Chapter 1

Test functions

Before we introduce distributions, we're first going to spend a bit of time studying some nice classes of *smooth* functions. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $C^k(\Omega)$ the space of all k-times continuously differentiable complex valued functions on Ω, and by

$$C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega),$$

the set of smooth functions on Ω.

When dealing with partial derivatives of high orders, the notation can get rather messy. To mitigate this, it's convenient to introduce *multi-indices*. We define a multi-index α to be an element of $(\mathbb{Z}_{\geq 0})^n$, i.e. a $n-$vector of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$. We define $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f,$$

in other words, we differentiate α_1 times with respect to x_1, α_2 times with respect to x_2 and so on. When it's unambiguous on which variables the derivative acts, we will also use the more compact notation:

$$D_i := \frac{\partial}{\partial x_i} \ ,$$

and

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x^\alpha} \ .$$

For a vector $x \in \mathbb{R}^n$, we will also use the notation:

$$x^\alpha := (x_1)^{\alpha_1}(x_2)^{\alpha_2} \cdots (x_n)^{\alpha_n}$$

The spaces $C^k(\Omega)$ and $C^\infty(\Omega)$ are vector spaces over $\mathbb{C}$, where addition and scalar multiplication are defined pointwise. If $\phi_1, \phi_2 \in C^k(\Omega)$ and $\lambda \in \mathbb{C}$, we define the maps $\phi_1 + \phi_2$, $\lambda\phi_1$ by

$$\begin{aligned} \phi_1 + \phi_2 \ : \ & \Omega \to \mathbb{C}, & \lambda\phi_1 \ : \ & \Omega \to \mathbb{C}, \\ & x \mapsto \phi_1(x) + \phi_2(x), & & x \mapsto \lambda\phi_1(x). \end{aligned} \tag{1.1}$$

Exercise($*$). Show that with the definitions (1.1) the space $C^k(\Omega)$ is a vector space over $\mathbb{C}$, and that $C^l(\Omega)$ is a vector subspace of $C^k(\Omega)$ provided $k \leq l \leq \infty$.

Definition 1.1. *If $\phi \in C^0(\Omega)$, the* support *of ϕ is the set:*

$$\operatorname{supp} \phi = \overline{\{x \in \Omega : \phi(x) \neq 0\}},$$

where the closure is understood to be relative[1] *to Ω. That is* $\operatorname{supp} \phi$ *is the closure of the set on which ϕ is not zero. We say that ϕ has compact support if* $\operatorname{supp} \phi$ *is compact.*

For $0 \leq k \leq \infty$, we define $C^k_0(\Omega)$ to be the subset of $C^k(\Omega)$ consisting of functions with compact support. $C^k_0(\Omega)$ is a vector subspace of $C^k(\Omega)$.

Theorem 1.1. *There exists a function $\psi \in C^\infty_0(\mathbb{R}^n)$ such that*

i) $\psi \geq 0$

ii) $\psi(0) \neq 0$

iii) $\operatorname{supp} \psi \subset B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}$

iv) We have:

$$\int_{\mathbb{R}^n} \psi(x) dx = 1.$$

Proof. First, we note that the function:

$$\chi(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{t}} & t > 0 \end{cases}$$

is smooth, i.e. $\chi \in C^\infty(\mathbb{R})$. Moreover, $\chi \geq 0$ and $\chi(1) \neq 0$. We define $\psi_0(x) = \chi\left(1 - |x|^2\right)$. Since the map $x \mapsto |x|^2$ is smooth, $\psi_0 \in C^\infty(\mathbb{R}^n)$. We set:

$$\psi(x) = \frac{\psi_0(x)}{\int_{\mathbb{R}^n} \psi_0(x) dx}.$$

It is easy to verify that ψ satisfies conditions $i) - iv)$. □

Corollary 1.2. *For $\Omega \subset \mathbb{R}^n$ open, $C^\infty_0(\Omega)$ is not the trivial subspace $\{0\} \subset C^\infty(\Omega)$.*

Proof. Since Ω is open, there exists ϵ, x such that the ball $B_\epsilon(x) = \{y \in \mathbb{R}^n : |y - x| < \epsilon\}$ is contained in Ω. The function $y \mapsto \psi\left[\epsilon^{-1}(y - x)\right]$ is easily seen to belong to $C^\infty_0(\Omega)$. □

Exercise 1.4. For $t \in \mathbb{R}$ let:

$$\chi(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{t}} & t > 0 \end{cases}$$

[1]If $\Omega \subset \mathbb{R}^n$ is open, and $A \subset \Omega$, then the closure of A relative to Ω is the intersection of Ω with the closure of A as a subset of $\mathbb{R}^n$. Note that the closure of A relative to Ω may not be closed as a subset of $\mathbb{R}^n$.

a) Show that $\chi \in C^\infty(\mathbb{R})$.

b) Show that there exists a function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

   i) $0 \leq \psi \leq 1$
   ii) $\operatorname{supp} \psi \subset B_2(0)$
   iii) $\psi(x) = 1$ for $|x| \leq 1$.

   Hint: First construct a positive smooth function $\tilde{\chi} : \mathbb{R} \to [0, 1]$ such that

$$\tilde{\chi}(t) = \begin{cases} 0 & t < -1 \\ 1 & t > 1 \end{cases}$$

Suppose $\Omega \subset \Omega'$, where both are open subsets of $\mathbb{R}^n$. If $\phi \in C_0^k(\Omega)$, then we can extend ϕ to a function on Ω' by setting $\phi = 0$ on $\Omega' \setminus \Omega$. This extended function will be smooth in Ω' and we do not alter the support, so in this way we see that $C_0^k(\Omega)$ is a vector subspace of $C_0^k(\Omega')$.

The following result is useful:

Lemma 1.3. *Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $K \subset \Omega$ is compact. Then $d(K, \partial\Omega) > 0$, where:*

$$d(K, \partial\Omega) := \inf_{x \in K, y \in \partial\Omega} |x - y| .$$

Proof. K is compact, so $K \subset B_R(0)$ for some $R > 0$. Let $\Omega_R = \Omega \cap B_R(0)$. Ω_R is open and bounded, with $K \subset \Omega_R$. It suffices to show that $d(K, \partial\Omega_R) > 0$. Since Ω_R is bounded, $\partial\Omega_R$ is compact. Therefore the map:

$$\begin{aligned} f \quad : \quad & K \times \partial\Omega_R \to \mathbb{R}_{\geq 0}, \\ & (x, y) \mapsto |x - y| , \end{aligned}$$

is a continuous map on a compact set, hence it achieves its minimum d at (x_0, y_0). Suppose that $d = 0$, then $x_0 = y_0$, but $x_0 \in K \subset \Omega$ and $y_0 \in \partial\Omega \in \Omega^c$ a contradiction. Thus $d > 0$ and we're done. □

Corollary 1.4. *If $\phi \in C_0^k(\Omega)$, extend ϕ to $\mathbb{R}^n$ by $\phi = 0$ on Ω^c. Define $\tau_x \phi$ by:*

$$\begin{aligned} \tau_x \phi \quad : \quad & \Omega \to \mathbb{C}, \\ & y \mapsto \phi(y - x). \end{aligned} \tag{1.2}$$

Then there exists $\epsilon > 0$ such that $\tau_x \phi \in C_0^k(\Omega)$ for all $x \in B_\epsilon(0)$.

