Chapter 2

Distributions

The theory of distributions (sometimes called generalised functions) gives us a way of performing a similar trick for PDEs to that we do for polynomials when we move from the reals to the complex numbers. Let's recall the general linear PDE of order k,:

$$Lu := \sum_{|\alpha|\leq k} a_\alpha D^\alpha u = f, \tag{2.1}$$

where we'll assume that $a_\alpha \in C^\infty(\Omega)$ for an open $\Omega \subset \mathbb{R}^n$. We want to extend the notion of a solution for this PDE to include the situation where u and f need not be of class $C^k(\Omega)$. Let's denote by[1] $\mathscr{D}'(\Omega)$ the space to which our generalised solution u and right hand side f should belong. What properties do we require for $\mathscr{D}'(\Omega)$ so that we can at least make sense of the PDE (2.1)? We can start to make a list of desirable properties:

i) The smooth functions $C^\infty(\Omega)$ should be included in $\mathscr{D}'(\Omega)$ in such a way that we can recover them (i.e. the inclusion map $\iota : C^\infty(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$ should be injective).

ii) $\mathscr{D}'(\Omega)$ should be a vector space over $\mathbb{C}$, and the vector space operations should be compatible with the inclusion ι.

iii) We need to be able to multiply elements of $\mathscr{D}'(\Omega)$ by elements of $C^\infty(\Omega)$.

iv) We need to be able to differentiate elements of $\mathscr{D}'(\Omega)$. Moreover whatever definition for 'differentiation' we come up with, it should be compatible with the inclusion ι.

v) We need some topology on $\mathscr{D}'(\Omega)$ such that the above operations are continuous.

The final point is somewhat technical, but it turns out we'll not get very far without some kind of topology for $\mathscr{D}'(\Omega)$, and given that we have one it should be in some way 'compatible' with the structure of $\mathscr{D}'(\Omega)$.

The idea that we shall pursue is to consider the space of distributions to be the dual space to that space of test functions. For any topological vector space V over $\mathbb{C}$ one can define in a natural way the *continuous dual space* which consists of continuous linear maps

$$V' = \{w : V \to \mathbb{C} \mid w \text{ linear, continuous}\}$$

[1] This is a bit of a cheat. You'll see shortly why we use this notation.

We're going to take as our space $\mathscr{D}'(\Omega)$ the continuous dual space of $\mathscr{D}(\Omega)$:

Definition 2.1. *A distribution $u \in \mathscr{D}'(\Omega)$ is a linear functional on the space of test functions*

$$u : \mathscr{D}(\Omega) \to \mathbb{C}$$

which is continuous with respect to the topology of $\mathscr{D}(\Omega)$.

We will state (but not prove) a criterion for continuity. Those interested in the proof of this result will find it in the Appendix.

Theorem 2.1. *Let $u : \mathscr{D}(\Omega) \to \mathbb{C}$ be a linear map. The following are equivalent:*

i) u is continuous with respect to the topology of $\mathscr{D}(\Omega)$.

ii) For each sequence $\{\phi_j\}_{j=1}^{\infty} \subset \mathscr{D}(\Omega)$ with $\phi_j \to 0$ in $\mathscr{D}(\Omega)$, we have:

$$\lim_{j\to\infty} u\left[\phi_j\right] = 0.$$

iii) For each compact $K \subset \Omega$, there exists $N \in \mathbb{N}$ and a constant C such that:

$$|u[\phi]| \leq C \sup_{x\in K} \sum_{|\alpha|\leq N} \left|D^\alpha \phi(x)\right|, \qquad \text{for all } \phi \in C_0^\infty(K). \tag{2.2}$$

Remark. *1. By the linearity of u, condition ii) is equivalent to:*

$$\lim_{j\to\infty} u\left[\phi_j\right] = u\left[\phi\right], \qquad \text{for all } \phi_j \to \phi \text{ in } \mathscr{D}(\Omega).$$

2. If there exists a single $N \in \mathbb{N}$ such that (2.2) holds for all compact $K \subset \Omega$ (possibly with C depending on K), then we say that u has finite order*. The least such N is called the* order *of u.*

For a general topological space (as opposed to a metric space) in order to establish that a function is continuous, it is necessary to consider open sets and their pull-backs etc. It is not usually enough to simply check continuity for sequences. The reason that we can get away with it in this case is somewhat complicated, but boils down to the fact that although the topology of $\mathscr{D}(\Omega)$ does not arise as a metric topology, it is in some sense the limit of a sequence of spaces which are metric.

We can think of a distribution as an operation which swallows a smooth function of compact support and produces a real number. Let's look at two important examples:

Example 7. *a) (The Dirac delta) For $x \in \Omega$ we define the distribution:*

$$\delta_x \phi := \phi(x) \qquad \forall\ \phi \in \mathscr{D}(\Omega).$$

b) If $f \in L^1_{loc.}$, we can define the distribution T_f by:

$$T_f \phi := \int_\Omega f(x)\phi(x)dx, \qquad \forall\ \phi \in \mathscr{D}(\Omega).$$

c) For $\phi \in \mathscr{D}(\mathbb{R})$, we define:

$$P.V.\left(\frac{1}{x}\right)[\phi] = \lim_{\epsilon \to 0}\left[\int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x}dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x}dx\right].$$

This is clearly linear (assuming the limit exists). By a change of variables, we can re-write:

$$P.V.\left(\frac{1}{x}\right)[\phi] = \lim_{\epsilon \to 0}\int_{\epsilon}^{\infty} \frac{\phi(x) - \phi(-x)}{x}dx$$

Note that

$$\frac{\phi(x) - \phi(-x)}{x} = \int_{-1}^{1} \phi'(xt)dt$$

so that:

$$\left|\frac{\phi(x) - \phi(-x)}{x}\right| = \left|\int_{-1}^{1} \phi'(xt)dt\right| \leq 2\sup_{\mathbb{R}^n}\left|\phi'\right|.$$

From this, we conclude that the limit $\epsilon \to 0$ above is well defined, and moreover, if $\phi \in C_0^\infty(B_R(0))$ then:

$$\left|P.V.\left(\frac{1}{x}\right)[\phi]\right| \leq 4R\sup_{\mathbb{R}^n}\left|\phi'\right|.$$

We conclude that $P.V.\left(\frac{1}{x}\right)$ defines a distribution of order at most one.

Exercise 2.6. a) Show that δ_x, as defined in Example 7 is continuous and linear, hence a distribution. Find the order.

b) Show that T_f, as defined in Example 7 is continuous and linear, hence a distribution. Find the order.

c) By constructing a suitable sequence of smooth functions show that the order of $P.V.\left(\frac{1}{x}\right)$ is one.

