$$\left|c_g - c'_g\right| \leq ||f - f_\epsilon||_{L^1(q)} < \frac{\epsilon}{2},$$

so we conclude that for $|g| > R$:

$$|c_g| = \left|c_g - c'_g + c'_g\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $c_g \to 0$ as $|g| \to \infty$.

iii) Since $f \in C^{k+1}(\mathbb{R}^n)$, we have that $D^\alpha f \in L^1_{loc.}(\mathbb{R}^n)$ for $|\alpha| < n+1$. Applying the previous two results we conclude that $|c_g| \leq K(1 + |g|)^{-n+1}$ for some $K > 0$. Thus the partial sums:

$$F_n(x) = \sum_{g\in\mathbb{Z}^n, |g|\leq n} c_g e^{2\pi i g\cdot x}$$

[5] See Exercise 5.4

converge uniformly to some continuous function F by the Weierstrass $M-$test. We have:

$$T_f = \lim_{n\to\infty} \sum_{g\in\mathbb{Z}^n, |g|\le n} c_g T_{e_{2\pi g}} = \lim_{n\to\infty} T_{F_n} = T_F$$

since uniform convergence implies convergence in $\mathscr{S}'$. By the injectivity of the mapping between continuous functions and distributions we conclude $f = F$.

iv) Suppose $f, h \in C^\infty(\mathbb{R}^n)$ are periodic. Then:

$$f(x) = \sum_{g\in\mathbb{Z}^n} f_g e^{2\pi i g\cdot x}, \qquad h(x) = \sum_{g\in\mathbb{Z}^n} h_g e^{2\pi i g\cdot x}$$

with $\sup_{g\in\mathbb{Z}^n}(1+|g|)^N |f_g| < \infty$ for all $N \in \mathbb{N}$, and similarly for h_g. We calculate:

$$\begin{aligned}\int_q \overline{f}(x)g(x)dx &= \int_q \left(\sum_{g\in\mathbb{Z}^n} \overline{f}_g e^{-2\pi i g\cdot x}\right)\left(\sum_{g'\in\mathbb{Z}^n} h_{g'} e^{2\pi i g'\cdot x}\right) dx \\ &= \sum_{g\in\mathbb{Z}^n}\sum_{g'\in\mathbb{Z}^n} \overline{f}_g h_{g'} \int_q e^{2\pi i (g'-g)\cdot x} dx \\ &= \sum_{g\in\mathbb{Z}^n}\sum_{g'\in\mathbb{Z}^n} \overline{f}_g h_{g'} \delta_{gg'} = \sum_{g\in\mathbb{Z}^n} \overline{f}_g h_g.\end{aligned}$$

In particular, we have that:

$$||f||_{L^2(q)} = ||f_g||_{\ell^2(\mathbb{Z}^n)},$$

where for a sequence $\{a_g\}_{g\in\mathbb{Z}^n}$, we define:

$$||a_g||_{\ell^2(\mathbb{Z}^n)} = \left(\sum_{g\in\mathbb{Z}^n} |a_g|^2\right)^{\frac{1}{2}}.$$

Now suppose $f \in L^2_{loc.}(\mathbb{R}^n)$. Given $k > 0$, we can find $f^{(k)}$ with Fourier coefficients $f^{(k)}_g$such that:

$$\left|\left|f - f^{(k)}\right|\right|_{L^2(q)} < \frac{1}{k}.$$

Since by Cauchy-Schwarz we have:

$$||f||_{L^1(q)} = \int_q |f(x)|\, dx \le \left(\int_q |f(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_q dx\right)^{\frac{1}{2}} = ||f||_{L^2(q)}$$

we have that:

$$\left|\left|f_g - f^{(k)}_g\right|\right|_{\ell^\infty(\mathbb{Z}^n)} := \sup_{g\in\mathbb{Z}^n} \left|f_g - f^{(k)}_g\right| < \frac{1}{k},$$

Now, $f^{(k)}$ is a Cauchy sequence in $L^2(q)$, so $\{f^{(k)}_g\}$ is a Cauchy sequence in $\ell^2(\mathbb{Z}^n)$. We conclude that $f^{(k)}_g$ converges in $\ell^2(\mathbb{Z}^n)$, however we also know that $f^{(k)} \to f$

in $\ell^\infty(\mathbb{Z}^n)$, thus we must have $f^{(k)} \to f$ in $\ell^2(\mathbb{Z}^n)$. Taking a similar sequence of $h^{(k)} \in C^\infty(\mathbb{R}^n)$ approximating h with Fourier coefficients $h_g^{(k)}$, and recalling that:

$$\left(f^{(k)}, h^{(k)}\right)_{L^2(q)} = \left(f^{(k)}, h^{(k)}\right)_{\ell^2(\mathbb{Z}^n)},$$

the result follows on sending $k \to \infty$. The convergence of the Fourier series in $L^2(q)$ follows by showing that the partial sums form a Cauchy sequence in $L^2(q)$.