Proof. We have

$$\operatorname{supp} \tau_x \phi = \operatorname{supp} \phi + x$$

Since $\operatorname{supp} \phi$ is compact, $\operatorname{supp} \tau_x \phi$ is just a translate of a compact set, so is compact as a subset of $\mathbb{R}^n$. We need to check that $\operatorname{supp} \tau_x \phi \subset \Omega$. We have $d(\operatorname{supp} \phi, \partial\Omega) = \delta > 0$. Set $\epsilon = \delta/2$. Then we have, by Lemma 1.3

$$\operatorname{supp} \phi + B_\epsilon(0) \subset \Omega$$

but if $x \in B_\epsilon(0)$, then $\operatorname{supp} \tau_x \phi \subset \operatorname{supp} \phi + B_\epsilon(0)$ and we're done. □

1.1 The space $\mathscr{D}(\Omega)$

So far we have defined the sets $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ as sets and shown that they have the algebraic structure of a vector space. We want to discuss notions of convergence and continuity in these spaces, and for this we shall require a topology. It turns out that the appropriate topology is somewhat subtle, and to describe it properly would take rather too long. Appendix B develops the topology in detail, for those who are interested. We shall simply quote the following result:

Theorem 1.5. *The set $C_0^\infty(\Omega)$ can be endowed with a topology τ, such that:*

i) The vector space operations of addition and scalar multiplication are continuous with respect to τ.

ii) A sequence $\{\phi_j\}_{j=1}^\infty \subset C_0^\infty(\Omega)$ tends to zero with respect to the topology τ if there exists a compact $K \subset \Omega$ such that $\operatorname{supp}\phi_j \subset K$ for all $j \in \mathbb{N}$ and for each multi-index α we have:

$$\sup_{x\in K} |D^\alpha \phi_j| \to 0,$$

as $j \to \infty$. Similarly, $\phi_j \to \phi$ with respect to τ if $\phi_j - \phi \to 0$.

We denote the set $C_0^\infty(\Omega)$ equipped with the topology τ by $\mathscr{D}(\Omega)$.

Note carefully that although we've said what it means for a sequence to converge, this is not in general enough to specify the topology completely. With some reasonable further conditions (see the Appendix) we can in fact pick out τ uniquely. For the purposes that we require, it will be enough to know about convergence of sequences.

Example 1. *Suppose $\phi \in \mathscr{D}(\Omega)$. Let δ be such that $\tau_x\phi \in \mathscr{D}(\Omega)$ for $|x| < \delta$. If $\{x_l\}_{l=1}^\infty \subset \mathbb{R}^n$ is a sequence with $|x| < \delta$, and $x_l \to 0$, then*

$$\tau_{x_l}\phi \to \phi, \qquad \textit{as } l \to \infty.$$

To see why this is so, recall that there exists $\epsilon > 0$ such that $\operatorname{supp}\phi + B_{2\epsilon}(0) \subset \Omega$. Suppose that $|x| < \epsilon$. Then

$$\operatorname{supp}\tau_x\phi = \operatorname{supp}\phi + x \subset \overline{\operatorname{supp}\phi + B_\epsilon(0)} \subset \operatorname{supp}\phi + B_{2\epsilon}(0) \subset \Omega.$$

Thus for i large enough, $\operatorname{supp}\tau_{x_l}\phi \subset K := \overline{\operatorname{supp}\phi + B_\epsilon(0)}$, where K is a compact subset of Ω. Now for any multi-index α, $D^\alpha\phi$ is a continuous function defined on a compact set, hence is uniformly continuous. In particular this implies that

$$\sup_K |D^\alpha\phi(y + x_l) - D^\alpha\phi(y)| \to 0, \qquad \textit{as } x_l \to 0,$$

which immediately gives us that $\tau_{x_l}\phi \to \phi$ in $\mathscr{D}(\Omega)$.

Example 2. *Suppose $\phi \in \mathscr{D}(\Omega)$. For $h > 0$ sufficiently small, we define the* forward difference quotient*:*

$$\Delta_i^h \phi = \frac{1}{h}\left(\tau_{-he_i}\phi - \phi\right)$$

with $\{e_i\}_{i=1}^n$ the standard basis on $\mathbb{R}^n$. Then

$$\Delta_i^h \phi \to D_i\phi, \qquad \text{as } h \to 0.$$

By the same argument as for the previous example, there exists a compact $K \subset \Omega$ such that $\operatorname{supp} \Delta_i^h \phi \subset K$ *for h sufficiently small. By the mean value theorem, for each $x \in K$, there exists $t_x \in (0, h)$ such that*

$$D^\alpha \Delta_i^h \phi(x) = \frac{D^\alpha \phi(x + he_i) - D^\alpha \phi(x)}{h} = D_i D^\alpha \phi(x + t_x e_i)$$

Fix $\epsilon > 0$. Since $D_i D^\alpha \phi$ is continuous on K (hence uniformly continuous), there exists $\delta > 0$, independent of x such that If $t_x < \delta$ we have

$$|D_i D^\alpha \phi(x + t_x e_i) - D_i D^\alpha \phi(x)| < \epsilon.$$

If we take $h < \delta$, then $t_x < \delta$ for all x and we conclude:

$$\sup_{x\in K} |D_i D^\alpha \phi(x + t_x e_i) - D_i D^\alpha \phi(x)| < \epsilon,$$

which implies

$$\sup_{x\in K} \left| D^\alpha \Delta_i^h \phi(x) - D^\alpha D_i \phi(x) \right| \to 0 \qquad \text{as } h \to 0.$$

Example 3. *Fix $\phi \in \mathscr{D}(\mathbb{R})$ with $\phi(x) \not\equiv 0$. The sequence:*

$$\phi_j(x) = \frac{1}{j}\phi(x - j), \qquad j = 1, 2, \ldots$$

does NOT *converge in $\mathscr{D}(\mathbb{R})$. We have that*

$$\sup_{\mathbb{R}} |D^\alpha \phi| \to 0, \qquad \text{as } j \to \infty,$$

but there is no compact set which contains the support of ϕ_j for all j.