2.1 Functions as distributions

Let's have a look at how we're doing with our 'wish list' of properties. For Property *i)* we will take inspiration from part *a)* of the example above and define

$$\begin{array}{rcl} \iota \ : \ C^\infty(\Omega) & \to & \mathscr{D}'(\Omega), \\ f & \mapsto & T_f. \end{array}$$

In order to check that this is injective, we need to show that if $T_f = T_g$ then $f = g$. We shall prove this by showing that for any distribution of the form T_f, it is possible to recover f by applying T_f to appropriately shifted and scaled bump functions. Recall that in the proof of Theorem 1.10, we introduced the bump functions ϕ_ϵ. The idea will be to make use of our previous results about convolutions. We require a bit of notation first. Recall that if $\phi : \mathbb{R}^n \to \mathbb{C}$, and $x \in \mathbb{R}^n$ we set $\tau_x\phi(y) = \phi(y - x)$. We introduce the spatial inversion operator $\check{\cdot}$ defined by $\check{\phi}(y) = \phi(-y)$. By convention, the translation operator acts first, so that $\tau_x\check{\phi}(y) = \phi(x - y)$.

Theorem 2.2. *Suppose $f \in C^k(\Omega)$, and let ϕ_ϵ be as in Theorem 1.13. Define for any x with $d(x, \partial\Omega) \geq \epsilon$:*

$$f_\epsilon(x) := T_f[\tau_x \check{\phi}_\epsilon].$$

Then for any compact subset $K \subset \Omega$ and any $|\alpha| \leq k$ we have:

$$\sup_{x \in K} |D^\alpha f_\epsilon(x) - D^\alpha f(x)| \to 0$$

as $\epsilon \to 0$.

Proof. Fix $K \subset \Omega$ compact. Recall that by Lemma 1.14, there exists $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ on $K + B_\delta(0)$ for some $\delta > 0$. Take $\epsilon < \delta$. Then, since $\operatorname{supp} \phi_\epsilon \subset B_\epsilon(0)$ we have for $x \in K$:

$$\begin{aligned} f_\epsilon(x) &= \int_\Omega f(y) \tau_x \check{\phi}_\epsilon(y) dy \\ &= \int_\Omega f(y) \phi_\epsilon(x - y) dy \\ &= \int_{\mathbb{R}^n} \chi(y) f(y) \phi_\epsilon(x - y) dy \\ &= \phi_\epsilon \star (\chi f). \end{aligned}$$

Here we have used the fact that when $|x - y| < \epsilon$ we have $\chi = 1$ to insert the cut-off function without altering the integral. Now, since χ is smooth, $\chi f \in C_0^k(\mathbb{R}^n)$, so as $\epsilon \to 0$, we have by Theorem 1.13 that for any $|\alpha| \leq k$:

$$D^\alpha(\phi_\epsilon \star (\chi f)) \to D^\alpha(\chi f)$$

uniformly on $\mathbb{R}^n$ as $\epsilon \to 0$. In particular we have uniform convergence on K, so that:

$$\sup_{x \in K} |D^\alpha(\phi_\epsilon \star (\chi f))(x) - D^\alpha(\chi f)(x)| \to 0.$$

Since for $x \in K$ we have $\phi_\epsilon \star (\chi f)(x) = f_\epsilon(x)$ and $\chi(x) = 1$, this is the result we require. □

This result immediately tells us that our map ι is injective.

Corollary 2.3. *Suppose $f, g \in C^0(\Omega)$. If $T_f = T_g$ then $f \equiv g$. In particular this implies $\iota : C^\infty(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$ is injective.*

> **Exercise 2.7.** Suppose $f \in L^1_{loc.}(\Omega)$. Take ϕ_ϵ as in Theorem 1.13, and define for x with $d(x, \partial\Omega) \geq \epsilon$:
>
> $$f_\epsilon(x) := T_f\left[\tau_x \check{\phi}_\epsilon\right].$$
>
> Show that for any compact $K \subset \Omega$:
>
> $$||f_\epsilon - f||_{L^1(K)} \to 0$$
>
> as $\epsilon \to 0$.
> *[Hint: follow the proof of Theorem 2.2, but use part b) of Theorem 1.13]*

2.2 Derivatives of distributions

Things are looking good for Property *ii*) because the dual space to a vector space is naturally a vector space. If $u_1, u_2 \in \mathscr{D}'(\Omega)$ we define the sum $u_1 + u_2 \in \mathscr{D}'(\Omega)$ by:

$$(u_1 + u_2)\phi = u_1\phi + u_2\phi \qquad \forall\ \phi \in \mathscr{D}(\Omega).$$

It's easy to check that this is linear and continuous from the properties of u_i.

How about Property *iii*)? First, let's notice that if $a \in C^\infty(\Omega)$ and $\phi \in C_0^\infty(\Omega)$ then $a\phi \in C_0^\infty(\Omega)$. We can therefore define for $u \in \mathscr{D}'(\Omega)$ the product $au \in \mathscr{D}'(\Omega)$ by

$$(au)\phi = u[a\phi] \qquad \forall\ \phi \in \mathscr{D}(\Omega).$$

Now let's consider Property *iv*). We want to find a definition for the derivative of a distribution which gives the right answer when the distribution in question arises from a smooth function f by the map $\iota : f \mapsto T_f$. If $f \in C^\infty(\Omega)$, then certainly $D_i f \in C^\infty(\Omega)$. Let's consider $T_{D_i f}$. We have:

$$T_{D_i f}\phi = \int_\Omega [D_i f](x)\phi(x)dx$$

Since $\phi \in C_0^\infty(\Omega)$, we can integrate by parts in this integral without picking up any boundary terms to find

$$\begin{aligned} T_{D_i f}\phi &= -\int_\Omega f(x)[D_i\phi](x)dx \\ &= -T_f\left[D_i\phi\right]. \end{aligned}$$

Motivated by this, we *define* the D^α derivative of a distribution u to be the distribution $D^\alpha u$ which acts on a test function $\phi \in C_0^\infty(\Omega)$ as:

$$(D^\alpha u)[\phi] := (-1)^{|\alpha|} u\left[D^\alpha \phi\right].$$

The derivative we have defined on distributions *extends* the usual derivative for functions defined as the linear approximation to the function at a point. The advantage of the distributional derivative is that it is defined for any distribution. We'll work out a couple of examples:

Example 8. *a) The derivative of the Dirac delta acts on a test function $\phi \in C_0^\infty(\Omega)$ by:*

$$\left(D_i\delta_x\right)\phi = -\delta_x\left[D_i\phi\right] = -\frac{\partial\phi}{\partial x_i}(x), \qquad \forall\ \phi \in C_0^\infty(\mathbb{R}^n).$$

b) Consider the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(x) := \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

This function is certainly not *differentiable at* $x = 0$. *We can define the distribution* $T_H \in \mathscr{D}'(\mathbb{R})$ *to be*

$$T_H \phi = \int_{\mathbb{R}} H(x)\phi(x)dx$$

for $\phi \in C_0^\infty(\mathbb{R})$. *We then compute:*

$$\begin{aligned} (D_x T_H)\,\phi &= -T_H\,[D_x\phi] \\ &= -\int_{\mathbb{R}} H(x)\phi'(x)dx \\ &= -\int_0^\infty \phi'(x)dx \\ &= \phi(0) \end{aligned}$$

Here, we've used the fact that ϕ *has compact support as well as the fact that* H *vanishes for* $x < 0$. *Thus, we can say that*

$$D_x T_H = \delta_0.$$

Thus we see that the theory of distributions allows us to give some sort of meaning to the derivative of a functions whose classical derivative does not exist.