□

Exercise 5.4. Suppose $f \in L^p_{loc.}(\mathbb{R}^n)$ is a periodic function. Fix $\epsilon > 0$, and let:

$$Q = \{x \in \mathbb{R}^n : |x_j| < 1, j = 1, \dots, n\}, \qquad q = \left\{x \in \mathbb{R}^n : |x_j| < \frac{1}{2}, j = 1, \dots, n\right\}$$

a) Show that there exists $h_\epsilon \in C^\infty(\mathbb{R}^n)$ with:

$$\operatorname{supp} h_\epsilon \subset Q$$

such that:

$$||f \mathbb{1}_q - h_\epsilon||_{L^p(\mathbb{R}^n)} < \epsilon.$$

Define

$$f_\epsilon = \sum_{g \in \mathbb{Z}^n} \tau_g h_\epsilon$$

b) Show that f_ϵ is smooth and periodic.

c) Show that there exists a constant c_n depending only on n such that:

$$||f - f_\epsilon||_{L^p(q)} < c_n \epsilon.$$

Exercise 5.5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = x \quad \text{for } |x| < \frac{1}{2}, \qquad f(x+1) = f(x).$$

Show that:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{2\pi n} e^{-2\pi i n x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n x),$$

with convergence in $L^2_{loc.}(\mathbb{R})$.

Exercise 5.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = \begin{cases} -1 & -\frac{1}{2} < x \leq 0 \\ 1 & 0 < x \leq \frac{1}{2} \end{cases}, \qquad f(x+1) = f(x).$$

a) Show that:

$$f(x) = \frac{1}{\pi i} \sum_{n=-\infty}^{\infty} \frac{2}{2n+1} e^{-2\pi i(2n+1)x} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[2\pi(2n+1)x\right]$$

With convergence in $L^2_{loc.}(\mathbb{R}^n)$.

Define the partial sum:

$$S_N(x) = 8 \sum_{n=0}^{N-1} \frac{1}{2\pi(2n+1)} \sin\left[2\pi(2n+1)x\right].$$

b) Show that:

$$S_N(x) = 8 \int_0^x \sum_{n=0}^{N-1} \cos\left[2\pi(2n+1)t\right] dt.$$

c) Show that:

$$\cos\left[2\pi(2n+1)t\right] \sin 2\pi t = \frac{1}{2} \left(\sin\left[2\pi(2n+2)t\right] - \sin\left[4\pi n t\right]\right)$$

And deduce:

$$S_N(x) = 8 \int_0^x \frac{\sin 4\pi N t}{2 \sin 2\pi t} dt.$$

d) Show that the first local maximum of S_N occurs at $x = \frac{1}{4N}$, and:

$$S_N\left(\frac{1}{4N}\right) \geq 8 \int_0^{\frac{1}{4N}} \frac{\sin 4\pi N t}{4\pi t} dt = \frac{2}{\pi} \int_0^{\pi} \frac{\sin s}{s} ds \simeq 1.179\ldots$$

e) Conclude that the sum in part $a)$ does not converge uniformly.

This lack of uniform convergence of a Fourier series at a point of discontinuity is known as Gibbs Phenomenon.

Exercise 5.7. (*) Suppose that $\lambda = \{\lambda_1, \ldots \lambda_n\}$ is a basis for $\mathbb{R}^n$. We define the lattice generated by λ to be:

$$\Lambda = \left\{ \sum_{j=1}^{n} z_j \lambda_j : z_j \in \mathbb{Z} \right\}.$$

Define the fundamental cell:

$$q_\Lambda = \left\{ \sum_{j=1}^{n} x_j \lambda_j : |x_j| < \frac{1}{2} \right\}.$$

We say that $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic if:

$$\tau_g u = u \qquad \text{for all } g \in \Lambda.$$

a) Show that there exists $\psi \in C_0^\infty(2q_\Lambda)$ such that $\psi \geq 0$ and

$$\sum_{g \in \Lambda} \tau_g \psi = 1.$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic and ψ, ψ' are both as in part a), then

$$\frac{1}{|q_\Lambda|} u[\psi] = \frac{1}{|q_\Lambda|} u[\psi'] =: M(u)$$

c) Define the *dual lattice* by:

$$\Lambda^* := \{x \in \mathbb{R}^n : g \cdot x \in 2\pi\mathbb{Z}, \ \forall g \in \Lambda\}$$

Show that there exists a basis $\lambda^* = \{\lambda_1^*, \ldots \lambda_n^*\}$ such that $\lambda_j^* \cdot \lambda_k = \delta_{jk}$, and Λ^* is the lattice induced by λ^*.

d) Show that if $g \in \Lambda^*$ then e_g is $\Lambda-$periodic.

e) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$\hat{u} = \sum_{g \in \Lambda^*} c_g \delta_g$$

for some $c_g \in \mathbb{C}$ satisfying $|c_g| \leq K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$.

f) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$u = \sum_{g \in \Lambda^*} d_g T_{e_g}$$

where $|d_g| \leq K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$ are given by:

$$d_g = M(e_{-g} u)$$