For those who are interested, $\mathscr{D}(\Omega)$ is en example of a *locally convex topological vector space*. The topology of $\mathscr{D}(\Omega)$ is not inherited from a norm or a metric, so this space does not have a Banach space structure, for example.

1.2 The space $\mathscr{E}(\Omega)$

The set $C^\infty(\Omega)$ can also be given a topology in a fairly natural way. Again, we shall not go into the details of its construction, but simply assert the existence of a suitable topology.

Theorem 1.6. *The set $C^\infty(\Omega)$ can be endowed with a topology τ, such that:*

i) The vector space operations of addition and scalar multiplication are continuous with respect to τ.

ii) A sequence $\{\phi_j\}_{j=1}^\infty \subset C^\infty(\Omega)$ tends to zero with respect to the topology τ if for every compact $K \subset \Omega$ and for each multi-index α we have:

$$\sup_{x\in K} |D^\alpha \phi_j| \to 0,$$

as $j \to \infty$. Similarly, $\phi_j \to \phi$ with respect to τ if $\phi_j - \phi \to 0$.

We denote the set $C_0^\infty(\Omega)$ equipped with the topology τ by $\mathscr{E}(\Omega)$.

Again, specifying the convergent sequences does not uniquely specify the topology, so for the full construction refer to the Appendix.

Example 4. *Recall that $C_0^\infty(\Omega) \subset C^\infty(\Omega)$. If $\{\phi_i\}_{i=1}^\infty \subset C_0^\infty(\Omega)$ tends to 0 in $\mathscr{D}(\Omega)$, then $\phi_i \to 0$ in $\mathscr{E}(\Omega)$. In fact, we can say more: the inclusion map $\iota : \mathscr{D}(\Omega) \hookrightarrow \mathscr{E}(\Omega)$ is continuous.*

Example 5. *Fix $\phi \in \mathscr{D}(\mathbb{R})$ with $\phi(x) \not\equiv 0$, and consider the sequence:*

$$\phi_j(x) = j\phi(x-j), \qquad j = 1, 2, \ldots$$

This converges to 0 in $\mathscr{E}(\mathbb{R})$. For any compact K, $\operatorname{supp} \phi_j \cap K = \emptyset$ for j sufficiently large, i.e., the support of ϕ_j eventually leaves any compact set. This shows that the topology of $\mathscr{D}(\Omega)$ is not *simply the induced topology of $C_0^\infty(\Omega)$ thought of as a subspace of $\mathscr{E}(\Omega)$.*

Exercise 1.5. a) Suppose $\phi \in \mathscr{E}(\mathbb{R}^n)$. Let $\{x_l\}_{l=1}^\infty \subset \mathbb{R}^n$ be a sequence with $x_l \to 0$. Show that

$$\tau_{x_l}\phi \to \phi, \qquad \text{as } l \to \infty.$$

in $\mathscr{E}(\mathbb{R}^n)$, where τ_x is the translation operator defined in equation (1.2).

b) Suppose $\phi \in \mathscr{E}(\mathbb{R}^n)$, show that

$$\Delta_i^h \phi \to D_i\phi, \qquad \text{as } h \to 0,$$

in $\mathscr{E}(\mathbb{R}^n)$, where Δ_i^h is the difference quotient defined in Example 2.

For those who are interested, $\mathscr{E}(\Omega)$ is also a locally convex topological space, but the topology on $\mathscr{E}(\Omega)$ is induced by a complete translation invariant metric. This makes $\mathscr{E}(\Omega)$ into what is known as a *Fréchet space*. The topology is not induced by a norm, so it cannot be given a Banach space structure.

1.3 The space $\mathscr{S}$

The spaces $\mathscr{D}(\Omega)$ and $\mathscr{E}(\Omega)$ are both defined on arbitrary open sets in $\mathbb{R}^n$. The final space of functions that we wish to consider is a subspace of $\mathscr{E}(\mathbb{R}^n)$ consisting of functions which are *rapidly decreasing* near infinity.

Definition 1.2. *A function $\phi \in C^\infty(\mathbb{R}^n)$ is said to be* rapidly decreasing *if:*

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha \phi(x)\right| < \infty$$

for all multi-indices α and all $N \in \mathbb{N}$.

Notice that rapidly decreasing functions and their derivatives decay faster than any inverse power of $|x|$ as $|x| \to \infty$.

Example 6. *i) Suppose $\phi \in C_0^\infty(\mathbb{R}^n)$, then ϕ is rapidly decreasing.*

ii) The function $x \mapsto e^{-|x|^2}$ is rapidly decreasing.

Theorem 1.7. *The set of rapidly decreasing functions can be endowed with a topology τ, such that:*

i) The vector space operations of addition and scalar multiplication are continuous with respect to τ.

ii) A sequence $\{\phi_j\}_{j=1}^\infty$ of rapidly decreasing functions tends to zero with respect to the topology τ if for every multi-index α and $N \in \mathbb{N}$ we have:

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha \phi_j(x)\right| \to 0,$$

as $j \to \infty$. Similarly, $\phi_j \to \phi$ with respect to τ if $\phi_j - \phi \to 0$.

We denote the set of rapidly decreasing functions equipped with the topology τ by $\mathscr{S}$. This is often known as the Schwartz class *of functions.*

As for $\mathscr{E}(\mathbb{R}^n)$, the topology on $\mathscr{S}$ is induced by a complete translation invariant metric, so that $\mathscr{S}$ is a *Fréchet space.* The topology is not induced by a norm, so it cannot be given a Banach space structure.

Lemma 1.8. *The spaces $\mathscr{D}(\mathbb{R}^n)$, $\mathscr{S}$ and $\mathscr{E}(\mathbb{R}^n)$ satisfy:*

$$\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S} \subset \mathscr{E}(\mathbb{R}^n).$$

Moreover, the inclusion map is continuous in each case.

Exercise 1.6. a) Show that $\mathscr{S}$ is a vector subspace of $\mathscr{E}(\mathbb{R}^n)$. Show that if $\{\phi_j\}_{j=1}^\infty$ is a sequence of rapidly decreasing functions which tends to zero in $\mathscr{S}$, then $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$.

b) Show that $\mathscr{D}(\mathbb{R}^n)$ is a vector subspace of $\mathscr{S}$. Show that if $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of compactly supported functions which tends to zero in $\mathscr{D}(\mathbb{R}^n)$ then $\phi_j \to 0$ in $\mathscr{S}$.

c) Give an example of a sequence $\{\phi_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^n)$ such that

i) $\phi_j \to 0$ in $\mathscr{S}$, but ϕ_j has no limit in $\mathscr{D}(\mathbb{R}^n)$.

ii) $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$, but ϕ_j has no limit in $\mathscr{S}$.