Exercise 2.8. a) Show that if $f_1, f_2 \in C^0(\Omega)$ and $a \in C^\infty(\Omega)$, then

$$aT_{f_1} + T_{f_2} = T_{af_1+f_2}$$

b) Show that if $f \in C^k(\Omega)$ then

$$D^\alpha T_f = T_{D^\alpha f}$$

for $|\alpha| \leq k$. Deduce that $\iota \circ D^\alpha = D^\alpha \circ \iota$.

c) Deduce that if $f \in C^k(\Omega)$ then

$$\sum_{|\alpha|\leq k} a_\alpha D^\alpha T_f = T_{Lf}.$$

where

$$Lf = \sum_{|\alpha|\leq k} a_\alpha D^\alpha f$$

A useful result allows us to infer regularity of a function from the regularity of its distributional derivatives:

Theorem 2.4. *Suppose that* $f \in C^0(\Omega)$ *defines the distribution* T_f *in the usual way and suppose moreover that for any multi-index* α *with* $|\alpha| \leq k$ *there exists* $g^\alpha \in C^0(\Omega)$ *such that*

$$D^\alpha T_f = T_{g^\alpha}$$

where D^α *is the distributional derivative. Then in fact* $f \in C^k(\Omega)$ *and* $D^\alpha f = g^\alpha$ *in the sense of classical derivatives.*

Proof. First we show the result for $k = 1$. Let us fix a compact subset K of Ω and let ϕ_ϵ be as in Theorem 1.13. Let us define for $x \in K$ and ϵ sufficiently small:

$$f_\epsilon(x) = T_f[\tau_x \check{\phi}_\epsilon] = \int_\Omega f(y)\phi_\epsilon(x-y)dy.$$

We know from Theorem 2.2 that $f_\epsilon \to f$ uniformly on K as $\epsilon \to 0$. Let us calculate

$$\begin{aligned} D_i f_\epsilon(x) &= \int_\Omega f(y)\frac{\partial}{\partial x_i}\phi_\epsilon(x-y)dy \\ &= -\int_\Omega f(y)\frac{\partial}{\partial y_i}\phi_\epsilon(x-y)dy \\ &= (D_i T_f)\,[\tau_x \check{\phi}_\epsilon] \\ &= (T_{g_i})\,[\tau_x \check{\phi}_\epsilon] \end{aligned}$$

Now, as $\epsilon \to 0$, we know that $(T_{g_i})\,[\tau_x \check{\phi}_\epsilon] \to g_i(x)$ uniformly on K. Thus we have

$$f_\epsilon \to f, \quad D_i f_\epsilon \to g_i$$

uniformly on K as $\epsilon \to 0$. This (by a result from analysis) implies that $f \in C^1(K)$, $D_i f = g_i$. Since this holds for any compact set we have that $f \in C^1(\Omega)$ with $D_i f = g_i$. By repeated application of the $k = 1$ result, we can establish that the result holds for all k. □

This tells us that the distributional derivative is essentially equivalent to the classical derivative wherever both are defined and continuous. One should be careful, however. There are examples of continuous functions whose derivative vanishes almost everywhere, but whose distributional derivative is not the zero distribution. Such strange beasts go by the name of 'Devil's Staircases'.

2.3 Convergence of distributions

Property v) of our 'wish list' is the only one that we've not yet addressed directly. We wish to give $\mathscr{D}'(\Omega)$ a topology. In fact, as the continuous dual of a topological vector space, it carries a natural topology. Again, we shan't discuss the full details of the topology, but rather describe the convergent subsequences:

Theorem 2.5. *The vector space $\mathscr{D}'(\Omega)$ inherits a topology from $\mathscr{D}(\Omega)$, called the weak-$\star$ topology (pronounced "weak star topology"), such that*

i) The vector space operations on $\mathscr{D}'(\Omega)$ are continuous.

ii) A sequence $\{u_j\}_{j=1}^\infty \subset \mathscr{D}'(\Omega)$ converges to zero in $\mathscr{D}'(\Omega)$ if

$$u_j[\phi] \to 0, \qquad \text{for all } \phi \in \mathscr{D}(\Omega).$$

as $j \to \infty$. Similarly $u_j \to u$ for $u \in \mathscr{D}'(\Omega)$ if $u_j - u \to 0$.

Example 9. *a) Suppose* $\{f_j\}_{j=1}^{\infty} \subset C^0(\Omega)$ *is a sequence of functions such that* $f_j \to 0$ *uniformly on any compact* $K \subset \Omega$*. Then* $T_{f_j} \to 0$ *in* $\mathscr{D}'(\Omega)$*. To see this, note that for any* $\phi \in \mathscr{D}(\Omega)$*, there exists a* K *such that* $\operatorname{supp}\phi \subset K$*. Then*

$$\left|T_{f_j}[\phi]\right| = \left|\int_K f_j(y)\phi(y)dy\right| \leq |K| \sup_K |f_j\phi|$$

but the right hand side is tending to zero, so $T_{f_j}[\phi] \to 0$ *for any* $\phi \in \mathscr{D}(\Omega)$*.*

b) A similar argument shows that if $\{f_j\}_{j=1}^{\infty} \subset L^1_{loc.}(\Omega)$ *is a sequence such that* $||f_j||_{L^1(K)} \to 0$ *for any compact* $K \subset \Omega$*, then* $T_{f_j} \to 0$ *in* $\mathscr{D}'(\Omega)$*.*

c) Let $\Omega = \mathbb{R}$*. Define a distribution as follows. For* $\phi \in \mathscr{D}(\Omega)$*, set:*

$$u[\phi] = \sum_{m=-\infty}^{\infty} \phi^{(|m|)}(m)$$

For any given ϕ *this sum will only have finitely many non-zero terms. It is straightforward to verify that* u *is itself a distribution, in fact it is an example of a distribution of infinite order. Consider the sequence of distributions*

$$u_M = \sum_{m=-M}^{M} D^{|m|}\delta_m$$

Let $\phi \in \mathscr{D}(\Omega)$ *be any test function. Then* $\operatorname{supp}\phi \subset B_R(0)$*, so for* $M \geq R$*, we have:*

$$u_M[\phi] = \sum_{m=-M}^{M} \phi^{(|m|))}(m) = u[\phi]$$

Thus we can write:

$$u = \sum_{i=-\infty}^{\infty} D^{|m|}\delta_m,$$

where the sum converges in $\mathscr{D}(\Omega)$*.*

d) Consider the bump functions $\tau_x\check{\phi}_\epsilon$ *for* $x \in \Omega$*, where* ϕ_ϵ *is as constructed in Theorem 1.13. Then:*

$$T_{\tau_x\check{\phi}_\epsilon} \to \delta_x$$

in $\mathscr{D}'(\Omega)$ *as* $\epsilon \to 0$*. To see this, recall that for* $\psi \in \mathscr{D}(\Omega)$*, and* ϵ *sufficiently small:*

$$T_{\tau_x\check{\phi}_\epsilon}[\psi] = \int_\Omega \phi_\epsilon(y)\psi(x-y)dy = \int_{\mathbb{R}^n} \phi_\epsilon(y)\psi(x-y)dy = \phi_\epsilon \star \psi(x).$$

By Theorem 1.13 we have $T_{\tau_x\check{\phi}_\epsilon}[\psi] \to \psi(x) = \delta_x[\psi]$*. Since* ψ *was arbitrary the result follows.*

2.4 Convolutions and the fundamental solution

What is the advantage of introducing distributions? Taking together various of the properties we've considered above, we can formulate the following proposition.