Exercise 1.7. a) Suppose $\phi \in \mathscr{S}$. Let $\{x_l\}_{l=1}^{\infty} \subset \mathbb{R}^n$ be a sequence with $x_l \to 0$. Show that

$$\tau_{x_l}\phi \to \phi, \qquad \text{as } l \to \infty.$$

in $\mathscr{S}$, where τ_x is the translation operator defined in equation (1.2).

b) Suppose $\phi \in \mathscr{S}$, show that

$$\Delta_i^h \phi \to D_i \phi, \qquad \text{as } h \to 0,$$

in $\mathscr{S}$, where Δ_i^h is the difference quotient defined in Example 2.

1.4 Convolutions

If f, g are functions mapping $\mathbb{R}^n$ to $\mathbb{C}$, then we define the convolution of f and g to be:

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y) dy,$$

provided the integral exists. This will happen if (for example) $f \in C_0^0(\mathbb{R}^n)$ and $f \in C^0(\mathbb{R}_n)$.

Lemma 1.9. *Suppose $f, g, h \in \mathscr{S}$. Then:*

$$f \star g = g \star f, \qquad f \star (g \star h) = (f \star g) \star h.$$

and

$$\int_{\mathbb{R}^n} (f \star g)(x) dx = \int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} g(x) dx.$$

Proof. With the change of variables $y = x - z$, we have[2]

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y) dy = \int_{\mathbb{R}^n} f(x-z) g(z) dz = (g \star f)(x)$$

[2] If you're worried about a missing minus sign from the change of variables when n is odd, observe:

$$\int_{-\infty}^{\infty} k(x)dx = \int_{\infty}^{-\infty} k(-y)d(-y) = -\int_{\infty}^{-\infty} k(-y)dy = \int_{-\infty}^{\infty} k(-y)dy.$$

Next, we calculate:

$$\begin{aligned}
[f \star (g \star h)](x) &= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} g(z) h(x-y-z) dz \right) dy \\
&= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} g(w-y) h(x-w) dw \right) dy \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(w-y) dy \right) h(x-w) dw \\
&= [(f \star g) \star h](x)
\end{aligned}$$

Above we have made the substitution $w = y + z$ to pass from the first to second line, and we have used the fact that $f, g, h \in \mathscr{S}$ to invoke Fubini's theorem (Theorem A.5) when passing from the second to third line. Finally, we calculate:

$$\begin{aligned}
\int_{\mathbb{R}^n} (f \star g)(x) dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(x-y) dy \right) dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(x-y) dx \right) dy \\
&= \int_{\mathbb{R}^n} \left(f(y) \int_{\mathbb{R}^n} g(z) dz \right) dy \\
&= \int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} g(z) dy.
\end{aligned}$$

where again, the fact that $f, g \in \mathscr{S}$ allows us to invoke Fubini. □

The assumption that the functions are Schwartz is certainly overkill in this theorem. It would be enough, for example, to consider functions in $C^0_0(\mathbb{R}^n)$, or even weaker spaces, provided we can justify the application of Fubini's theorem.

Exercise 2.1 (*)**.** Suppose that we work over $\mathbb{R}^n$ and that $f, g, h \in \mathscr{S}$.

a) Show that for any multi-index α, we have that $D^\alpha f \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, i.e. that

$$||D^\alpha f||_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |D^\alpha f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

b) Define

$$\begin{aligned}
F \quad : \quad & \mathbb{R}^n \times \mathbb{R}^n, \\
& (x, y) \mapsto f(x) g(y-x).
\end{aligned}$$

Show that $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

c) For each $x \in \mathbb{R}^n$, set

$$\begin{aligned}
G_x \quad : \quad & \mathbb{R}^n \times \mathbb{R}^n, \\
& (y, z) \mapsto f(y) g(z) h(x-y-z).
\end{aligned}$$

Show that $G_x \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

1.4.1 Differentiating convolutions

A remarkable property of the convolution is that the regularity of $f \star g$ is determined by the regularity of the *smoother* of f and g. This is a result of the following Lemma:

Theorem 1.10. *Suppose $f \in L^1_{loc.}(\mathbb{R}^n)$ and $g \in C^k_0(\mathbb{R}^n)$ for some $k \geq 0$. Then $f \star g \in C^k(\mathbb{R}^n)$ and*

$$D^\alpha(f \star g) = f \star D^\alpha g,$$

for any multiindex with $|\alpha| \leq k$.

Before we prove this, it's convenient to prove a technical Lemma which will streamline the proof.

Lemma 1.11. *a) Suppose $f \in C^0_0(\mathbb{R}^n)$ and $\{z_i\}_{i=1}^\infty \subset \mathbb{R}^n$ is a sequence with $z_i \to 0$ as $i \to \infty$. Then for any $x \in \mathbb{R}^n$:*

i) $\tau_{z_j} f(x) \to f(x)$ as $j \to \infty$.

ii) $\left|\tau_{z_j} f(x)\right| \leq (\sup_{\mathbb{R}^n} |f|) \mathbb{1}_{B_R(0)}(x)$, for some $R > 0$ and all j.

b) Suppose $f \in C^1_0(\mathbb{R}^n)$ and $\{h_j\}_{j=1}^\infty \subset \mathbb{R}$ is a sequence with $h_j \to 0$ as $j \to \infty$. Then for any $x \in \mathbb{R}^n$:

i) $\Delta_i^{h_j} f(x) \to D_i f(x)$ as $j \to \infty$.

ii) $\left|\Delta_i^{h_j} f(x)\right| \leq (\sup_{\mathbb{R}^n} |D_i f|) \mathbb{1}_{B_R(0)}(x)$, for some $R > 0$ and all j.

Proof. a) i) Recall $\tau_{z_j} f(x) = f(x - z_j)$. Clearly since $z_j \to 0$, $f(x - z_j) \to f(x)$ as $j \to \infty$ by the continuity of f.

ii) Since $z_j \to 0$, there exists some $\rho > 0$ such that $z_j \in \overline{B_\rho(0)}$ for all j. Now

$$\operatorname{supp} \tau_{z_j} f = \operatorname{supp} f + z_j \subset \operatorname{supp} f + \overline{B_\rho(0)}.$$

Since the sum of two bounded set is bounded, we conclude that there exists $R > 0$ such that $\operatorname{supp} \tau_{z_j} f \subset B_R(0)$. Thus $\tau_{z_j} f = \tau_{z_j} f \mathbb{1}_{B_R(0)}$ and we estimate:

$$\left|\tau_{z_j} f(x)\right| = \left|\tau_{z_j} f(x)\right| \mathbb{1}_{B_R(0)}(x) \leq \sup_{\mathbb{R}^n} |f| \mathbb{1}_{B_R(0)}(x).$$

b) Suppose $f \in C^1_0(\mathbb{R}^n)$ and $\{h_j\}_{j=1}^\infty \subset \mathbb{R}$ is a sequence with $h_j \to 0$ as $j \to \infty$. Then for any $x \in \mathbb{R}^n$:

i) From the definition of the difference quotient and of the partial derivative:

$$\Delta_i^{h_j} f(x) = \frac{f(x + h_j e_i) - f(x)}{h} \to D_i f(x), \qquad \text{as } j \to \infty.$$

ii) Since $h_j \to 0$, there is some $k > 0$ such that $|h_j| \leq k$ for all j. We have:

$$\begin{aligned}\operatorname{supp} \Delta_i^{h_j} f &\subset \operatorname{supp} \tau_{-he_i} f \cup \operatorname{supp} f = (\operatorname{supp} f - h_j e_i) \cup \operatorname{supp} f \\ &\subset \left(\operatorname{supp} f + \overline{B_\rho(0)}\right) \cup \operatorname{supp} f \\ &\subset B_R(0)\end{aligned}$$

for some $R > 0$ since the union of two bounded sets is bounded. Thus $\Delta_i^{h_j} f = \Delta_i^{h_j} f \mathbb{1}_{B_R(0)}$. We also observe that by the mean value theorem, for any $h \in \mathbb{R}$, there exists $s \in \mathbb{R}$ with $|s| < |h|$ such that

$$\frac{f(x + h_j e_i) - f(x)}{h} = D_i f(x + s e_i)$$

thus

$$\left|\Delta_i^{h_j} f(x)\right| \leq \sup_{\mathbb{R}^n} |D_i f|.$$

Putting these two facts together, we readily find:

$$\left|\Delta_i^{h_j} f(x)\right| = \left|\Delta_i^{h_j} f(x)\right| \mathbb{1}_{B_R(0)}(x) \leq \sup_{\mathbb{R}^n} |D_i f| \, \mathbb{1}_{B_R(0)}(x).$$

□

Now, with this technical result in hand we can attack the proof our original theorem.

Proof of Theorem 1.10. 1. First we establish the result for $k = 0$. We need to show (M)
that if $f \in L^1_{loc.}(\mathbb{R}^n)$ and $g \in C^0_0(\mathbb{R}^n)$ then $f \star g$ is continuous. To show this, it suffices to show that $f \star g(x - z_j) \to f \star g(x)$ for any sequence $\{z_j\}_{j=1}^\infty$ with $z_j \to 0$. Now, note that

$$f \star g(x - z_j) = \int_{\mathbb{R}^n} f(y) g(x - z_j - y) dy = \int_{\mathbb{R}^n} f(y) \tau_{z_j} g(x - y) dy.$$

Now, sending $j \to \infty$, we are done, so long as we can justify interchanging the limit and the integral. Note that for any fixed x and all j:

$$\left|f(y) \tau_{z_j} g(x - y)\right| \leq \sup_{\mathbb{R}^n} |g| \, \mathbb{1}_{B_R(0)}(x - y) \, |f(y)|$$

for some R by the previous Lemma. Since $f \in L^1_{loc.}(\mathbb{R}^n)$ the right hand side is integrable, and so by the dominated convergence theorem:

$$\lim_{j\to\infty} f \star g(x - z_j) = \int_{\mathbb{R}^n} \lim_{j\to\infty} f(y) \tau_{z_j} g(x - y) dy = \int_{\mathbb{R}^n} f(y) g(x - y) dy = f \star g(x).$$

2. Now suppose that $f \in L^1_{loc.}(\mathbb{R}^n)$ and $g \in C^1_0(\mathbb{R}^n)$. Clearly $f \star D_i g$ is continuous by the previous argument. To show $f \star g \in C^1(\mathbb{R}^n)$, it suffices to show that for any $x \in \mathbb{R}^n$ and any sequence $\{h_j\}_{j=1}^\infty \subset \mathbb{R}$ with $h_j \to 0$ we have:

$$\lim_{j\to\infty} \Delta_i^{h_j} f \star g(x) = f \star D_i g(x).$$

Note that

$$\begin{aligned}\Delta_i^{h_j} f \star g(x) &= \frac{f \star g(x + h_j e_i) - f \star g(x)}{h} \\ &= \int_{\mathbb{R}^n} f(y) \left(\frac{g(x + h_j e_i - y) - g(x-y)}{h} \right) dy \\ &= \int_{\mathbb{R}^n} f(y) \Delta_i^{h_j} g(x-y) dy\end{aligned}$$

so that again we are done provided we can send $j \to \infty$ and interchange the limit and the integral. An argument precisely analogous to the previous case allows us to invoke the DCT and deduce that:

$$\lim_{j\to\infty} \Delta_i^{h_j} f \star g(x) = \int_{\mathbb{R}^n} \lim_{j\to\infty} f(y) \Delta_i^{h_j} g(x-y) dy = f \star D_i g(x).$$

3. The case where $f \in C_0^k(\mathbb{R}^n)$ with $k > 1$ now follows by a simple induction. □

Exercise 2.2. Show that Theorem 1.10 holds under the alternative hypotheses:

a) $f \in L^1(\mathbb{R}^n)$, $g \in C^k(\mathbb{R}^n)$ with $\sup_{\mathbb{R}^n} |D^\alpha g| < \infty$ for all $|\alpha| \leq k$.

b) $f \in L^1(\mathbb{R}^n)$ with $\operatorname{supp} f$ compact, $g \in C^k(\mathbb{R}^n)$.

We have shown that when two functions are convolved, loosely speaking the resulting function is at least as regular as the *better* of the two original functions. It is also important to know how convolution modifies the support of a function.

Lemma 1.12. *Suppose $f \in L^1_{loc.}(\mathbb{R}^n)$ and $g \in C_0^k(\mathbb{R}^n)$ for some $k \geq 0$. Then*[3]

$$\operatorname{supp} (f \star g) \subset \operatorname{supp} f + \operatorname{supp} g.$$

Proof. Recall:

$$f \star g(x) = \int_{\mathbb{R}^n} f(y) g(x-y) dy.$$

Clearly, if $f \star g(x) \neq 0$, then there must exist $y \in \mathbb{R}^n$ such that $y \in \operatorname{supp} f$ and $x - y = z \in \operatorname{supp} g$. Thus $x = y + z$ with $y \in \operatorname{supp} f$ and $z \in \operatorname{supp} g$. This tells us that:

$$\{x \in \mathbb{R}^n : f \star g(x) \neq 0\} \subset \operatorname{supp} f + \operatorname{supp} g.$$

Since $\operatorname{supp} f$ is closed and $\operatorname{supp} g$ is compact, we know that $\operatorname{supp} f + \operatorname{supp} g$ is closed, thus

$$\operatorname{supp} f \star g = \overline{\{x \in \mathbb{R}^n : f \star g(x) \neq 0\}} \subset \operatorname{supp} f + \operatorname{supp} g,$$

which is the result we require. □

[3] Strictly speaking, we haven't defined the support of a measurable function. We can do this in several ways, but the simplest is to define:

$$\operatorname{supp} f = \bigcap \{E \subset \mathbb{R}^n : E \text{ is closed, and } f = 0 \text{ a.e. on } E^c\}.$$

In other words $\operatorname{supp} f$ is the smallest closed set such that f vanishes almost everywhere on its complement.