Proposition 1. *Suppose that $f \in C^0(\Omega)$ and that $T \in \mathscr{D}'(\Omega)$ satisfies the distributional equation:*

$$\sum_{|\alpha|\leq k} a_\alpha D^\alpha T = T_f,$$

for $a_\alpha \in C^\infty(\Omega)$ and that moreover there exist functions $u_\alpha \in C^0(\Omega)$ such that $D^\alpha T = T_{u_\alpha}$. Then $u := u_0 \in C^k(\Omega)$ is a classical solution of the equation

$$\sum_{|\alpha|\leq k} a_\alpha D^\alpha u = f.$$

This gives us a new approach to finding a classical solution for a linear PDE. We first show that there exists a distribution which solves the PDE and then worry about whether it is in fact a classical solution.

Now we are going to specialise to the case of linear operators of constant coefficients defined on $\mathbb{R}^n$. We will take L to be the partial differential operator

$$L := \sum_{|\alpha|\leq k} a_\alpha D^\alpha,$$

where a_α are now assumed to be constant. We wish to find solutions to the distributional equation

$$Lu = T_f$$

since by Proposition 1 we hope that this will lead to classical solutions $w \in C^k(\Omega)$ of the equation

$$\sum_{|\alpha|\leq k} a_\alpha D^\alpha w = f,$$

A powerful approach to finding a distributional solution to a PDE is to first construct a fundamental solution. For this, we will require the notion of a convolution. We will specialise to the case where $\Omega = \mathbb{R}^n$, i.e. to distributions defined on all of space.

It's useful to introduce the notion of the support of a distribution.

Definition 2.2. *A distribution $u \in \mathscr{D}'(\Omega)$ is supported in the closed set $K \subset \Omega$ if*

$$u[\phi] = 0 \qquad \forall\ \phi \in C_0^\infty(\Omega \setminus K)$$

in other words if u gives zero when applied to any test function that vanishes on K. The support of u, supp u is the set:

$$\text{supp } u = \bigcap \{K : u \text{ supported in } K\}.$$

As an intersection of closed sets, this is closed. If there exists a compact K such that u is supported in K then we say that u has compact support.

With this definition it is easy to see that for $f \in C^0(\Omega)$, $\operatorname{supp} T_f = \operatorname{supp} f$ where the support of the function f is defined in the usual way as the closure of the set on which f is non-zero. Also, one can easily check that if u is supported in K, then so is $D^\alpha u$ for any multi-index α.

Exercise($*$). Show that

$$\operatorname{supp} \delta_x = \{x\}.$$

Deduce that there is no function $f \in C^0(\Omega)$ such that $\delta_x = T_f$.

Suppose that $f \in C^0(\mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$. Recall that the convolution of the two functions is defined to be

$$(f \star \phi)(x) = \int_{\mathbb{R}^n} f(y)\phi(x-y)dy.$$

So that

$$(f \star \phi)(x) = T_f \left[\tau_x \check{\phi}\right].$$

Now, we can see that it is straightforward to define the convolution of any distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ with a test function $\phi \in \mathscr{D}(\mathbb{R}^n)$ by the formula:

$$(u \star \phi)(x) := u \left[\tau_x \check{\phi}\right].$$

Lemma 2.6 (Properties of convolutions)**.** *Suppose $u, u_i \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$. Then*

i) If

$$u_1 \star \phi = u_2 \star \phi \qquad \text{for all } \phi \in \mathscr{D}(\mathbb{R}^n),$$

then $u_1 = u_2$.

ii) $u \star \phi \in C^\infty(\mathbb{R}^n)$ and

$$D^\alpha(u \star \phi) = u \star D^\alpha \phi = D^\alpha u \star \phi. \tag{2.3}$$

iii) We have:

$$\operatorname{supp} u \star \phi \subset \operatorname{supp} u + \operatorname{supp} \phi.$$

In particular, if u has compact support, then $u \star \phi$ has compact support, and $u \star \phi \in \mathscr{D}(\mathbb{R}^n)$.

Proof. i) Notice that $u[\phi] = (u \star \check{\phi})(0)$, so we deduce that

$$u_1\phi = (u_1 \star \tilde{\phi})(0) = (u_2 \star \tilde{\phi})(0) = u_2\phi$$

for any test function ϕ, thus $u_1 = u_2$.

ii) We calculate

$$\begin{aligned}\Delta_i^h\left[u\star\phi\right](x)=\frac{(u\star\phi)(x+he_i)-(u\star\phi)(x)}{h} &= \frac{1}{h}\left(u\left[\tau_{(x+he_i)}\check{\phi}\right]-u\left[\tau_x\check{\phi}\right]\right)\\ &= u\left[\frac{1}{h}\left(\tau_{(x+he_i)}\check{\phi}-\tau_x\check{\phi}\right)\right]\\ &= u[\Delta_i^h\tau_x\check{\phi}]\end{aligned}$$

here we use the linearity of u. Now, if $\phi\in C_0^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}\lim_{h\to 0}\Delta_i^h\tau_x\check{\phi} &= \lim_{h\to 0}\frac{\phi(x+he_i-y)-\phi(x-y)}{h}\\ &= D_i\phi(x-y)\\ &= \left(\tau_x(\check{D_i\phi})\right)(y)\end{aligned}$$

with convergence in the topology of $\mathscr{D}(\mathbb{R}^n)$. As a result, using the continuity of the distribution, we have that

$$\lim_{h\to 0}\Delta_i^h\left[u\star\phi\right](x)=u\left[\tau_x(\check{D_i\phi})\right]=\left(u\star D_i\phi\right)(x).$$

Repeating the argument for higher derivatives, we conclude that the first equality of (2.3) holds. To get the second equality, we calculate:

$$\begin{aligned}D^\alpha\left[\tau_x\check{\phi}\right](y) &= \frac{\partial^{|\alpha|}}{\partial y^\alpha}\left[\phi(x-y)\right]\\ &= (-1)^{|\alpha|}\,(D^\alpha\phi)(x-y)\\ &= (-1)^{|\alpha|}\left[\tau_x(\check{D^\alpha\phi})\right](y)\end{aligned}$$

Now, using the definition of the derivative of a distribution:

$$\begin{aligned}(D^\alpha u\star\phi)(x) &= D^\alpha u\left[\tau_x\check{\phi}\right]=(-1)^{|\alpha|}u\left[D^\alpha\left(\tau_x\check{\phi}\right)\right]\\ &= (-1)^{|\alpha|}u\left[(-1)^{|\alpha|}\left[\tau_x(\check{D^\alpha\phi})\right]\right]\\ &= u\left[\tau_x(\check{D^\alpha\phi})\right]\\ &= (u\star D^\alpha\phi)(x).\end{aligned}$$

iii) Suppose for $x\in\mathbb{R}^n$ that $u\star\phi(x)\neq 0$. Then we must have

$$\operatorname{supp}u\cap\operatorname{supp}\tau_x\check{\phi}\neq\emptyset.$$

In particular, there is $z\in\operatorname{supp}u$ such that $\tau_x\check{\phi}(z)=\phi(x-z)\neq 0$. Thus $x-z\in\operatorname{supp}\phi$ for $z\in\operatorname{supp}u$, and we conclude:

$$\{x: u\star\phi(x)\neq 0\}\subset\operatorname{supp}u+\operatorname{supp}\phi.$$