Exercise 2.3. a) Prove the following identities for $r, s > 0$ and $x \in \mathbb{R}^n$:

i) $B_r(x) + B_s(0) = B_{r+s}(x)$
ii) $\overline{B_r(x)} + B_s(0) = B_{r+s}(x)$
iii) $\overline{B_r(x)} + \overline{B_s(0)} = \overline{B_{r+s}(x)}$

Suppose that $A, B \subset \mathbb{R}^n$. Show that:

b) If one of A or B is open, then so is $A + B$.

c) If A and B are both bounded, then so is $A + B$.

d) If A is closed and B is compact, then $A + B$ is closed.

e) If A and B are both compact, then so is $A + B$.

Exercise 2.4. Show that if $f \in C_0^k(\mathbb{R}^n)$ and $g \in C_0^l(\mathbb{R}^n)$ then $f \star g \in C_0^{k+l}(\mathbb{R}^n)$. Conclude that $\mathscr{D}(\mathbb{R}^n)$ is closed under convolution.

1.4.2 Approximation of the identity

An important use of the convolution is to construct smooth approximations to functions in various function spaces. The following theorem is very useful in constructing approximations:

Theorem 1.13. *Suppose $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfies:*

i) $\phi \geq 0$

ii) $\operatorname{supp} \phi \subset B_1(0)$

iii) $\int_{\mathbb{R}^n} \phi(x) dx = 1$

Such a ϕ exists by Theorem 1.1. Define:

$$\phi_\epsilon(y) = \frac{1}{\epsilon^n} \phi\left(\frac{y}{\epsilon}\right).$$

Then:

a) If $f \in C_0^k(\mathbb{R}^n)$, then $\phi_\epsilon \star f$ is smooth, and

$$D^\alpha (\phi_\epsilon \star f) \to D^\alpha f$$

uniformly on $\mathbb{R}^n$ for any multi-index with $|\alpha| \leq k$.

b) If $g \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, then $\phi_\epsilon \star g$ is smooth, and

$$\phi_\epsilon \star g \to g \qquad \text{in } L^p(\mathbb{R}^n).$$

c) *Suppose $f \in C^k(\mathbb{R}^n)$ with $\sup_{\mathbb{R}^n} |D^\alpha f| < \infty$ for $|\alpha| \le k$, and suppose $g \in L^1(\mathbb{R}^n)$ with $g \ge 0$, $\int_{\mathbb{R}^n} g(x)dx = 1$. Set $g_\epsilon(y) = \epsilon^{-n} g\left(\epsilon^{-1} y\right)$. Then $f \star g_\epsilon \in C^k(\mathbb{R}^n)$, and*

$$D^\alpha \left(f \star g_\epsilon\right)(x) \to D^\alpha f(x)$$

for any $x \in \mathbb{R}^n$ for any multi-index with $|\alpha| \le k$.

Proof. a) Note that the rescaling of ϕ to produce ϕ_ϵ is such that a change of variables gives:

$$\int_{\mathbb{R}^n} \phi_\epsilon(y)dy = 1.$$

By Theorem 1.10, we have that $D^\alpha(\phi_\epsilon \star f) = \phi_\epsilon \star D^\alpha f$ for any $|\alpha| \le k$. Using these two facts, we calculate:

$$\begin{aligned} D^\alpha(\phi_\epsilon \star f)(x) - D^\alpha f(x) &= \int_{\mathbb{R}^n} \phi_\epsilon(y) D^\alpha f(x-y)dy - D^\alpha f(x) \int_{\mathbb{R}^n} \phi_\epsilon(y)dy \\ &= \int_{\mathbb{R}^n} \phi_\epsilon(y) \left[D^\alpha f(x-y) - D^\alpha f(x)\right] dy \\ &= \int_{B_1(0)} \phi(z) \left[D^\alpha f(x-\epsilon z) - D^\alpha f(x)\right] dz \end{aligned}$$

where in the last line we made the substitution $y = \epsilon z$, and noted that ϕ has support in $B_1(0)$, so we can restrict the range of integration. Now, since $\phi \ge 0$, we can estimate:

$$\begin{aligned} \left|D^\alpha(\phi_\epsilon \star f)(x) - D^\alpha f(x)\right| &\le \int_{B_1(0)} \phi(z) \left|D^\alpha f(x-\epsilon z) - D^\alpha f(x)\right| dz \\ &\le \sup_{z \in B_1(0)} \left|D^\alpha f(x-\epsilon z) - D^\alpha f(x)\right| \times \int_{B_1(0)} \phi(z)dx \\ &= \sup_{z \in B_1(0)} \left|D^\alpha f(x-\epsilon z) - D^\alpha f(x)\right| \end{aligned}$$

since $\int_{\mathbb{R}^n} \phi = 1$. Now, since $D^\alpha f$ is continuous and of compact support, it is uniformly continuous on $\mathbb{R}^n$. Fix $\tilde{\epsilon} > 0$. There exists δ such that for any $v, w \in \mathbb{R}^n$ with $|x - y| < \delta$, we have

$$\left|D^\alpha f(v) - D^\alpha f(w)\right| < \tilde{\epsilon}$$

For any $x \in \mathbb{R}^n$, taking $\epsilon < \delta$, and $v = x + \epsilon z$, $w = x$ with $z \in B_1(0)$ we have $|v - w| < \delta$, so:

$$\left|D^\alpha f(x - \epsilon z) - D^\alpha f(x)\right| < \tilde{\epsilon}$$

holds for any $x \in \mathbb{R}^n, z \in B_1(0)$. We have therefore shown that for any $\tilde{\epsilon} > 0$, there exists δ such that for any $\epsilon < \delta$ we have:

$$\sup_{x \in \mathbb{R}^n} \left|D^\alpha(\phi_\epsilon \star f)(x) - D^\alpha f(x)\right| < \tilde{\epsilon}.$$

This is the statement of uniform convergence on $\mathbb{R}^n$.

(U) b) For this proof, we shall require certain facts from Measure Theory. First we require Minkowski's Integral Identity (see Exercise 2.5). This states[4] that for $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ a measurable function, we have the estimate:

$$\left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} F(x,y)dx\right|^p dy\right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x,y)|^p \, dy\right]^{\frac{1}{p}} dx$$

Now, following the calculation in the previous proof, we readily have that:

$$|(\phi_\epsilon \star g)(x) - g(x)| \leq \int_{\mathbb{R}^n} \phi(z) \, |g(x - \epsilon z) - g(x)| \, dz$$

Integrating and applying Minkowski's integral inequality, we have:

$$\begin{aligned} ||\phi_\epsilon \star g - g||_{L^p(\mathbb{R}^n)} &= \left[\int_{\mathbb{R}^n} |(\phi_\epsilon \star g)(x) - g(x)|^p \, dx\right]^{\frac{1}{p}} \\ &\leq \left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \phi(z) \, |g(x-\epsilon z) - g(x)| \, dz\right|^p dx\right]^{\frac{1}{p}} \\ &\leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \phi(z)^p \, |g(x-\epsilon z) - g(x)|^p \, dx\right]^{\frac{1}{p}} dz \\ &= \int_{\mathbb{R}^n} \phi(z) \, ||\tau_{\epsilon z} g - g||_{L^p(\mathbb{R}^n)} \, dz \end{aligned} \tag{1.3}$$

To establish our result it will suffice to set $\epsilon = \epsilon_j$, where $\{\epsilon_j\}_{j=1}^\infty \subset \mathbb{R}$ is any sequence with $\epsilon_j \to 0$, and show that $\left|\left|\phi_{\epsilon_j} \star g - g\right|\right|_{L^p(\mathbb{R}^n)} \to 0$. Note that since $\left|\left|\tau_{\epsilon_j z} g\right|\right|_{L^p(\mathbb{R}^n)} = ||g||_{L^p(\mathbb{R}^n)}$ we have:

$$\phi(z) \left|\left|\tau_{\epsilon_j z} g - g\right|\right|_{L^p(\mathbb{R}^n)} \leq 2\phi(z) \, ||g||_{L^p(\mathbb{R}^n)}$$

so the integrand is dominated uniformly in j by an integrable function. Now we claim (another Measure Theoretic / Functional Analysis fact, see Lemma 1.15) that as y varies, $\tau_y : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a continuous family of bounded linear operators. This means that for each $z \in \mathbb{R}^n$ we have:

$$\lim_{j\to\infty} \left|\left|\tau_{\epsilon_j z} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

Thus we can apply the Dominated Convergence Theorem (Theorem A.4) to the integral on the right hand side of 1.3, and conclude that

$$\lim_{j\to\infty} \left|\left|\phi_{\epsilon_j} \star g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

[4] There is more general statement for a map $F : X \times Y \to \mathbb{C}$, which is measurable with respect to the product measure $\mu \times \nu$ where (X, μ) and (Y, ν) are measure spaces.

c) Again, by Theorem 1.10, we have that $D^\alpha(f \star g_\epsilon) = D^\alpha f \star g_\epsilon$ for any $|\alpha| \leq k$. By a change of variables, we calculate:

$$D^\alpha(f \star g_\epsilon)(x) = \int_{\mathbb{R}^n} g_\epsilon(y) D^\alpha f(x-y)dy = \int_{\mathbb{R}^n} g(y) D^\alpha f(x - \epsilon z)dz$$

Now, clearly for each fixed $x \in \mathbb{R}^n$:

$$g(z) D^\alpha f(x - \epsilon z) \to g(z) D^\alpha f(x)$$

for $z \in \mathbb{R}^n$ as $\epsilon \to 0$. Furthermore,

$$|g(z) D^\alpha f(x - \epsilon z)| \leq g(z) \sup_{\mathbb{R}^n} |D^\alpha f|$$

which is an integrable function of z, so by the Dominated convergence theorem, we conclude:

$$D^\alpha(f \star g_\epsilon)(x) \to D^\alpha f(x) \int_{\mathbb{R}^n} g(z)dz = D^\alpha f(x)$$

as $\epsilon \to 0$.

□

The final application of convolutions is to the construction of cut-off functions. These are often extremely useful for localising a problem to a particular region of interest for some reason or other.

Lemma 1.14. *Suppose $\Omega \subset \mathbb{R}^n$ is open, and $K \subset \Omega$ is compact. Then there exists $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ in a neighbourhood of K.*

Proof. By Lemma 1.3, there exists $\epsilon > 0$ such that $d(K, \partial\Omega) > 4\epsilon$. We define $K_\epsilon = K + \overline{B_{2\epsilon}(0)}$. As the sum of two compact sets, K_ϵ is compact. Moreover, $K_\epsilon \subset \Omega$. Suppose ϕ_ϵ is as in Theorem 1.13. Consider:

$$\chi := \phi_\epsilon \star \mathbb{1}_{K_\epsilon}.$$

We have by Theorem 1.10 that $\chi \in C^\infty(\mathbb{R}^n)$ and from Lemma 1.12 we deduce:

$$\operatorname{supp} \chi = K_\epsilon + \operatorname{supp} \phi_\epsilon \subset K + \overline{B_{2\epsilon}(0)} + \overline{B_\epsilon(0)} = K + \overline{B_{3\epsilon}(0)} \subset \Omega.$$

Thus $\chi \in C_0^\infty(\Omega)$. Now, suppose $x \in K + B_\epsilon(0)$. Then $x + B_\epsilon(0) \subset K_\epsilon$ and so:

$$\begin{aligned}\chi(x) &= \int_{\mathbb{R}^n} \phi_\epsilon(y) \mathbb{1}_{K_\epsilon}(x-y)dy \\ &= \int_{B_\epsilon(0)} \phi_\epsilon(y) \mathbb{1}_{K_\epsilon}(x-y)dy \\ &= \phi_\epsilon(y)dy = 1.\end{aligned}$$

Thus $\chi(x) = 1$ for $x \in K + B_\epsilon(0)$, which is a neighbourhood of K. □

The following exercise and Lemma are included for completeness, as they establish results required for the proof of Theorem 1.13. The material is not examinable, and should be considered a 'bonus' for those interested in measure theoretic aspects of the theorem.