Since $\operatorname{supp}\phi$ is compact and $\operatorname{supp}u$ is closed, $\operatorname{supp}u+\operatorname{supp}\phi$ is closed, and so:

$$\operatorname{supp}u\star\phi=\overline{\{x: u\star\phi(x)\neq 0\}}\subset\operatorname{supp}u+\operatorname{supp}\phi.$$ □

We would like to define the convolution of two distributions. To do this, we recall that if $f, g, h \in C^0_0(\mathbb{R}^n)$ then

$$(f \star g) \star h = f \star (g \star h), \tag{2.4}$$

so that the convolution is associative. Motivated by this, we define:

Definition 2.3. *Suppose $u_1, u_2 \in \mathscr{D}'(\mathbb{R}^n)$ are distributions and that u_2 has compact support. The convolution $u_1 \star u_2$ is the unique distribution which satisfies*

$$(u_1 \star u_2) \star \phi = u_1 \star (u_2 \star \phi)$$

for all test functions $\phi \in \mathscr{D}(\mathbb{R}^n)$.

Notice that the fact that u_2 has compact support is required for this definition to make sense, as it ensures $(u_2 \star \phi)$ is a test function. Since we can recover a distribution from its convolution with an arbitrary test function, this defines $u_1 \star u_2$.

Exercise 2.9. a) Show that for $f, g \in C^0_0(\mathbb{R}^n)$:

$$T_{f\star g} = T_f \star T_g.$$

b) Show that convolution is linear in both of its arguments, i.e. if $u_i \in \mathscr{D}'(\mathbb{R}^n)$ and u_3, u_4 have compact support then

$$(u_1 + au_2) \star u_3 = u_1 \star u_3 + au_2 \star u_3$$

and

$$u_1 \star (u_3 + au_4) = u_1 \star u_3 + au_1 \star u_4$$

where $a \in \mathbb{C}$ is a constant.

Theorem 2.7. *Suppose that $u_1, u_2 \in \mathscr{D}'(\mathbb{R}^n)$ are distributions and that u_2 has compact support. Then*

$$D^\alpha(u_1 \star u_2) = u_1 \star D^\alpha u_2 = D^\alpha u_1 \star u_2 \tag{2.5}$$

Proof. We will consider the convolution of the distribution $D^\alpha(u_1 \star u_2)$ with an arbitrary test function ϕ and use the definition of the convolution, together with the previous Lemma to shuffle the derivatives around:

$$\begin{aligned} D^\alpha(u_1 \star u_2) \star \phi &= (u_1 \star u_2) \star D^\alpha \phi \\ &= u_1 \star (u_2 \star D^\alpha \phi) \\ &= u_1 \star (D^\alpha u_2 \star \phi) \\ &= (u_1 \star D^\alpha u_2) \star \phi, \end{aligned}$$

which establishes the first equality in (2.5) since ϕ was arbitrary. For the second equality, we note

$$\begin{aligned} (u_1 \star D^\alpha u_2) \star \phi &= u_1 \star (D^\alpha u_2 \star \phi) \\ &= u_1 \star D^\alpha(u_2 \star \phi) \\ &= D^\alpha u_1 \star (u_2 \star \phi) \\ &= (D^\alpha u_1 \star u_2) \star \phi, \end{aligned}$$

again since ϕ was arbitrary we're done. □

Exercise 2.10. a) Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ then

$$\delta_0 \star \phi = \phi$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ has compact support, then

$$\delta_0 \star u = u$$

Now let us introduce the notion of a fundamental solution to a linear PDE. We say that a distribution G is a *fundamental solution* of the partial differential operator with constant coefficients L

$$L := \sum_{|\alpha|\leq k} a_\alpha D^\alpha$$

where a_α are constants if it satisfies the distributional equation:

$$LG = \delta_0$$

The reason that this is useful is the following:

Lemma 2.8. *Suppose that $G \in \mathscr{D}'(\mathbb{R}^n)$ is a fundamental solution of L and let $u_0 \in \mathscr{D}'(\mathbb{R}^n)$ be a distribution of compact support. Then the distribution $u := G \star u_0$ solves the distributional equation*

$$\sum_{|\alpha|\leq k} a_\alpha D^\alpha u = u_0.$$

Proof. First note that the linearity and differentiability properties of the convolution imply that

$$\begin{aligned} L(G \star u_0) &= \sum_{|\alpha|\leq k} a_\alpha D^\alpha (G \star u_0) \\ &= \sum_{|\alpha|\leq k} a_\alpha (D^\alpha G \star u_0) && \text{[Lemma 2.6]} \\ &= \left(\sum_{|\alpha|\leq k} a_\alpha D^\alpha G \right) \star u_0 && \text{[Linearity]} \\ &= (LG) \star u_0 \end{aligned}$$

Now, use the definition of the fundamental solution to obtain

$$L(G \star u_0) = \delta_0 \star u_0 = u_0$$

since the convolution of a distribution with δ_0 gives back the distribution (Exercise 2.10). □

Thus, once we can find a fundamental solution, we can essentially solve the equation $Lu = u_0$ for an arbitrary right hand side. Rather crudely, we can think of the Dirac delta distribution as an identity. Then the fundamental solution provides an inverse to the operator L.

2.4.1 An example: Poisson's equation

Consider the following classical PDE problem. Given $f \in C_0^2(\mathbb{R}^3)$, find a $w \in C^2(\mathbb{R}^3)$ such that:

$$\Delta w = f.$$

Here the Laplace operator is given by:

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{2.6}$$

Following the procedure outlined above, we will first turn the equation into the distributional PDE, and seek $u \in \mathscr{D}'(\mathbb{R}^3)$ such that:

$$\Delta u = T_f. \tag{2.7}$$

If we can find a G such that

$$\Delta G = \delta_0,$$

then we can write down a solution of (2.7) by convolution. There are several ways to find such a G. We will note two facts (which we will not attempt to prove at this stage)

- The Laplace operator and δ_0 are both invariant under rotations about the origin.
- The Laplace operator is elliptic. In particular, it has the property of elliptic regularity. Roughly speaking this means that if a distribution u satisfies $\Delta u = 0$ on some open set, then u is a smooth function (thought of as a distribution).

Based on these observations, it is reasonable to suspect that we can write $G = T_g$ for $g \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ a radial function $g = g(r)$. Since g must satisfy the Laplace equation away from the origin, and is spherically symmetric, we have (on changing to polar coordinates):

$$\frac{d^2}{dr^2}(rg) = 0, \qquad r > 0.$$

Thus:

$$g = \frac{A}{r} + B.$$

Clearly, since we can add a constant to any solution of (2.6), B is arbitrary and we choose it to be 0. Let us proceed leaving A arbitrary. We shall eventually see that $A = -(4\pi)^{-1}$. Note that $g \in L^1_{loc.}(\mathbb{R}^3)$, so defines a distribution in the natural way.

We wish to show that for a suitable choice of A, the distribution $G = T_g$ satisfies (2.7). To show this, we take $\phi \in \mathscr{D}(\mathbb{R}^3)$ to be arbitrary, and choose $R > 0$ such that $\operatorname{supp} \phi \subset B_R(0)$. We calculate:

$$\begin{aligned}
\Delta T_g[\phi] &= T_g \Delta[\phi] \\
&= \int_{\mathbb{R}^3} g(x) \Delta\phi(x) dx \\
&= \lim_{\epsilon \to 0} \int_{B_R(0) \setminus B_\epsilon(0)} g(x) \Delta\phi(x) dx.
\end{aligned}$$

In the last line, we use the dominated convergence theorem to justify the limit. The reason that we have inserted this limit is that on $B_R(0) \setminus B_\epsilon(0)$ the integrand is smooth, so we are entitled to apply the divergence theorem. Note that for ψ_1, ψ_2 smooth functions, we have the identity:

$$\nabla \cdot (\psi_1 \nabla \psi_2 - \psi_2 \nabla \psi_1) = \psi_1 \Delta \psi_2 - \psi_2 \Delta \psi_1.$$

Integrating this identity over $B_R(0) \setminus B_\epsilon(0)$ with $\psi_1 = g$, $\psi_2 = \phi$ and applying the divergence theorem, we have:

$$\begin{aligned}\int_{B_R(0)\setminus B_\epsilon(0)} g(x)\Delta\phi(x)dx &= \int_{\partial(B_R(0)\setminus B_\epsilon(0))} \left[g\frac{\partial \phi}{\partial n} - \phi \frac{\partial g}{\partial n}\right] d\sigma + \int_{B_R(0)\setminus B_\epsilon(0)} \Delta g(x)\phi(x)dx \\ &= \int_{\partial B_\epsilon(0)} \left[\phi\frac{\partial g}{\partial n} - g\frac{\partial \phi}{\partial n}\right] d\sigma\end{aligned}$$

In passing from the first to second line we have noted that $\Delta g = 0$ away from the origin, and also that $\phi = 0$ in a neighbourhood of $\partial B_R(0)$. The change of sign comes from the fact that $\partial B_\epsilon(0)$ is an inner boundary. Now, we estimate:

$$\sup_{y\in\partial B_\epsilon(0)} \left| g(y)\frac{\partial\phi}{\partial n}(y)\right| = \sup_{y\in\partial B_\epsilon(0)} \frac{|A|}{\epsilon}\left|\frac{\partial\phi}{\partial n}(y)\right| \leq \frac{|A|}{\epsilon}\sup_{\mathbb{R}^3} |D\phi| .$$

Thus:

$$\left| \int_{\partial B_\epsilon(0)} g\frac{\partial\phi}{\partial n} d\sigma\right| \leq 4\pi\epsilon^2 \times \frac{|A|}{\epsilon}\sup_{\mathbb{R}^3}|D\phi| \to 0,$$

as $\epsilon \to 0$. For the other term, note that on $\partial B_\epsilon(0)$, $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$, so that for $y \in B_\epsilon(0)$ we have:

$$\phi(y)\frac{\partial g}{\partial n}(y) = \frac{-A}{\epsilon^2}\phi(y).$$

We therefore have:

$$\int_{\partial B_\epsilon(0)} \phi\frac{\partial g}{\partial n} d\sigma = (-4\pi A) \times \frac{1}{|\partial B_\epsilon(0)|}\int_{\partial B_\epsilon(0)} \phi(y) d\sigma_y.$$

Now, for any $\delta > 0$, since ϕ is continuous, there exists $\epsilon > 0$ such that $|\phi(y) - \phi(0)| < \delta$ for all $y \in B_\epsilon(0)$. We estimate:

$$\begin{aligned}\left|\frac{1}{|\partial B_\epsilon(0)|}\int_{\partial B_\epsilon(0)} \phi(y)d\sigma_y - \phi(0)\right| &= \left|\frac{1}{|\partial B_\epsilon(0)|}\int_{\partial B_\epsilon(0)} (\phi(y) - \phi(0))\, d\sigma_y\right| \\ &\leq \frac{1}{|\partial B_\epsilon(0)|}\int_{\partial B_\epsilon(0)} |\phi(y)-\phi(0)|\, d\sigma_y \\ &< \delta \times \frac{1}{|\partial B_\epsilon(0)|}\int_{\partial B_\epsilon(0)} d\sigma_y = \delta.\end{aligned}$$

Thus, we conclude:

$$\lim_{\epsilon\to 0}\frac{1}{|\partial B_\epsilon(0)|}\int_{\partial B_\epsilon(0)} \phi(y)d\sigma_y = \phi(0).$$

Putting this all together, we have:

$$\Delta T_g[\phi] = (-4\pi A) \times \phi(0) = (-4\pi A)\,\delta_0[\phi].$$

Thus if $A = -(4\pi)^{-1}$, we deduce that $\Delta T_g = \delta_0$, and so $G = T_g$ is a fundamental solution.

We conclude that if $f \in C_0^0(\mathbb{R}^3)$, then a solution of the distributional equation (2.7) is given by:

$$u = T_g \star T_f = T_{g\star f}.$$

If $f \in C_0^2(\mathbb{R}^3)$, the we know that $g \star f \in C^2(\mathbb{R}^3)$, and moreover that $w = g \star f$ is a solution of the classical equation (2.6). Thus, the solution we seek is:

$$w(x) = \frac{-1}{4\pi}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}dy.$$

By examining this integral more carefully, it is possible to show that assuming $f \in C_0^2(\mathbb{R}^3)$ in order to get $w \in C^2(\mathbb{R}^3)$ is overkill. In fact, it suffices to have $f \in C_0^{0,\alpha}(\mathbb{R}^3)$ for some $0 < \alpha \le 1$, where a function belongs to the Hölder space $C_0^{0,\alpha}(\mathbb{R}^3)$ if it has compact support, and there exists a constant C such that:

$$|f(x) - f(y)| \le C\,|x-y|^\alpha\,,$$

holds for any $x, y \in \mathbb{R}^3$. This is the subject of the Schauder estimates for elliptic PDE (see "Elliptic Partial Differential Equations of Second Order", Gilbarg and Trudinger).