Exercise 2.5 (*)**.** Suppose that $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a positive integrable simple function,

a) Show that Minkowski's integral inequality holds for the case $p = 1$:

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x, y) dx \right| dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x, y)| \, dy dx$$

b) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hint: set $t = p^{-1}$, consider the function $\log [ta^p + (1 - t)b^q]$ and use the concavity of the logarithm

c) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $||f||_p = 1$ and $||g||_q = 1$ then

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq 1.$$

Deduce Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \ \leq ||f||_p \, ||g||_q \, , \qquad \text{for all } f \in L^p(\mathbb{R}^n), \ \ g, \in L^q(\mathbb{R}^n).$$

d) Set $G(y) = \left(\int_{\mathbb{R}^n} F(x, y) dx \right)^{p-1}$

i) Show that if $q = \frac{p}{p-1}$:

$$||G||_{L^q(\mathbb{R}^n)} = \left| \left| \int_{\mathbb{R}^n} F(x, \cdot) dx \right| \right|_{L^p(\mathbb{R}^n)}^{p-1}$$

ii) Show that:

$$\left| \left| \int_{\mathbb{R}^n} F(x, \cdot) dx \right| \right|_{L^p(\mathbb{R}^n)}^{p} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G(y) F(x, y) dy \right) dx$$

iii) Applying Hölder's inequality, deduce:

$$\left| \left| \int_{\mathbb{R}^n} F(x, \cdot) dx \right| \right|_{L^p(\mathbb{R}^n)}^{p} \leq ||G||_{L^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} ||F(x, \cdot)||_{L^p(\mathbb{R}^n)} \, dx$$

e) Deduce that Minkowski's integral inequality

$$\left[\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} F(x,y)dx\right|^p dy\right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |F(x,y)|^p\, dy\right]^{\frac{1}{p}} dx$$

holds for any measurable function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, where $1 \leq p < \infty$.

(U) **Lemma 1.15.** *Suppose $p \in [1,\infty)$ and $g \in L^p(\mathbb{R}^n)$. Let $\{z_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ be a sequence of points such that $z_j \to 0$ as $j \to \infty$. Then:*

$$\left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} \to 0.$$

Proof. 1. First, suppose $g = \mathbb{1}_Q$, where $Q = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$ is a n-box, with side-lengths $I_m = b_m - a_m$ for $m = 1, \ldots, n$. Now, since when a box is translated by a vector z_j each side is translated by a distance of at most $|z_j|$, and has area at most $I^2_{max.}$, where I_{max} is the longest side-length we can crudely estimate

$$\left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} \leq 2n\,|z_j|\, I^2_{\max}.$$

Note that this estimate requires $p < \infty$: it does not hold for $p = \infty$. We conclude that:

$$\lim_{j\to\infty} \left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

2. Now suppose $g = \mathbb{1}_A$, where A is a measurable set of finite measure. Fix $\epsilon > 0$. By the Borel regularity of Lebesgue measure, there exists a compact $K \subset A$ and an open $U \supset A$ such that $|U \setminus K| < \epsilon$. Since U is open, we can write U as a collection of open n-boxes:

$$U = \bigcup_{\alpha\in\mathscr{A}} Q_\alpha$$

Since K is compact, it is covered by a finite subset of these:

$$K \subset \bigcup_{i=1}^{N} Q_i := B.$$

Now, note that $K \subset B \subset U$, so the symmetric difference $A\Delta B \subset U \setminus K$. Thus[5] $||\mathbb{1}_A - \mathbb{1}_B||_{L^p(\mathbb{R}^n)} = |A\Delta B| < \epsilon$. By the paragraph 1 above, we know that there exists J such that for all $j \geq J$ we have:

$$\left|\left|\tau_{z_j}\mathbb{1}_B - \mathbb{1}_B\right|\right|_{L^p(\mathbb{R}^n)} < \epsilon$$

Therefore:

$$\begin{aligned}\left|\left|\tau_{z_j}\mathbb{1}_A - \mathbb{1}_A\right|\right|_{L^p(\mathbb{R}^n)} &= \left|\left|\tau_{z_j}\mathbb{1}_A - \tau_{z_j}\mathbb{1}_B + \tau_{z_j}\mathbb{1}_B - \mathbb{1}_B + \mathbb{1}_B - \mathbb{1}_A\right|\right|_{L^p(\mathbb{R}^n)} \\ &\leq \left|\left|\tau_{z_j}\mathbb{1}_A - \tau_{z_j}\mathbb{1}_B\right|\right|_{L^p(\mathbb{R}^n)} + \left|\left|\tau_{z_j}\mathbb{1}_B - \mathbb{1}_B\right|\right|_{L^p(\mathbb{R}^n)} + ||\mathbb{1}_B - \mathbb{1}_A||_{L^p(\mathbb{R}^n)} \\ &= 2\,||\mathbb{1}_A - \mathbb{1}_B||_{L^p(\mathbb{R}^n)} + \left|\left|\tau_{z_j}\mathbb{1}_B - \mathbb{1}_B\right|\right|_{L^p(\mathbb{R}^n)} \\ &< 3\epsilon\end{aligned}$$

[5] This is another point at which $p \neq \infty$ is crucial.

for all $j \geq J$. Thus

$$\lim_{j\to\infty} \left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

3. Now suppose g is a simple function, i.e. $g = \sum_{i=1}^N g_i \mathbb{1}_{A_i}$ for $g_i \in \mathbb{C}$ and A_i measurable sets of finite measure. Then we have:

$$\left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} \leq \sum_{i=1}^N |g_i| \left|\left|\tau_{z_j} \mathbb{1}_{A_i} - \mathbb{1}_{A_i}\right|\right|_{L^p(\mathbb{R}^n)}$$

   so as $j \to \infty$ we have:

$$\lim_{j\to\infty} \left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

4. Now suppose that $g \in L^p(\mathbb{R}^n)$. Fix $\epsilon > 0$. Recall that there exists a simple function $\tilde{g}$ such that $||g - \tilde{g}||_{L^p(\mathbb{R}^n)} < \epsilon$. By the previous part, we can find J such that $\left|\left|\tau_{z_j}\tilde{g} - \tilde{g}\right|\right|_{L^p(\mathbb{R}^n)} < \epsilon$ for all $j \geq J$. Now:

$$\begin{aligned}\left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} &= \left|\left|\tau_{z_j} g - \tau_{z_j}\tilde{g} + \tau_{z_j}\tilde{g} - \tilde{g} + \tilde{g} - g\right|\right|_{L^p(\mathbb{R}^n)} \\ &\leq \left|\left|\tau_{z_j} g - \tau_{z_j}\tilde{g}\right|\right|_{L^p(\mathbb{R}^n)} + \left|\left|\tau_{z_j}\tilde{g} - \tilde{g}\right|\right|_{L^p(\mathbb{R}^n)} + ||\tilde{g} - g||_{L^p(\mathbb{R}^n)} \\ &= 2\,||g - \tilde{g}||_{L^p(\mathbb{R}^n)} + \left|\left|\tau_{z_j}\tilde{g} - g\right|\right|_{L^p(\mathbb{R}^n)} \\ &< 3\epsilon\end{aligned}$$

   Thus, we conclude that

$$\lim_{j\to\infty} \left|\left|\tau_{z_j} g - g\right|\right|_{L^p(\mathbb{R}^n)} = 0.$$

   and we're done. □