2.5 Distributions of compact support

Recall that as well as the space $\mathscr{D}(\Omega)$ of test functions, we also defined the space $\mathscr{E}(\Omega)$ consisting of smooth functions on Ω where the topology is such that a sequence $\{\phi_j\}_{j=1}^\infty \subset \mathscr{E}(\Omega)$ converges to zero in $\mathscr{E}(\Omega)$ if for any compact $K \subset \Omega$ and any multiindex α we have:

$$\sup_{x\in K} |D^\alpha \phi_j(x)| \to 0.$$

It is natural to define $\mathscr{E}'(\Omega)$ to be the set of continuous linear maps $\mathscr{E}(\Omega) \to \mathbb{C}$. Since the topology of $\mathscr{E}(\Omega)$ is induced by a metric, continuity is equivalent to sequential continuity. For a linear map $u : \mathscr{E}(\Omega) \to \mathbb{C}$ to belong to $\mathscr{E}'(\Omega)$, it is enough that:

$$\lim_{j\to\infty} u[\phi_j] = 0$$

for any sequence $\{\phi_j\}_{j=1}^\infty \subset \mathscr{E}(\Omega)$ which converges to zero in $\mathscr{E}(\Omega)$. Notice that if $u \in \mathscr{E}'(\Omega)$, then since $\mathscr{D}(\Omega)$ is a subspace of $\mathscr{E}(\Omega)$, $u : \mathscr{D}(\Omega) \to \mathbb{C}$ is a linear map. Moreover, we know that if $\{\phi_j\}_{j=1}^\infty$ is a sequence tending to zero in $\mathscr{D}(\Omega)$, then it also tends to zero in $\mathscr{E}(\Omega)$. Thus $u \in \mathscr{E}'(\Omega)$ is naturally an element of $\mathscr{D}'(\Omega)$, we have:

$$\mathscr{E}'(\Omega) \subset \mathscr{D}'(\Omega).$$

We are justified then in referring to elements of $\mathscr{E}'(\Omega)$ as distributions.

We can give a useful characterisation of continuity for linear maps from $\mathscr{E}(\Omega)$ to $\mathbb{C}$ as follows:

Lemma 2.9. *Suppose $u : \mathscr{E}(\Omega) \to \mathbb{C}$ is a linear map. Then u is continuous if and only if there is some compact $K \subset \Omega$, $N \in \mathbb{N}$ and $C > 0$ such that:*

$$|u[\phi]| \leq C \sup_{x\in K; |\alpha|\leq N} |D^\alpha \phi(x)|, \quad \text{for all } \phi \in \mathscr{E}(\Omega). \tag{2.8}$$

Proof. First we show that (2.10) implies that u is continuous. Pick a sequence $\{\phi_j\}_{j=1}^\infty \subset \mathscr{E}(\Omega)$ which converges to zero in $\mathscr{E}(\Omega)$. This means that for all α and any compact $K' \subset \Omega$ we have

$$\sup_{x\in K'} |D^\alpha \phi(x)| \to 0.$$

In particular, this holds with $K' = K$ and for all α with $|\alpha| \leq N$, so as $j \to \infty$ we have:

$$|u[\phi_j]| \leq C \sup_{x\in K; |\alpha|\leq N} |D^\alpha \phi_j(x)| \to 0,$$

and so u is continuous.

To show the opposite implication, we assume that (2.10) does not hold for any K, N, C. We take an exhaustion of Ω by compact sets K_i, $i = 1, 2, \ldots$ such that $K_i \subset K_{i+1}^\circ$ and $\Omega = \cup_i K_i$ (see Lemma B.6). Then since (2.10) does not hold for any K, N, C, in particular it does not hold for $K = K_j$, $N = j$ and $C = j$. Thus there must exist $\phi_j \in \mathscr{E}(\Omega)$ such that:

$$|u[\phi_j]| > j \sup_{x\in K_j; |\alpha|\leq j} |D^\alpha \phi_j(x)|.$$

We define:

$$\psi_j(x) = \frac{\phi_j(x)}{|u[\phi_j]|}$$

Clearly $\psi_j \in \mathscr{E}(\Omega)$. We claim that $\psi_j \to 0$ in $\mathscr{E}(\Omega)$. To see this, fix a compact $K \subset \Omega$ and a multiindex α. For sufficiently large j, $K \subset K_j$ and $j \geq |\alpha|$. Thus:

$$\sup_{x\in K} |D^\alpha \phi_j(x)| \leq \sup_{x\in K_j; |\beta|\leq j} \left| D^\beta \phi_j(x) \right|$$

as a result, we can estimate:

$$\begin{aligned} \sup_{x\in K} |D^\alpha \psi_j(x)| &= \frac{1}{|u[\phi_j]|} \sup_{x\in K} |D^\alpha \phi_j(x)| \\ &< \frac{\sup_{x\in K} |D^\alpha \phi_j(x)|}{j \sup_{x\in K_j; |\beta|\leq j} |D^\beta \phi_j(x)|} < \frac{1}{j} \end{aligned}$$

We conclude that $D^\alpha \psi_j$ tends to zero on K, but since α and K were arbitrary, this implies $\psi_j \to 0$ in $\mathscr{E}(\Omega)$. However, $u[\psi_j] \not\to 0$ since $|u[\psi_j]| = 1$ by construction. Thus u is not continuous. This establishes that if u is continuous, then (2.10) must hold for some K, N, C. □

With this result in hand, we can give some examples of distributions $u \in \mathscr{E}'(\Omega)$.

Example 10. *i) If* $f \in C_0^0(\Omega)$*, then defining as usual:*

$$T_f[\phi] = \int_\Omega \phi(x) f(x) dx, \qquad \textit{for all } \phi \in \mathscr{E}(\Omega)$$

we have $T_f \in \mathscr{E}'(\Omega)$*, since:*

$$|T_f[\phi]| \le \int_\Omega |f(x)|\, dx \sup_{y \in \operatorname{supp} f} |\phi(y)|\,.$$

If $f \in C^0(\Omega)$ *but* $\operatorname{supp} f$ *is not compact, then* $T_f \notin \mathscr{E}(\Omega)$*.*

ii) If $x \in \Omega$*, then then setting:*

$$\delta_x[\phi] = \phi(x) \qquad \textit{for all } \phi \in \mathscr{E}(\Omega),$$

we have $\delta_x \in \mathscr{E}'(\Omega)$*. If* K *is any compact set containing* x*, then:*

$$|\delta_x[\phi]| \le \sup_{y \in K} |\phi(y)|\,.$$

iii) The map $u : \mathscr{D}(\mathbb{R}) \to \mathbb{C}$

$$u[\phi] = \sum_{m=-\infty}^{\infty} \phi(m) \tag{2.9}$$

does not define a distribution in $\mathscr{E}'(\mathbb{R})$*. Indeed, the sum need not converge for any given element of* $\mathscr{E}(\mathbb{R})$*. For example, the constant function* $\phi(x) = 1$ *belongs to* $\mathscr{E}(\mathbb{R})$*, but the sum in (2.9) does not converge for this test function.*

In these examples we see that elements of $\mathscr{E}'(\Omega)$ have compact support, while distributions with non-compact support do not appear to make sense when applied to elements of $\mathscr{E}(\Omega)$. In fact this is a more general result:

Theorem 2.10. *Suppose* $u \in \mathscr{E}'(\Omega)$*. Then* $u \in \mathscr{D}'(\Omega)$ *and* u *has compact support. Conversely, suppose that* $u \in \mathscr{D}(\Omega)$ *has compact support. Then there exists a unique* $\tilde{u} \in \mathscr{E}'(\Omega)$ *such that*

$$\tilde{u}[\phi] = u[\phi] \qquad \textit{for all } \phi \in \mathscr{D}(\Omega).$$

We say that $\tilde{u}$ *is the extension of* u *as a linear map on* $\mathscr{E}(\Omega)$*.*

Proof. Suppose $u \in \mathscr{E}'(\Omega)$. We have already argued that $u \in \mathscr{D}'(\Omega)$ in a natural fashion, so it remains to show that $\operatorname{supp} u$ is compact. By Lemma 2.9 there exists some compact $K \subset \Omega$ and $N \in \mathbb{N}$, $C > 0$ such that:

$$|u[\phi]| \le C \sup_{x \in K; |\alpha| \le N} |D^\alpha \phi(x)|\,, \qquad \text{for all } \phi \in \mathscr{E}(\Omega).$$

Now suppose that $\operatorname{supp} \phi \subset \Omega \setminus K$. From the estimate above, we have that $u[\phi] = 0$. Thus $\operatorname{supp} u \subset K$ and we must have that $\operatorname{supp} u$ is compact.

Now suppose that $u \in \mathscr{D}'(\Omega)$ has compact support. By Lemma 1.14 we know that there exists $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ on $\operatorname{supp} u$. For $\phi \in \mathscr{E}(\Omega)$, we define:

$$\tilde{u}[\phi] = u[\chi\phi].$$

This makes sense because $\chi\phi$ is compactly supported in Ω, so $u[\chi\phi]$ is defined. If $\phi_j \to 0$ in $\mathscr{E}(\Omega)$, then $\chi\phi_j \to 0$ in $\mathscr{D}(\Omega)$ (see Exercise 3.1). Thus $\tilde{u} \in \mathscr{E}'(\Omega)$. We also note that if $\phi \in \mathscr{D}(\Omega)$, then $\chi\phi - \chi$ has support in $\Omega \setminus \operatorname{supp} u$. Thus:

$$0 = u\left[\chi\phi - \chi\right] = \tilde{u}[\phi] - u[\phi],$$

so that $\tilde{u}$ and u agree on $\mathscr{D}(\Omega)$. It remains to show that $\tilde{u}$ is unique. Suppose $\tilde{v} \in \mathscr{E}'(\Omega)$ satisfies

$$\tilde{u}[\phi] = \tilde{v}[\phi] \qquad \text{for all } \phi \in \mathscr{D}(\Omega).$$

Let $\psi \in \mathscr{E}(\Omega)$ be arbitrary. We can find $\phi_j \in \mathscr{D}(\Omega)$ such that $\phi_j \to \psi$ in $\mathscr{E}(\Omega)$ (see Exercise 3.1). We have, using the continuity of $\tilde{u}$, $\tilde{\psi}$:

$$\tilde{u}[\psi] = \lim_{j\to\infty} \tilde{u}[\phi_j] = \lim_{j\to\infty} \tilde{v}[\phi_j] = \tilde{v}[\psi].$$

Thus $\tilde{u} = \tilde{v}$, since ψ was arbitrary. □

Exercise 3.1. a) Suppose that $\{\phi_j\}_{j=1}^\infty \subset \mathscr{E}(\Omega)$ is a sequence such that $\phi_j \to \phi$ in $\mathscr{E}(\Omega)$, and $\chi \in \mathscr{D}(\Omega)$. Show that

$$\chi\phi_j \to \chi\phi \quad \text{in } \mathscr{D}(\Omega).$$

b) Show that if $\psi \in \mathscr{E}(\Omega)$, then there exists a sequence $\{\phi_j\}_{j=1}^\infty \subset \mathscr{D}(\Omega)$ such that $\phi_j \to \psi$ in $\mathscr{E}(\Omega)$.
[Hint: Take an exhaustion of Ω by compact sets and apply Lemma 1.14]

2.6 Tempered distributions

The final class of distributions that we shall consider are the *tempered* distributions. The space of tempered distributions arises as the continuous dual of $\mathscr{S}$, the Schwartz space of rapidly decreasing functions. Recall that $\phi \in \mathscr{S}$ if $\phi \in C^\infty(\mathbb{R}^n)$ and for any multiindex and any $N \in \mathbb{N}$ we have:

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha \phi(x)\right| < \infty.$$

We say that a sequence $\{\phi_j\}_{j=1}^\infty \subset \mathscr{S}$ tends to zero if:

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha \phi_j(x)\right| \to 0$$

for all $N \in \mathbb{N}$ and all multiindices α.

We define $\mathscr{S}'$ to be the continuous dual space of $\mathscr{S}$. That is to say, $u \in \mathscr{S}'$ if $u : \mathscr{S} \to \mathbb{C}$ is a continuous linear map. Since the topology of $\mathscr{S}$ can be induced by a metric, again sequential continuity is equivalent to continuity. For a linear map $u : \mathscr{S} \to \mathbb{C}$ to belong to $\mathscr{S}'$, it is enough that:

$$\lim_{j\to\infty} u[\phi_j] \to 0,$$

for any sequence $\{\phi_j\}_{j=1}^{\infty} \subset \mathscr{S}$ which converges to zero in $\mathscr{S}$.

Lemma 2.11. *Suppose $u : \mathscr{S} \to \mathbb{C}$ is a linear map. Then u is continuous if and only if there exist $N, k \in \mathbb{N}$ and $C > 0$ such that:*

$$|u[\phi]| \leq C \sup_{x\in\mathbb{R}^n;|\alpha|\leq k} \left|(1 + |x|)^N D^\alpha \phi(x)\right|, \quad \textit{for all } \phi \in \mathscr{S}. \tag{2.10}$$

Proof. See Exercise 3.2 □

Example 11. *i) Suppose $f \in L^1_{loc.}(\mathbb{R}^n)$ and there exist $C > 0, N \in \mathbb{N}$ such that*

$$|f(x)| \leq C(1 + |x|)^N.$$

Then $T_f \in \mathscr{S}'$

ii) The map:

$$\phi \mapsto \int_{\mathbb{R}^n} e^{|x|^2} \phi(x)dx$$

does not define a tempered distribution.

iii) For $\phi \in C^\infty(\mathbb{R})$, and $N \in \mathbb{N}$ we set:

$$u[\phi] = \sum_{m=-\infty}^{\infty} m^N \phi(m).$$

The sum converges for $\phi \in \mathscr{S}$, and defines a tempered distribution.

From these examples, and Lemma 2.11, we see that the tempered distributions are those that don't grow too much near infinity.

Exercise 3.2. a) Prove Lemma 2.11.
[Hint: You should argue in a similar fashion to the proof of Lemma 2.9]

b) Show that we have the inclusions:

$$\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}' \subset \mathscr{D}'(\mathbb{R}^n).$$

Exercise 3.3. Let $\{a_j\}_{j=1}^{\infty} \subset \mathbb{R}$ be a sequence of real numbers. Define for $\phi \in C^\infty(\mathbb{R})$:

$$u[\phi] = \sum_{j=1}^{\infty} a_j \phi(j)$$

provided that the sum converges. Give necessary and sufficient conditions on a_j such that:

a) $u \in \mathscr{E}'(\mathbb{R})$.

b) $u \in \mathscr{S}'$.

c) $u \in \mathscr{D}'(\mathbb{R})$.